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ALGEBRA MADE EASY

(MATRICULATION ALGEBRA)

FOR

MATRICULATION STUDENTS OF THE INDIAN UNIVERSITIES.

(With an appendix completing the syllabuses prescribed for the Matriculation Examination of the Calcutta and the other Indian Universities)

BY

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AND

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PREFACE.

The present work is intended as a text-book on Algebra for all classes of students in our schools It differs, however, in several respects from the existing text-books on the subject at present in use

Algebra like every other branch of Mathematics should be studied more as a subject for mental discipline than for anything else. An intelligent grasp of principles therefore is to be chiefly aimed at and not the mere learning by rote of a certain number of rules with some readiness in their application This is the ideal I have ever kept in view in the preparation of this work

The elementary principles of the subject have been dwelt upon at considerable length in the earlier chapters of the book. The full import of negative quantities has been explained, it is believed, with some degree of clearness, almost at the very outset, and rules for their addition and subtraction have subsequently been deduced therefrom by a very simple mode of reasoning.

The proposition of each article after being clearly demonstrated has been copiously illustrated by a number of select examples, a much larger number of other examples, arranged progressively, has then been added as an exercise for the student The last article of each chapter consists of a number of miscellaneous examples fully worked out as interesting illustrations of special artifices, these again are followed by similar others for exercise.

The chapters on Formulæ and Factors will, it is hoped, be particularly acceptable to the young learner The subject of factorisation has been treated exhaustively as far as the limits of this work would allow. The last chapter, on Elimination and Miscellaneous Artifices, will, I hope, be of considerable use to the more advanced student

Entrance Examination Papers of the Calcutta University from 1858 to 1890 will be found at the very end. The more important and difficult problems from these papers are fully worked out in the body of the work in illustration of the principles upon which their solutions depend, whilst others, comparatively simpler, have been suitably introduced among the exercises just to give the student an opportunity of reassuring himself, when successful in working them out with unaided exertion, that his knowledge has, to some extent at least, come up to the University standard With the examination papers are also given references to the pages where these problems are to be found in the body of the work

Instead of ending the book with a collection of miscellaneous examples promiscuously arranged, I have added a number of miscellaneous examples in the form of separate examination papers, any one of which may be regarded as a good exercise for the student at a sitting of about two hours and a half

The entire book contains nearly 3000 examples in all, of which over 400 are fully worked out Many of these examples have been specially devised for this work whilst for the rest I am indebted to several of the standard works of English authors as also to many of the examination papers of the Indian and English Universities

I have attempted to make the work useful to the school student as a means of acquiring algebraical skill along with a sound knowledge of principles, and I have spared no

pains for it. It is now for all experienced teachers of Mathematics to judge as to how far I have been successful in my endeavour. To gentlemen interested in the cause of education I shall be much obliged if they will kindly communicate to me any corrections or suggestions that they may consider necessary for the improvement of the work.

DACCA, *March*, 1890. K. P. BASU.

PREFACE TO THE SECOND EDITION.

A few words of explanation seem to be necessary in connection with the publication of this edition The First Edition having been published rather unseasonably last year, I did not at all anticipate that a second edition would be in demand so soon. Accordingly the work of re-publication was not taken in hand earlier than January last But the book beginning to be received with increased favour in different educational circles with the commencement of the new academic session, the First Edition, consisting of 2250 copies, was found to be exhausted before the end of the last month Hence, in the interests of the students of all those schools in which the book has been adopted as a text-book, my publisher had no other alternative than to hasten the work by all possible means. In consequence of this, I am sorry, I have not been able to give the book as thorough a revision as I intended, nor to effect such improvements as have been kindly suggested by some friends.

DACCA. *March*, 1891 K. P. BASU.

PREFACE TO THE FIFTH EDITION

In this Edition the bulk of the work has increased by about 60 pages The additions that have been made are as follows —(1) An increase in the number of examples of exercise in the earlier chapters of the book, (2) the insertion of examples with *fractional indices* in the chapters on *Multiplication and Division*, (3) The introduction of three sets of Miscellaneous Exercises in suitable places in the body of the work, (4) an article on the method of finding the *Cube Root* of a Compound Algebraical Expression, and (5) a chapter on *Quadratic Equations* For several of these improvements I am indebted to the kind and repeated suggestions of friends who are practical workers in the field of education. It is therefore hoped that the present Edition will be found considerably more useful than its predecessors

DACCA *January*, 1894. K. P. BASU.

PREFACE TO THE SIXTH EDITION

In this Edition the book has been thoroughly revised and answers to the examples in all the exercises have been carefully verified, Some additions and alterations have also been occasionally made, but they do not deserve any special mention. I am indebted to several friends for their kindness in pointing out errors and misprints My special thanks are due to Babu Bipinbihary Ganguli, B A, Teacher, Jubilee School, Dacca and to Moulvie Abdullah Khan, Teacher, D B. School, Dipalpur (Montgomery).

DACCA . *April*, 1895. K. P BASU.

PREFACE TO THE SEVENTEENTH EDITION.

The syllabuses for the compulsory and Additional Papers in Mathematics, prescribed for the Matriculation Examination, under the new regulations of the Calcutta University, respectively comprise the following subjects in Algebra —

Compulsory Paper: The Four Simple Rules; Proportion, Simple Equations, Greatest Common Measure; Least Common Multiple, Graphs of Simple Equations.

Additional Paper Quadratic Equations with one unknown quantity, Extraction of Square root, Graphs of Pure Quadratic Equations (excluding constructions with different scales along two axes), Arithmetical and Geometrical Progressions, The Elementary Laws of Indices.

With the addition of an appendix, containing Chapters on Graphs, and Arithmetical and Geometrical Progressions this treatise now contains all that is included in the Algebraical Curriculum for the Matriculation Examination. It is hoped, therefore, that in its present form, the book will completely serve the purpose of a "Matriculation Algebra."

DACCA *April*, 1908 K. P. BASU.

PREFACE TO THE TWENTY-FOURTH EDITION.

In this Edition, in place of old University Papers, recent up to-date Examination Papers of all the Indian Universities have been inserted, and to enhance the usefulness of the book the syllabuses of all the Universities have been added. A Chapter on *Variation* has also been given in the Appendix to complete the syllabus of the Bombay University The book in its present form satisfies the syllabuses of all the Indian Universities.

CALCUTTA *January*, 1916

TABLE OF CONTENTS.

INTRODUCTION.

CHAPTER I.

DEFINITIONS AND EXPLANATION OF SIGNS

CHAPTER II

POSITIVE AND NEGATIVE QUANTITIES

CHAPTER III.

ADDITION

CHAPTER IV

SUBTRACTION—REMOVAL OF BRACKETS

CHAPTER V.

MULTIPLICATION

CHAPTER VIII

FACTORS

CHAPTER IX.

HIGHEST COMMON FACTORS

CHAPTER X.

LOWEST COMMON MULTIPLE

CHAPTER XI.

FRACTIONS

CHAPTER XII.

SIMPLE EQUATIONS

CHAPTER XIII.

PROBLEMS LEADING TO SIMPLE EQUATIONS

CHAPTER XIV.

SIMULTANEOUS EQUATIONS OF THE FIRST DEGREE.

CHAPTER XVIII.

SQUARE AND CUBE ROOTS

CHAPTER XIX.

RATIO AND PROPORTION

1. *Ratio.*

2 *Proportion*

CHAPTER XX

ELIMINATION AND MISCELLANEOUS ARTIFICES

CHAPTER XXI.

QUADRATIC EQUATIONS

APPENDIX

ALGEBRA MADE EASY.

INTRODUCTION.

1. How things are measured and represented by number. This will be best explained by taking up some particular instances familiar to the student

(i) If we want to know the length of a piece of cloth we are satisfied when we find how often this length contains a smaller length called a *cubit* (the distance between the elbow and the tip of the middle finger).

(ii) If we want to know the distance between Dacca and Calcutta we are satisfied when we are told how often this distance contains a smaller distance called a *mile*.

(iii) If we want to know the value of a sum of money we are satisfied when we are told how often this sum contains a smaller sum called a *rupee*.

(iv) If we want to know the weight of a quantity of rice we are satisfied when we find how often this weight contains a smaller weight called a *seer*

From the above instances it is clear that whenever we have to measure a thing we do so by finding how often it contains a smaller thing of the same kind The "smaller thing" chosen for this purpose is called the *unit*, and the *number* which shows how often this unit is contained in the thing measured is called the *numerical measure* (or simply, the *measure*) of the latter —thus in the first instance, the *unit of length* is a cubit, in the second, the *unit of distance* is a *mile*, in the third, the *unit of money* is a *rupee*; and in the fourth instance, the *unit of weight* is a *seer*. Again, if we know that the piece of cloth is 10 cubits long, that the distance between Dacca and Calcutta is 260 miles, that the sum of money is 500 rupees. and that the weight of the

rice is 25 seers, then, 10 is the *measure* of the length of the cloth, 260 is the *measure* of the distance between Dacca and Calcutta, 500 is the *measure* of the sum of money, and 25 is the *measure* of the weight of the rice

A thing is said to be *represented* by the number which shows how often that thing contains the unit of its kind, thus in the above instances, the length of the piece of cloth is represented by 10, the distance between the two places is represented by 260, and so on

Note (1) Such expressions as "a sum of money estimated in pounds = 30," "a distance estimated in miles = 25" and the like, respectively mean "the numerical measure of a sum of money when a £ is the unit, is 30" "the numerical measure of a distance when the unit is a mile, is 25," &c

Note (2) It must be clearly understood that one and the same thing will be represented by different numbers when the units are different, thus taking a foot as the unit, a length of 10 feet is represented by 10, but if the unit be 2 feet, the same length is represented by 5

Example 1 If the *unit of length* be a foot, what will be the measure of 5 yards and 2 feet ?

5 yards and 2 feet, being equivalent to 17 feet, evidently contains the unit of length (*i.e*, a foot) 17 times

Hence the required *measure* is 17.

Example 2. If a minute and a half be represented by 30, what is the *unit of time* ?

A minute and a half is equivalent to 90 seconds

Now, since 30 is the *measure* of 90 seconds, it is clear that the *unit of time* is contained 30 times in 90 seconds

Hence, the *unit of time* is $\frac{1}{30}$th part of 90 seconds, and is therefore equal to 3 seconds

Exercise (1).

1. What will be the *measure* of a *maund* and 25 *seers*, when a *seer* is the *unit of weight* ?

2. What will be the measure of the same weight, when 5 *seers* is the *unit* ?

3. If a distance of 300 miles be represented by 25, what is the *unit of distance* ?

4. If the same distance be represented by 40, what is the *unit* ?

5. If a sum of 400 rupees be represented by 16, what will be the *measure* of Rs 225 ?

6. If a length of 8 feet 4 inches be represented by 25, what will be the *measure* of 4 feet ?

7. What must be the *unit of time* in order that 2 hours and 15 minutes may be represented by 3 ?

8. If the *unit of time* be 15 seconds, what time will be represented by 60 ?

9. If the *unit of weight* be $7\frac{1}{2}$ lbs, what number will represent $2\frac{1}{7}$ cwt ?

10. If 4 square feet be the *unit of area*, what number will represent an area of 9 square inches, and what will represent 9 square yards ?

11. If an area of 125 sq ft be represented by $8\frac{1}{3}$, how many square yards are there in 3 times the unit area ?

12 What is the unit of money if a sum of £3. 7*s.* 6*d.* be represented by 9 ?

13. If 7*s* 8*d* be the unit of money, what will be the measure of £7 13*s* 4*d.* ?

14 If Rs 2 13*a* 7*p.* be the unit of money, what will be the measure of Rs 25 10*a* 3*p* ?

15. If 23 seers 5 chattacks be the unit of weight, what will be the measure of 16 maunds $12\frac{3}{4}$ seers ?

16. If Rs 20. 10*a.* be represented by $5\frac{1}{2}$, what will be the measure of Rs 45, supposing the new unit to be 3 times the former ?

17. If 273 be the measure of 19 cwt. 2 qrs, what number will represent one ton, supposing the new unit to be one sixteenth of the former ?

18. If 84 be the measure of 39 yds 2 ft, what number will represent 75 yards, supposing the new unit to be three-seventeenths of the former ?

19. If 26 days 10 hours and 26 minutes be represented by 120, what number will represent a leap year, supposing the new unit to be 47 minutes 13 seconds less than the former ?

20. In the preceding example what would be the answer if the latter unit exceeded the former by 6 hours 54 minutes 47 seconds.

2 Different uses of the word Quantity.

(i) Any thing that can be represented by *number* is called a *quantity* Thus time, weight, money, distance, &c., which all admit of numerical representation, as shown in the preceding article, are quantities.

(ii) *Quantity* is also often used in the sense of *number*, integral or fractional

(iii) An Algebraical *expression* also is sometimes called a *quantity* [We shall refer to this again in its proper place]

N B Quantities like weight, money, distance, area, &c, are often spoken of as *concrete quantities*, as distinguished from *numerical quantities* which mean only Arithmetical *numbers*, integral or fractional

[Note Any *whole* number is called an *integer* or an *integral* number]

3. What is Algebra ? Algebra, like Arithmetic is a science of *numbers* with this distinction that the numbers in Algebra are *generally* denoted by *letters* instead of by *figures*

Hence, whenever concrete quantities come under the domain of Algebra, it is *only* their numerical *measure* (*i e*, the abstract numbers which represent them) with which we must concern ourselves

4. Symbols The letters used to denote numbers, as well as the several signs that are used either for indicating the operations to be performed upon the numbers to which they are attached or as abbreviations, are called *symbols*

The letters as distinguished from the signs, are called *symbols of quantity*

CHAPTER I

DEFINITIONS AND EXPLANATION OF SIGNS.

1 The Plus Sign The sign + is read *plus* and when placed before a number indicates that the number is to be *added* to what precedes it Thus $a+b$ (which is read *a plus b*) means that the number denoted by b is to be *added* to that denoted by a, hence, if a denotes 5 and b denotes 3, $a+b$ denotes 8 Again, $a+b+c$ means that the number denoted by b is to be added to that denoted by a, and to the

result, thus obtained, is to be added the number denoted by c, hence, if a, b, c denote 5, 3, 2 respectively, $a+b+c$ denotes 10

2. The Minus Sign The sign $-$ is read *minus* and when placed before a number indicates that the number is to be *subtracted* from what precedes it Thus $a-b$ (which is read *a minus b*) means that the number denoted by b is to be *subtracted* from that denoted by a, hence, if a denotes 8 and b denotes 3, $a-b$ denotes 5. Again, $a-b-c$ means that the number denoted by b is to be subtracted from that denoted by a, and from the result, thus obtained, the number denoted by c is to be subtracted, hence, if a, b, c denote 8, 3, 1 respectively, $a-b-c$ denotes 4

N B When any number of quantities are connected with one another by the signs *plus* and *minus*, the order of the operations is *from left to right* Thus $a-b+c$ means that the number denoted by b is to be subtracted from that denoted by a, and to the result, thus obtained, is to be added the number denoted by c

3. The Sign Plus or Minus. The sign $\pm$ is read *plus* or *minus* and when placed before a number indicates that the number is to be *either added to* or *subtracted from* what precedes it Thus if a denote 7 and b denote 2, $a \pm b$ (which is read *a plus* or *minus b*) denotes *either* 9 *or* 5.

4. The Sign of difference. The sign $\sim$ when placed between two numbers indicates that the less of the two is to be subtracted from the greater. Thus, if a denote 5 and b denote 8, $a \sim b$ denote 3

5. The Sign of Multiplication The sign $\times$ is read *into* and when placed between two numbers indicates that the number on the right of it is to be *multiplied* by that on the left.

Thus $a \times b$ (which is read *a into b*) means that the number denoted by b is to be multiplied by that denoted by a; hence, if a denote 5 and b denote 3, $a \times b$ denotes 5 *times* 3, or, 15

The sign of multiplication is generally omitted when its position is between two numbers either (1) *both* of which are denoted by letters, or (2) the *first* of which is denoted by a *figure* and the second by a letter. Thus ab is used for $a \times b$, and $4a$ for $4 \times a$

Note The reason why 83 cannot be used for 8×3 is clear, because in Arithmetic 83 has already been understood to mean $80+3$

Sometimes the sign $\times$ is replaced by a dot; thus $a.b$ and 5.4 respectively mean the same as $a\times b$ and 5×4. The dot so used is always placed as shown in the above instances in order to distinguish it from the decimal point which is put a little higher up; thus $5\cdot 4$ is read *five into four* whereas $5\cdot4$ is read *five decimal four*.

6. The Sign of Division. The sign $\div$ is read *by* and when placed between two numbers indicates that the number on the left of it is to be *divided* by that on the right. Thus $a\div b$ (which is read *a* by *b*) means that the number denoted by a is to be divided by that denoted by b; hence, if a denote 6 and b denote 3, $a\div b$ denotes 2. Similarly, $a\div b\div c$ means that the number denoted by a is to be divided by that denoted by b, and the result, thus obtained, is to be divided by the number denoted by c.

N. B. When any number of quantities are connected together by the signs of multiplication and division, the order of the operations is always *from left to right*. Thus $a\times b\div c$ means that the number denoted by b is to be multiplied by that denoted by a, and the result thus obtained, is to be divided by the number denoted by c. Similarly, $a\div b\times c$ means that the number denoted by a is to be divided by that denoted by b, and by the result thus obtained, is to be multiplied the number denoted by c.

Note. *a divided by b* is also often expressed as $\frac{a}{b}$; thus $\frac{a}{b}$ means the same as $a\div b$.

7. Expression; Term. Any intelligible collection of letters, figures, and *signs of operation* is called an *Algebraical Expression*. Such a collection is also sometimes called an *Algebraical Quantity*, or briefly, a *Quantity*.

[See Art. 2 Introduction.]

[Note. Signs like $+$, $-$, $\times$, $\div$, which indicate the operations to be performed upon the numbers to which they are attached, are called *signs of operation*.]

The parts of an Algebraical Expression that are connected by the sign $+$ or $-$ are called its *terms*.

Thus $5a+ab\div c\times d-8c\times f\div g$ is an algebraical expression of which the terms are $5a$, $ab\div c\times d$, $8c\times f\div g$.

Expressions are either **simple** or **compound**. A *simple* expression is one which has not parts connected by the sign $+$ or $-$, *i.e.*, which consists of only one term, as $3ab$, and is also called a *Monomial*. A *compound* expression consists of two or more terms; if it consists of two terms as $2a+5bcd$, it is called a *Binomial*; if of three terms, as $a+bc+8efg$, a *Trinomial*; and if of more than three terms, a *Multinomial* or a *Polynomial*.

8. Sign of Equality. The sign $=$ is read "equals" or "is equal to" and when placed between two *expressions* indicates that they are equal to one another. Thus $b+c=a$ (which is read *b plus c equals a*) means that the number denoted by $b+c$ is equal to that denoted by a.

Examples.

N B (1) A distinction must be observed between $a-b\times c$ and $a-bc$ The latter means that the number denoted by a is to be divided by that denoted by bc, whereas the former means that the number denoted by a is to be divided by that denoted by b, and by the result, thus obtained, is to be multiplied the number denoted by c That is to say, when the sign of multiplication is omitted between any number of quantities, the result obtained by multiplying them together is to be regarded as a *single quantity*

N B (2) In finding the value of any *expression* the values of the several *terms* which it contains must be first determined by the process mentioned in the note of art 6 and *afterwards* the value of the whole expression is to be found by the process mentioned in the note of art 2 Thus in finding the value of the expression $a\times b-c\div d\times e+f\times g$ we must first of all find the values of the three terms, namely, $a\times b$, $c\div d\times e$, and $f\times g$, then subtract the value of the second term from that of the first, and to the result thus obtained, add the value of the third

The above principles will be sufficiently illustrated by the following examples :—

Example 1. If $a=2$, $b=3$, $c=5$, find the value of $5a+8b+7c$

$$5a = 5\times a = 5\times 2 = 10\ ,$$
$$8b = 8\times b = 8\times 3 = 24\ ,$$
$$7c = 7\times c = 7\times 5 = 35.$$

Therefore, $5a+8b+7c = 10+24+35$
$= 34+35$
$= 69.$

Example 2. If $a=8$, $b=5$, $c=2$, find the value of $6a-5b+4c$

$$6a = 6\times a = 6\times 8 = 48\ ;$$
$$5b = 5\times b = 5\times 5 = 25\ ;$$
$$4c = 4\times c = 4\times 2 = 8.$$

Therefore, $6a-5b+4c = 48-25+8$
$= 23+8 = 31.$

Example 3. If $m=3$, $n=7$, $t=9$, $v=4$, find the value of $7m\div 2n\times 8t\div 3v$.

As the order of the operations is from *left to right*, we must proceed as follows —Divide $7m$ by $2n$, multiply $8t$ by the result, and then divide the result, thus obtained, by $3v$

Now, (1) $7m-2n = \frac{7m}{2n} = \frac{7\times 3}{2\times 7} = \frac{3}{2}$,

(2) $\frac{3}{2}\times 8t = \frac{3}{2}\times 8\times 9 = 3\times 4\times 9$,

(3) $3\times 4\times 9-3v = \frac{3\times 4\times 9}{3\times 4} = 9$

Hence, the required value $= 9$

Example 4. If $a=1, b=2, c=3, d=6, e=5, f=0$, find the value of $abc-d-b\times a+def+b-a\times c-d\div bc$

The given expression consists of 5 terms, namely, abc, $d-b\times a$, def, $b-a\times c$, and $d-bc$

Now, (1) $abc = a\times b\times c = 1\times 2\times 3 = 6$;

(2) $d-b\times a = 6-2\times 1 = 3\times 1 = 3$;

(3) $def = d\times e\times f = 6\times 5\times 0 = 0$;

(4) $b\div a\times c = 2-1\times 3 = 2\times 3 = 6$;

(5) $d-bc = \frac{d}{bc} = \frac{6}{2\times 3} = 1.$

Hence, the required value $= 6-3+0+6-1$

$= 3+6-1 = 8$

Exercise (2).

If $a=2, b=4, c=8$, find the numerical values of the following expressions,—

1. $a+b\times c$
2. $c-a\times b$.
3. $c-b\times a$.
4. $c-ba$.
5. $c-3\times a$
6. $c-3a$
7. $c-b-a$
8. $a+c-b$.
9. $3c-4b+2a$.
10. $c-b-a+c-b$.
11. $c-b\div 2\times a$.
12. $c-b-2a$.
13. $5c-2b$
14. $5\times c-2\times b$.
15. $4ab-c-4\times a+b-2a$.
16. $80-b\times ca+80-bc\times a$.
17. $64\div c\times b\times a-64-cba$.

18. $3bc \div 16a + 5c \div 16 \times a - c \div 2b \times b \div a \times 4$

19. $10c - ab \times 2 + 32b \div 2ac + 15c - 3b \times c \div 5$

20. $48c \div b \div a \times 6 - 4b - 3c - 2b \div 4 \times 3 \div a \times 8$
$+ 6a \div c \div 2 \times b - 3 \times 5.$

If $m = 2$, $n = 3$, $p = 4$, $q = 0$, $r = 7$, $s = 10$, find the numerical values of the following expressions —

21. $8m - 3p \div mn + q \times 3r + 5s \div 2 \times p$

22. $s \times 6 \div 5m \times 8p \div 16n$ **23.** $24 \div 3p \times 4s \div 5r \times 14m$

24. $mnr + 5qs - 3s \div m \div 5n + 4r \div 3p \times 6m$

25. $3 \times r \div 5 \times s - 7 \times p - 8rs \div m \div 3 \times n - 7p + 5m \div 2r \times 7$

9. Factor. If any number be equal to the product of two or more numbers, each of the latter is called a *factor* of the former

[Note The *product* of two or more numbers is the result obtained by multiplying them together]

Thus 3, 5, and 7 are the factors of 105, because $105 = 3 \times 5 \times 7$;

similarly, 3, a, b and x are the factors of $3abx$, because $3abx = 3 \times a \times b \times x$.

10. Co-efficient. The number expressed in figures or symbols, which stands before an algebraical quantity as a multiplier, is called its *co-efficient*. Thus, in $5abc$, 5 is the co-efficient of abc, $5a$ is the co-efficient of bc, and $5ab$ is the co-efficient of c.

A co-efficient which is a purely numerical quantity is called a *numerical co-efficient*; thus, in $5abc$, the co-efficient of abc is numerical.

A co-efficient which is not wholly numerical is called a *literal* co-efficient; thus, in $5abc$, co-efficients of bc and c are *literal*.

[Note When no arithmetical number stands before a quantity the number 1 is understood thus a is understood to mean $1a$]

11. Power; Index; Exponent. If a quantity be multiplied by itself any number of times, the product is called a *power* of that quantity. Thus $a \times a$, $a \times a \times a$, $a \times a \times a \times a$, &c, are powers of a.

$a \times a$ is called the *second power* or *square* of a and is written a^2;

$a \times a \times a$ is called the *third power* or *cube* of a and is written a^3,

$a \times a \times a \times a$ is called the *fourth power* of a and is written a^4,

$a \times a \times a \times a \times a \times$ &c to n factors is called the *n^{th} power* of a and is written a^n

The small figure or letter placed above a quantity and to the right of it to express its power is called the **Index** or **exponent** of that power Thus 2, 3, 5, m are respectively the *indices* or *exponents* of a^2, a^3, a^5, a^m

[Note a^2 is usually read "a squared", a^3 is read "*a cubed*", a^4 is read "*a to the fourth*" or simply, "*a fourth*", and so on Thus a^n is read "*a to the nth*" or "*a nth*"

The quantity a itself is called the *first* power of a and thus a is understood to mean a^1]

Examples.

Example 1. If $a = 3$, find the numerical value of $a^5 - 5a$

We have
$$a^5 = a \times a \times a \times a \times a = 3 \times 3 \times 3 \times 3 \times 3 = 243,$$
and
$$5a = 5 \times a = 5 \times 3 = 15$$

Hence, the given expression $= 243 - 15 = 228$.

Example 2. If $a = 4$, find the numerical value of $2a^5 - 5a^2$

We have
$$2a^5 = 2 \times a \times a \times a \times a \times a = 2 \times 4 \times 4 \times 4 \times 4 \times 4 = 2048,$$
and
$$5a^2 = 5 \times a \times a = 5 \times 4 \times 4 = 80$$

Hence, the given expression $= 2048 - 80 = 1968.$

Example 3. If $a = 2$, $b = 3$, $c = 4$, $d = 5$, find the numerical value of $\dfrac{a^5b^3d}{c^2}$.

$$\text{The given expression} = \frac{a\times a\times a\times a\times a\times b\times b\times b\times d}{c\times c}$$
$$= \frac{2\times 2\times 2\times 2\times 2\times 3\times 3\times 3\times 5}{4\times 4}$$
$$= 2\times 3\times 3\times 3\times 5 = 270.$$

Exercise (3).

If $a = 2$, $b = 3$, $c = 4$, $m = 6$, $n = 7$, $x = 8$, $y = 12$, find the value of :—

1. $3a^3$.
2. $7x^2 - y$
3. $2a^7 - 7m^2$.
4. $8b^3c - ab^2x$.
5. $5c^5 - 3x^3$.
6. $7a^3b^2y \div m^4n$.
7. $x^a - c^b$
8. $9c^4x^3y^2 - 8a^8b^2m^2 - by^2 - a$
9. $2a^xy \div x^2y^a$
10. $3c^ma^y - x^nb^3$
11. Find the value of $40 + 66x^2 - 21x + x^6 - 65x^4$, when $x = 8$
12. Find the value of $8x^4 + 6x^3 + 11x^2 + 13x + 29$, when $x = 75$
13. Find the value of $35m^5 - 4m^2 + 7m + 15m^3 - 34m^4 - 3$, when $m = \frac{4}{7}$.
14. Find the value of $25n^8 - 27 + 20n + 78n^5 - 199n^6$, when $n = 2\cdot 6$.
15. Find the value of $50y^7 - 51y^4 + 35y - 563y^5 - 19$, when $y = 3\cdot 4$.
16. Find the value of $64k^{10} - 55k^4 + 32k^6 - 121k^8 + 64k^2 - 4k^5 + 79$, when $k = 1\cdot 375$

Find the value of $a^3 + b^3 + c^3 - 3abc$:—

17. When $a = 24$, $b = 27$, $c = 29$.
18. When $a = 3\cdot 625$, $b = 4\cdot 625$, $c = 5\cdot 625$.
19. When $a = 44\frac{3}{7}$, $b = 51\frac{3}{7}$, $c = 58\frac{3}{7}$.
20. When $a = 1659$, $b = 1667$, $c = 1674$.

12. Roots. That quantity, whose square (or second power) is equal to any given quantity a, is called the *square root* of a, and is denoted by the symbol $\sqrt[2]{a}$, or, more simply, by $\sqrt{a}$, thus $3 = \sqrt{9}$, because $3^2 = 9$.

That quantity, whose cube (or third power) is equal to any given quantity a, is called the *cube root* of a, and is denoted by the symbol $\sqrt[3]{a}$, thus $2 = \sqrt[3]{8}$, because $2^3 = 8$

Generally, that quantity, whose n^{th} power, where n is any whole number, is equal to any given quantity a, is called the n^{th} *root* of a, and is denoted by the symbol $\sqrt[n]{a}$ Thus $2 = \sqrt[5]{32}$, because $2^5 = 32$, $3 = \sqrt[4]{81}$, because $3^4 = 81$, and so on

The sign $\sqrt{}$ is often called the *Radical Sign* It is said to be a corruption of the letter r, the first letter of the word *radix*

Note $\sqrt{a}$, which means the square root of a, is often read simply as "root a"

13. Brackets Each of the symbols ( ) { }, and [ ] is called a *pair of brackets* When an algebraical expression is enclosed within brackets it is to be regarded as a *single* quantity by itself Thus $(a+b)x$ means that the number denoted by x is to be multiplied by that denoted by $a+b$, whereas $a+bx$ means that x is to be multiplied by b and the product added to a

Hence, the expression $d+(a+b)x$ must be regarded as a *binomial*, the two terms being d and $(a+b)x$ Similarly, $c-\{d+(a+b)x\}$ also must be regarded as a *binomial*, the terms being c and $\{d+(a+b)x\}$ whereas, if the brackets be taken off, $c-d+a+bx$ is a *multinomial* consisting of four terms, namely, c, d, a and bx

Sometimes instead of enclosing an expression within a pair of brackets a line called a *Vinculum* is drawn over it

Thus $a-\overline{b-c}$ and $a-(b-c)$ have the same meaning

N B From the above it is easy to understand the distinction between $\sqrt{a+b}$ or $\sqrt{(a+b)}$ and $\sqrt{a}+b$ either of the first two expressions means the square root of the number denoted by $a+b$, whereas the last means that b is to be added to the square root of a. Similarly, $\sqrt{ab}$ or $\sqrt{(ab)}$, means the square root of the number denoted by ab, whereas $\sqrt{a}b$ means the product of b and the square root of a

Note The three different kinds of brackets, ( ), { }, [ ], are often called respectively *parentheses*, *braces* and *crotchets*

Examples.

Example 1. If $a = 2$, $b = 4$, $c = 9$, find the values of

$$\sqrt{c}b + \sqrt{b} + 5, \quad \sqrt{cb} + \sqrt{(b+5)} \text{ and } \sqrt[3]{2b} + \sqrt{4}a$$

(i) $\sqrt{c}b + \sqrt{b} + 5 = \sqrt{9} \times 4 + \sqrt{4} + 5$
$= 3 \times 4 + 2 + 5$
$= 12 + 2 + 5 = 19.$

(ii) $\sqrt{cb} + \sqrt{(b+5)} = \sqrt{9 \times 4} + \sqrt{(4+5)}$
$= \sqrt{36} + \sqrt{9}$
$= 6 + 3 = 9$

(iii) $\sqrt[3]{2b} + \sqrt{4}a = \sqrt[3]{2 \times 4} + \sqrt{4} \times 2$
$= \sqrt[3]{8} + 2 \times 2$
$= 2 + 4 = 6$

Example 2. If $a = 3$, $b = 5$, $c = 8$, $d = 12$, $e = 20$, find the difference between the numerical values of $a\{c + b^2 - a(e-d)\}$ and $a\{c + \overline{b^2 - a}(e-d)\}$.

The first expression $= 3 \times \{8 + 5^2 - 3 \times (20 - 12)\}$
$= 3 \times \{8 + 25 - 3 \times 8\}$
$= 3 \times \{8 + 25 - 24\}$
$= 3 \times 9 = 27$,

and the second expression $= 3 \times \{8 + (5^2 - 3) \times (20 - 12)\}$
$= 3 \times \{8 + 22 \times 8\}$
$= 3 \times \{8 + 176\}$
$= 3 \times 184 = 552.$

Thus the required difference $= 552 - 27 = 525.$

Example 3. If $m = 10$, $n = 8$, $p = 2$, $q = 12$, $r = 15$, find the difference between the numerical values of the expressions $[\{rm - 2q - n(pq - m)\} - p] \times (r - m - p)$ and

$$[\{rm - \overline{2.q - n(pq - m)}\} - p] \times r - \overline{m - p}$$

The first expression

$= [\{15 \times 10 - 2 \times 12 - 8 \times (2 \times 12 - 10)\} - 2] \times (15 - 10 - 2)$

$$= [\{150-24-8\times 14\}-2]\times 3$$
$$= [\{126-112\}-2]\times 3$$
$$= [14-2]\times 3 = 7\times 3 = 21$$

and the second expression

$$= [\{15\times 10-2\times (12-8)(2\times 12-10)\}-2]\times 15-(10-2)$$
$$= [\{150-2\times 4\times 14\}-2]\times 15-8$$
$$= [\{150-112\}-2]\times 15-8$$
$$= [38-2]\times 15-8$$
$$= 19\times 15-8 = 285-8 = 277$$

Thus the required difference $= 277-21 = 256$

Exercise (4).

If $a = 5$, $b = 2$, $c = 1$, $d = 4$, $e = 9$, $f = 0$, $m = 7$, $n = 3$, $p = 8$, find the values of —

1. $\sqrt[3]{bdp}$ 2. $\sqrt[5]{pd}$ 3. $\sqrt[3]{pn}$ 4. $6\sqrt[6]{(cn^{\frac{4}{3}})}$

5. $4\sqrt[4]{4d}$ 6. $4\sqrt[4]{4^4}$. 7. $2\sqrt{4p^2}$. 8. $2\sqrt{4}p^2$.

9 $a+b\sqrt{e}$ 10 $\overline{a+b}\sqrt{e}$ 11. $3\sqrt{c}+p$ 12. $3\sqrt{c+p}$

13. $\sqrt[3]{3(c+p)}$ 14. $3\sqrt[3]{8}(n+3p)$ 15. $3\sqrt[3]{8(n+3p)}$.

16. $f\sqrt{m}+e$ 17. $f\sqrt{m+e}$. 18. $3e-(2d-b)$.

19. $3e-2(d-b)$ 20. $3(e-2d)-b$ 21. $(3e-2)d-b$.

22 $(3e-2)\overline{d-b}$ 23. $3\{e-(2d-b)\}$ 24. $3(e-2)(d-b)$.

25. $7p-(n^2-b^2)$ 26. $(7p-n)^2-b^2$. 27. $7p-(n^2-b)^2$

28. $7(p-n)^2-b^2$. 29. $\{7p-(n^2-b)\}^2$.

30. $\sqrt[3]{p+3c+4d}(c+n)^3$ 31. $\sqrt[3]{p}+3c+4d(c+n)^3$.

32. $\sqrt[3]{p}+\overline{3c+4d}(c+n)^3$ 33. $\sqrt[3]{p}+(3c+4d)c+n^3$.

34. $\sqrt[3]{p}+3\{(c+4)dc+n^3\}$.

If $x = 2$, $y = 3$, $z = 4$, $a = 6$, $d = 8$, $c = 5$, $n = 9$, $p = 1$, find the values of —

35. $a(x+y)^2(a-\overline{c-z})^3$.

36. $4\{n-a(d-\overline{a+p})\}\sim 4\{\overline{n-a}(d-a)+p\}$.

37. $5\{c+\overline{x^2+y}(n-\overline{d-z})\} \sim 5\{\overline{c+x^2+y)}\overline{n-d}-z\}$

38. $[x+y^2\{ap-z(c-\overline{a-x})\}] \sim [\overline{x+y^2}\{(ap-z)c-a\}-x]$

14. Like and Unlike Terms. Terms or simple expressions are said to be *like* when they do not differ at all or differ only in their numerical co-efficients, otherwise they are called *unlike.* Thus $3ax^2y^5$ and $5ax^2y^5$ are *like* terms, whereas $3ax^2y^5$ and $5ax^2y^4$ are *unlike*, similarly abc, $5axbd$, $7a^2b^3$ and c^2d^3x are all *unlike*

15. Special meaning of the word Sign, Like and Unlike Signs. The word *sign* is often used to denote exclusively the signs + and − Thus when we speak of the *sign* of a term we mean the *plus* or *minus* sign which stands before it

Two signs are called *like* when they are *both* + or *both* −, otherwise they are called *unlike* Thus in the expression $ax^2+bx-cy+d^2-f$, the signs of the 3rd and 5th terms are *like* as also those of the 2nd and 4th, whereas the signs of the 2nd and 3rd terms as well as those of the 4th and 5th are *unlike.*

16. Dimensions and Degree of a Product Each of the letters which occur as factors of an algebraical product is called a *dimension* of the product, and the number of the letters is called the *degree* of the product Thus a^2x^5y which is equivalent to $a\times a\times x\times x\times x\times x\times x\times y$, is said to be of *eight dimensions*, or of the *eighth degree*; similarly $ab^2c^4d^5$ is said to be of *twelve dimensions* or of the *twelfth degree*

A numerical co-efficient is not counted Thus $5ab^2c^3$ and ab^2c^3 are both said to be of *six dimensions* or of the *sixth degree*

When an algebraical expression contains terms of different dimensions, the degree of the term which is of the highest dimensions is also called the *degree of the expression*.

17. Homogeneous Expression. An algebraical expression is said to be *homogeneous* when all its terms are of the same dimensions Thus the expression $5a^3b-7a^2bc+8b^2c^2$ is homogeneous, for each of its terms is of four dimensions.

18. Functions; Variables. Any expression involving a letter is called a *function* of that letter Thus

x^3+5x+8 is a function of x, a^2+ab+b^2 is a function of a and b, $a^3+b^3+c^3+2abc$ is a function of a, b and c, and so on

The letters of which a function consists are called its *variables* Thus $x^2+5xy+y^2$ is a function of which the variables are x and y.

19. The signs $>$, $<$, $\because$ and $\therefore$

The sign $>$ when placed between two quantities indicates that the quantity on the left of it is *greater than* that on the right Thus $a+b > c+d$ means that $a+b$ is greater than $c+d$

The sign $<$ when placed between two quantities indicates that the quantity on the left of it is *less than* that on the right Thus $a+x < b+y$ means that $a+x$ is less than $b+y$.

The sign   is used as an abbreviation for the word *because* or *since*

The sign   is used as an abbreviation for the word *therefore* or *hence*

CHAPTER II

POSITIVE AND NEGATIVE QUANTITIES.

1. Quantities of the Same Class but of Opposite Character. When we speak of a quantity of money, it may be either a *gain* or a *loss*, a receipt or a payment. Now it is quite clear that whilst a gain adds to our stock, a loss lessens it, moreover, gain and loss are so related that if we gain as much as we lose the effect on our stock is nothing Hence a quantity of money which forms a *gain* is said to be *opposite in character* to a quantity which forms a *loss*.

When we speak of a distance measured from a point, it may be in either of two opposite directions, either towards the north or towards the south of the point, either towards the east or towards the west of the point, either towards the north-east or towards the south-west of the point, and so on. It is also clear that distances measured towards the east are so related to those measured towards the west that if we

first walk any distance towards the east and then walk an equal distance towards the west there will be no change in our position with respect to the starting point Hence, a distance measured in any direction is said to be *opposite in character* to that measured in the opposite direction

Thus, in the first illustration, in so far as a gain and a loss are both looked upon as portions of money, they are said to be quantities of the same class, but as they affect our stock in directly opposite ways (a gain increasing and a loss diminishing it) they are said to be *of opposite character.* In the second illustration, a distance measured towards the south of the point as well as one measured towards the north may both be styled *distance*, and thus far they are said to be quantities *of the same class* ; but when we consider the directions in which they are measured they must be regarded as *opposite in character.*

2. The Signs Plus and Minus under a new aspect.—It has been shown in the Introduction how concrete quantities are represented by numbers. It now remains to be seen how quantities of the same class but of *opposite* character are distinguished in their numerical representation

When we consider any pair of such quantities, we prefix the sign + before the numerical measures of one, and the sign − before those of the other It is quite immaterial which of the two quantities we select for representation by numbers preceded by the sign +, but when we have once made our choice, we must stick to it throughout any connected series of operations The following examples will illustrate the principle —

(i) *Income* and *debt* are evidently quantities of opposite character If then we choose to represent incomes by numbers preceded by the sign +, we must represent debts by numbers preceded by the sign −, and *vice versa*

Hence, if in any problem we choose the sign + for incomes and the sign − for debts, + 30, +45, +90 will respectively represent incomes of £30, £45, and £90, whereas −30, −45, −90 will represent debts of £30, £45 and £90 respectively, a £ being the unit But if the contrary choice be made +10, +25, +36 will respectively represent debts of £10, £25, and £36, and −10, −25, −36 will represent incomes of £10, £25, and £36 respectively.

Hence, generally, if a represent a portion of any quantity, $-a$ will represent an equal portion of the quantity opposite in character to it.

(ii) A D O C B

Suppose AB is a road If a person starting from any point O on it *travels towards* B to any point C and then *travels back* to O, it is evident that his position on the road is just the same at the end of his journey as at the commencement Thus it is clear that distances measured along the road from *left to right* are opposite in character to those measured from *right to left* Accordingly, if distances measured from left to right be represented by numbers preceded by the sign +, those measured from right to left must be represented by numbers preceded by the sign −, and *vice versa*

Hence, if we choose the sign + for distances measured *from right to left*, a distance of −3 miles from any point O will mean a distance of 3 miles measured from O *towards the right*, again, if a mile be the unit of distance, and if C and D be two points on opposite sides of O at distances of 5 miles and 4 miles respectively then the distances OD, OC, CD and DC will be respectively represented by +4, −5, +9 and −9

From the above instances it is quite clear that the signs + and −, besides being used as signs of the operations of addition and subtraction, are also used as *signs of distinction* between quantities *of opposite character* The signs when used in this sense are often called *signs of affection*

N B When no sign is prefixed to a number, the sign + is understood, thus a and $+a$ have the same meaning

3. Positive and Negative Quantities. Numbers or symbols preceded by the sign + or no sign are called *positive quantities*, whilst those preceded by the sign − are called *negative quantities* Thus each of the expressions 4, +6, a, $+b$, $+c$ is a *positive quantity*, whilst each of −4, −6, $-a$, $-b$, $-c$ is a *negative quantity*

Hence, the signs + and − are often respectively called the *positive* and *negative signs*

Note 1 In "positive and negative quantities" the word quantity is used in the sense of number There is no difficulty however in understanding a *negative number*, when the explanation given in Art 2 is remembered.

Note 2 The *absolute value* of a positive or a negative quantity is its value considered apart from its sign. Thus if a stands for 5 and b for 3, $+(ab)$ and $-(ab)$ have the *same* absolute value, namely, 15

N B It is important to bear in mind the meanings of such expressions as "a *gain* of −£20," "a *rise* of −8 inches," "a distance of −5 miles to the *north*" &c. The expressions respectively mean "a *loss* of £20," "a *fall* of 8 inches, "a distance of 5 miles to the *south*," &c

Exercise (5).

1. If £2 be the unit, what is meant by "A's gain = −50" ?

2. If a trader's loss of £60 be represented by 60, what will represent a gain of £70 ?

3. If an income of £80 be represented by 20, what will represent a debt of £100 ?

4. If a debt of £200 be represented by 25, what will represent an income of £800 ?

5 If a distance of 90 miles to the north of a point be represented by 18, what will represent a distance of 150 miles to the south of it ?

6. If a river level rises 8 inches on any day, falls 6 inches the next day, and again rises 10 inches on the third, how would you represent the *rises* on successive days, taking 2 inches as the unit of length ?

7. A man gains Rs 80 in one year, loses Rs 20 in the second year, loses Rs 40 in the third year, and gains Rs. 60 in the fourth year · how would you represent his *gains* in the successive years, taking Rs 2 as the unit ?

8. In the preceding question, how would the man's losses be represented ?

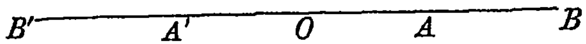
CHAPTER III.

ADDITION.

1. Definition. When two or more quantities are united together, the result is called their *sum* and the process of finding the result is called *addition.*

Note. As negative numbers are not recognised in Arithmetic there is clearly a difference between the Arithmetical and the Algebraical significance of the word *addition* Hence, when we speak of an *Algebraic sum,* we mean that quantities added together are not necessarily all positive

2 The result when one positive quantity is added to another. Suppose $B'B$ is a road and that dis-



tances measured from left to right are reckoned positive whilst those measured in the opposite direction, negative.

Suppose, O, A and B are three points on the road such that OA is 2 miles and AB is 3 miles, then if a mile be the unit of distance and if A and B be situated as shown in the figure, OA and AB will be respectively represented by $+2$ and $+3$

If then a man starting from O travels to A in the first hour and from A to B in the second hour, his distance from O at the end of two hours is evidently OB and will therefore be represented by $+5$

Hence, since (the distance travelled in the 1st hour)+(the distance travelled in the 2nd. hour) = (the distance travelled *in two hours*), we have $(+2)+(+3) = +5$.

Hence, generally speaking, $(+a)+(+b) = +(a+b)$; or more simply $(a)+(b) = (a+b)$

Thus, *when two positive quantities are added together, the sum is a positive quantity whose absolute value is equal to the arithmetical sum of the absolute values of those quantities.*

3 The result when one negative quantity is added to another Suppose in the above figure $OA' = 2$ miles and $A'B' = 3$ miles, and that A' is on the left of O and B' on the left of A', as shown in the figure Then the distances OA' and $A'B'$ are respectively represented by -2 and -3

If a man starting from O travels to A' in the first hour and from A' to B' in the second hour, his distance from O at the end of the second hour, will evidently be OB' and will therefore be represented by -5

Hence, since (the distance travelled in the first hour)+(the distance travelled in the 2nd hour) = (the distance travelled *in two hours*), we have $(-2)+(-3) = -5$.

Hence, generally speaking, $(-a)+(-b) = -(a+b)$.

Thus, *when two negative quantities are added together, the sum is a negative quantity whose absolute value is equal to the arithmetical sum of the absolute values of those quantities*

Example 1. Find the sum of $-a$, $-bc$, $-a^2b$, when $a = 2$, $b = 3$, $c = 5$

We have $a = 2$, $bc = 3\times 5 = 15$, $a^2b = 2^2\times 3 = 12$.

Hence, $(-a)+(-bc)+(-a^2b) = (-2)+(-15)+(-12)$

$$= -(2+15+12)$$

$$= -29.$$

Example 2. Find the value of $(-3c)+(-a^3d)+(b+f+g)$, when $a = 3$, $b = -2$, $c = 4$, $d = 5$, $f = -6$, $g = -8$.

We have $b+f+g = (-2)+(-6)+(-8)$

$$= -(2+6+8) = -16\ ;$$

also, $3c = 12$,

and $a^3d = 3^3\times 5 = 27\times 5 = 135.$

Hence, the given expression

$$= (-12)+(-135)+(-16)$$

$$= -(12+135+16)$$

$$= -163$$

Exercise (6)

1. Find the sum of -3, -7 and -12.
2. Find the sum of a, $-b$ and $-5c$, when $a = -5$, $b = 3$, $c = 2$.
3. Find the sum of -5, a and b, and find the result of adding it to -10, when $a = -6$ and $b = -20$
4. Find the value of $-a+(b+c)$, when $a = 5$, $b = -8$, $c = -6$
5. Find the value of $(-a^4b^2)+(-b^4c^2)+\{-(a^2-b^2)\}$, when $a = 4$, $b = 2$, $c = 3$
6. Find the sum of $-3a^3b^3$, d, e, $-20c^2$ and $(d+e)$, when $a = 1$; $b = 2$, $c = 3$, $d = -4$, $e = -5$
7. Find the sum of $-a^4(b-c)$, $-b^4(c-a)$ and $-c^4(b-a)$, when $a = 2$, $b = 5$, $c = 4$

8. Find the value of $\{-(x^2-y^2)\}+\{-(x^3-y^3)\}+\{-(x^4-y^4)\}$, when $x=5$, $y=3$

9. Find the sum of $-x^3(y^2-z^2)$, $-y^3(z^2-x^2)$ and $-z^3(y^2-x^2)$, when $x=3$, $y=6$, $z=5$.

10. Find the sum of $-\{a^4-b^4-c^4\}$, $-\{a^4\div(b^4-c^4)\}$, $-\{a^4-b^4\times c^4\}$ and $-\{(a^4-b^4)\times c^4\}$,
when $a=60$, $b=4$, $c=2$.

4. **The result when a Negative Quantity is added to a Positive Quantity.** In the figure of Art 2 suppose a man starting from O travels to B in the first hour and from B to A in the second hour, then the distances travelled in the first and second hours will be respectively represented by $+5$ and -3, and therefore the distance from O at the end of the second hour will be represented by $(+5)+(-3)$ But the distance of the man from O at the end of the second hour (*i e*, OA) is also evidently represented by $+2$. Hence we have $(+5)+(-3)=+2$, that is $=+(5-3)$

Again, if the man starting from O travels to B in the 1st hour and from B to A' in the second hour, then the distances travelled by him in the first and 2nd hours will be respectively represented by $+5$ and -7, and therefore his distance from O at the end of the second hour will be represented by $(+5)+(-7)$ But his distance from O at the end of the second hour (*i e*, OA') is also represented by -2 Hence we have $(+5)+(-7)=-2$, that is $=-(7-5)$

Thus, generally speaking, we have $(+a)+(-b)=+(a-b)$ or, $-(b-a)$ according as b is less or greater than a In other words, *if a positive and a negative quantity be added together, the sign of the result is positive or negative according as the absolute value of the negative quantity is less or greater than that of the positive quantity and the absolute value of the result is always equal to the difference between the absolute values of the quantities*

Cor. 1. Since $a+(-b)=-(b-a)$ when b is greater than a, putting $a=0$, we have $+(-b)=-b$, that is, *to add a negative quantity is the same as to subtract its absolute value, and conversely, to subtract a positive quantity is the same as to add a negative quantity having the same absolute value.*

Note Hence there is no difficulty in finding the value of $a-b$ when b is greater than a; for $a-b$ can *always* be taken to be equivalent to $a+(-b)$, and this latter is equal to $-(b-a)$ when b is greater than a. Thus $3-8=3+(-8)=-(8-3)=-5$

Cor. **2.** From Cor 1, it is evident that the sum of any number of quantities can be expressed by writing down the quantities one after the other with their respective signs. Thus $a-b+c-d$ means the same as $a+(-b)+c+(-d)$

Example 1. Find the value of $a-3b+2c-7d$, when $a=2$, $b=4$, $c=3$, $d=1$

$$\begin{aligned} a-3b+2c-7d &= a+(-3b)+2c+(-7d) \\ &= 2+(-12)+6+(-7) \\ &= -10+6+(-7) \\ &= -4+(-7) = -11. \end{aligned}$$

Example 2. Find the value of $a^2b-b^2c+c^2d-d^2a-bc^2$, when $a=1$, $b=2$, $c=3$, $d=4$

The given expression

$$\begin{aligned} &= (1^2\times 2)-(2^2\times 3)+(3^2\times 4)-(4^2\times 1)-(2\times 3^2) \\ &= 2-12+36-16-18 \\ &= -10+36-16-18 \\ &= 26-16-18 = 10-18 = -8 \end{aligned}$$

Exercise (7).

1. Find the sum of 7 and -4
2. Find the sum of 8 and -13.
3. Find the value of $b-c+d$, when $b=13$, $c=25$, $d=8$.
4. Find the sum of $3a$, $-5b$, c, d and $4e$, when $a=2$, $b=3$, $c=4$, $d=-7$, $e=3$
5. Find the value of $3m-5n+6q+r$, when $m=4$, $n=6$, $q=2$, $r=-8$
6. Find the value of $2x^2y-3y^3x-5x^2y^2+x^3y^4$, when $x=y=2$.
7. Find the sum of $-a^2c$, bd^2, $-cb^2$ and $-a^2d^2$, when $a=2$, $b=5$, $c=3$, $d=6$.

8. Find the value of $a^3-3a^2b+3ab^2-b^3$, when $a=3$ and $b=5$

9. Find the value of $m^4-4m^3n+6m^2n^2-4mn^3+n^4$; when $m=2$ and $n=4$

10. Find the value of $x^5-5x^4y+10x^3y^2-10x^2y^3+5xy^4-y^5$, when $x=5$ and $y=7$

5. When any number of quantities are added together, the result will be the same in whatever order the quantities may be taken.

Suppose a man starting from a place travels 6 miles to the north and then travels back along the same path 8 miles to the south Then his position at the end of the journey is 2 miles to the south of that place.

Again, if the man first travels 8 miles to the south and then travels 6 miles to the north, then also at the end of the journey he is still 2 miles to the south of the place

Thus, we have $6+(-8)=(-8)+6$, each being equal to -2, or, more briefly, we have $6-8=-8+6$, and a similar result in every other case

Hence, generally, $a-b=-b+a$

Again, since $2-10+6=-8+6=-2$,

and also $-10+6+2=-4+2=-2$,

we have $2-10+6=-10+6+2$, and a similar result in every other case,

hence, generally, $a-b+c=-b+c+a$.

Similarly, it may be shown that

$$\begin{aligned} a-b+c-d+e-f &= a+c+e-b-d-f \\ &= -b+e-d-f+c+a \\ &= \&c \qquad \&c \qquad \&c \end{aligned}$$

6. When any number of quantities are added together, they can be divided into groups and the result expressed as the sum of those groups.

We have $3-7-8+6-4+2$

$$\begin{aligned} &= -4-8+6-4+2 \\ &= -12+6-4+2 \\ &= -6-4+2=-10+2=-8; \end{aligned}$$

$$(3-7)+(-8+6)+(-4+2)$$
$$=-4+(-2)+(-2) = -8\ ;$$
$$3+(-7-8+6)+(-4+2)$$
$$= 3+(-9)+(-2) = -8\ ,$$
$$3+(-7-8)+(6-4)+2$$
$$= 3+(-15)+2+2 = -8$$

Thus we have $3-7-8+6-4+2$

$$= (3-7)+(-8+6)+(-4+2)$$
$$= 3+(-7-8+6)+(-4+2)$$
$$= 3+(-7-8)+(6-4)+2,$$

and similar results in all other cases.

Hence, generally, the expression $a+b-c-d+e-f+g$ can be put in any one of the following forms —

(1) $(a+b)+(-c-d)+e+(-f+g)$
(2) $a+(b-c)-d+(e-f+g)$
(3) $(a+b-c)+(-d+e-f)+g$
(4) $a+(b-c-d)+e+(-f+g)$
(5) $(a+b-c-d)+(e-f+g)$
&c &c. &c

Cor. 1. Conversely, we have $(a+b)+(-c-d)+e+(-f+g) = a+b-c-d+e-f+g$ Hence the following rule:— *To add together two or more algebraical expressions write down the terms in succession with their proper signs*

Cor 2. Since $a-b+c-d+e-f = a+c+e-b-d-f$ [Art 5] $= (a+c+e)+(-b-d-f)$, we have the following rule —

When any number of quantities are to be added some of which are positive and others negative, collect the positive terms in one group and the negative terms in another, and express the result as the sum of these two groups Thus $3-7+8-9+5-6 = (3+8+5)+(-7-9-6) = 16+(-22) = -6$

Example 1. Simplify $5a-3b+2c-4a+2b-7c$

The given expression

$$= 5a-4a-3b+2b+2c-7c \qquad \text{[Art. 5]}$$
$$= (5a-4a)+(-3b+2b)+(2c-7c) \qquad \text{[Art 6.]}$$
$$= a+(-b)+(-5c)$$
$$= a-b-5c.$$

Example 2. Simplify $3a^2b+5b^2c-6c^2a-10a^2b-7b^2c+8c^2a+4a^2b-b^2c+c^2a$

The given expression

$$= 3a^2b-10a^2b+4a^2b+5b^2c-7b^2c-b^2c-6c^2a+8c^2a+c^2a$$

$$= (3a^2b-10a^2b+4a^2b)+(5b^2c-7b^2c-b^2c)+(-6c^2a+8c^2a+c^2a)$$

$$= (-7a^2b+4a^2b)+(-2b^2c-b^2c)+(2c^2a+c^2a)$$

$$= (-3a^2b)+(-3b^2c)+(3c^2a)$$

$$= -3a^2b-3b^2c+3c^2a$$

Note In the process above, it must be noticed that when like terms are added together, the result is obtained by annexing the common letters to the sum of the numerical co-efficients For instance, we find that $5b^2c-7b^2c-b^2c=-3b^2c$, and evidently -3 is the sum of the co-efficients 5, -7 and -1

Example 3. Add together $3a-2b+c$ and $-5d+6e-f$, and find the numerical value of the sum, when $a=2$, $b=1$, $c=3$, $d=4$, $e=7$, $f=5$

We have $(3a-2b+c)+(-5d+6e-f)$

$$= 3a-2b+c-5d+6e-f$$

$$= 6-2+3-20+42-5$$

$$= (6+3+42)+(-2-20-5)=51+(-27)=24.$$

Exercise (8)

Simplify the following —

1. $a+b-c-2a-3b+4c$
2. $5a^2-7b^2+8c^2+5b^2-7c^2-6a^2$
3. $8a-ab-7a+5c-3a+5ab$
4. $5mnp^2-6abr-7c^2-8mnp^2+9c^2+4abr-5c^2$
5. $-7a^3b-5b^2c^2+10a^3b-3b^2c^2+3df-a^3b-b^2c^2-5df.$
6. $8x^4y-5xyz-17x^4y+20x^2y^2-2xyz-35x^2y^2+3x^4y-4xyz+5x^2y^2$
7. $-13a^2bc+15ab^2c-27abc^2-5a^2bc+13abc^2-23ab^2c+7a^2bc+6abc^2+19ab^2c$
8. $20x^3mn-23m^3nx+14n^3xm-37x^3mn-47n^3xm+54m^3nx-8x^3mn+13n^3xm-15m^3nx+20n^3xm$

If $a=2$, $b=3$, $c=4$, $d=5$, $e=6$, $f=7$, $g=8$, $k=9$, $m=10$, $n=12$, find the numerical value of the sum of :—

9 $-a+3b+5c$ and $5d-4e-6f$.

10 $5b-2f$, $-7m-8n$ and $3d+c-10h$

11. $3a^2$, $-5b^2+7c^2$, $-2e+5m-n$ and $10d-7h$

12. $-2a+3b-4c$, $-d-5e+6f$ and $3g-5h-3m+5n$.

13. $-ab+gk$, $mn-4bd+f$, $-c^2-d^2+mk$ and $6ac-5f-7e+abc$.

14 a^3b-cde, $f^2m-abn-e^2d$, $-mg+3h^2-2b^4d$ and $5c^2-7dmg+2a^2h-3m^2d$

15. $3b^4m-5h^2e-4m^2g$, $-13a^2m+4g^2d-7d^2c$, $-5n^2c+8fm^2+9d^3$ and $5g^2h-7m^2n-4e^3m+8cdk$

7. The ordinary Rule for adding together compound expressions. Put the expressions under one another so that the different sets of like terms may stand in vertical columns and draw a line below the last expression, then add up each vertical column and put the result below it The following examples will illustrate the method —

Example 1 Add together $3a-5b+7c-9d$, $-8c+5a-3d+7b$, $4d+2c-a$ and $2b-3c+6d$

The first expression $= 3a-5b+7c-9d$

The 2nd expression $= 5a+7b-8c-3d$ [Art. 5]

The 3rd expression $= -a \quad +2c+4d$

The 4th expression $= \quad 2b-3c+6d$

∴ The sum $= 7a+4b-2c-2d$

Example 2. Find the numerical value of the sum of $20a^2b^3-25b^3c^4+d^7$, $-22a^2b^3+19b^3c^4-3d^7$ and $2a^2b^3+7b^3c^4-2d^7$, when $a=498$, $b=3$, $c=2$, $d=19$.

The first expression $= 20a^2b^3-25b^3c^4+d^7$

The 2nd expression $= -22a^2b^3+19b^3c^4-3d^7$

The 3rd expression $= 2a^2b^3+7b^3c^4+2d^7$

∴ The sum $= b^3c^4$

$= 3^3\times 2^4 = 27\times 16 = 432.$

Exercise (9).

Add together —

1. $a-2b+5c$ and $-7a+3b-8c$

2. $-3x+5y-9z$, $5x-3y+7z$ and $-2y+z$

3. x^3+3x^2-5x+4, $2x^3-6x^2+7x-8$, $-x^3+7x^2-2x+9$ and $5x^2+2$

4. $3a-2b+7c-8d$, $2c+6d-5a$, $3b+d-10c$, $c-4b+a$ and $-7d+5b$

5 $x^2+2xy+3y^2-x+y+2$, $-5x^2+y^2+2x-5$, $-3xy-7y^2+3y+1$ and $6x^2+xy-x-4y+2$

6 $2x^2-5xy+y^2$, $4y^2-7x^2-5x+2y$, $3xy-5+y-6y^2$ and $3-4y+3x$

7. $abc+a^2b-b^2c^2$, $5a^2b-12b^2c^2-3abc$, $8b^2c^2-4a^2b+2abc$ and $2a^2b+5b^2c^2$

8. $m^3n^2-3mnp+2m^2n^3+6m^2n^2$, $7mnp-10m^2n^2+5m^3n^2+m^2n^3$, $2m^3n^2-5mnp+3m^2n^3$ and $-7m^3n^2+m^2n^2-4m^2n^3$

9. $12a^3b^2x-29b^3x^2a+37x^3a^2b+45a^2b^2x^2$, $25b^3x^2a-16a^2b^2x^2-18a^3b^2x-5x^3a^2b$, $32a^2b^2x^2-23x^3a^2b+20a^3b^2x-28b^3x^2a$ and $-9x^3a^2b-14a^3b^2x-60a^2b^2x^2+32b^3x^2a$

10. $-15a^4b^4c^4+7c^4a^3b^5-24b^4c^3a^5+27a^4b^3c^5$, $19c^4a^3b^5-15a^4b^3c^5+23a^4b^4c^4-8b^4c^3a^5$, $29b^4c^3a^5+11a^4b^4c^4-9a^4b^3c^5-16c^4a^3b^5$ and $-3a^4b^3c^5-10c^4a^3b^5+3b^4c^3a^5-18a^4b^4c^4$

11. $25a^3b^3-8b^3c^3-23c^3a^3+19a^2b^2c^2$, $16c^3a^3-14a^2b^2c^2-19a^3b^3-12b^3c^3$, $27a^2b^2c^2+13a^3b^3+17c^3a^3-20b^3c^3$, $29b^3c^3-6a^2b^2c^2-21a^3b^3-13c^3a^3$ and $10b^3c^3+3a^3b^3+4c^3a^3-27a^2b^2c^2$

12 $5a^3-18b^3-53c^3-25abc$, $38c^3-87a^3-7abc+25b^3$, $26abc-17c^3+11b^3+43a^3$, $13b^3-18abc+4a^3+21c^3$ and $-14a^3+12c^3+21abc-24b^3$

If $a = 5$, $b = 4$, $x = 8$, $y = 7$, find the numerical value of —

13. $(3x^3 + 5y^5 - 20a^2 + 49b^3) + (17a^2 - 27b^3 - 23x^3) + (-y^5 + 3b^3 - 3a^2) + (-23b^3 - 4y^5 + 7a^2 + 20x^3)$

14. $(10a^2 - 26x^5y^4 + 30x^3b^5 + 17a^6y^7) + (35x^5y^4 + 16a^6y^7 - 304a^2 - 28x^3b^5) + (-8a^6y^7 - 9x^5y^4 - 7x^3b^5) + (5x^3b^5 - 25a^6y^7 + 289a^2)$

15. $(2a^2 - 7b^2 + 9x^2 - 13y^2 + 15ab - 21xy) + (5y^2 + 8b^2 + 17xy - 6a^2 - 8ab - 20x^2) + (13x^2 - 20ab + 5a^2 - 16xy - 10y^2 - 2b^2) + (13ab - 2x^2 + 3b^2 + 23xy - a^2 + 18y^2)$.

16. $(29abx - 39bxy + 49xya - 59yab) + (29bxy + 49yab - 19abx - 39xya) + (2abx - 12xya + 6bxy + 24yab) + (3xya + 4bxy - 13abx - 14yab)$.

17. $(18a^2b^2 - 43b^2x^2 + 62x^2y^2 - 23abxy) + (39abxy + 28b^2x^2 - 25a^2b^2 - 42x^2y^2) + (19b^2x^2 + 37a^2b^2 - 25abxy + 35x^2y^2) + (9abxy - 29a^2b^2 - 55x^2y^2 - 4b^2x^2)$.

18. $(46a^4 + 38b^4 - 87abx^2 - 105y^4) + (47abx^2 + 85y^4 - 56a^4 - 58b^4) + (57y^4 + 75b^4 + 23a^4 + 63abx^2) + (-33b^4 + 8y^4 - 27abx^2 - 39a^4) + (26a^4 - 45y^4 - 22b^4 + 5abx^2)$

19. $(35xy^4 + 207ab^4 - 98bx^4 - 62ya^4 - 83abx^2y) + (68bx^4 + 102ya^4 - 65xy^4 - 87ab^4 + 53abx^2y) + (26abx^2y - 75ab^4 - 25ya^4 + 43bx^4 + 53xy^4) + (28ya^4 - 29xy^4 - 65abx^2y + 45ab^4 + 26bx^4) + (-89ab^4 - 43ya^4 + 69abx^2y + 6xy^4 - 39bx^4)$.

20. $(57a^4bx + 25b^4xy - 143x^4ya + 37y^4ab - 253a^2b^2x^2) + (63x^4ya - 92y^4ab - 63a^4bx + 73a^2b^2x^2 - 85b^4xy) + (35y^4ab + 132b^4xy + 82a^2b^2x^2 + 36x^4ya + 96a^4bx) + (-50a^2b^2x^2 - 78a^4bx + 27y^4ab - 17x^4ya - 52b^4xy) + (61x^4ya - 20b^4xy + 148a^2b^2x^2 - 7y^4ab - 12a^4bx)$

CHAPTER IV.

SUBTRACTION—REMOVAL OF BRACKETS.

1. Definition Any quantity b is said to be subtracted from any other quantity a when a third quantity c is found such that the *sum* of b and c is equal to a In other words, $c = a-b$ *when* c is such that $b+c = a$

The quantity *from* which another quantity is subtracted is called the *minuend* and the *quantity subtracted* is called the *subtrahend* The result is called the *difference* or the *remainder* Thus if $a-b = c$, a is the minuend, b the subtrahend, and c the remainder

2 To subtract a positive quantity is the same as to add a negative quantity having the same absolute value, and to subtract a negative quantity is the same as to add a positive quantity having the same absolute value.

Since $3+4 = 7$, we have $7-3 = 4 = 7+(-3)$, again, since $6+(-2) = 4$, we have $4-6 = -2 = 4+(-6)$

Hence, generally $a-b = a+(-b)$, i e, *to subtract a positive quantity is the same as to add a negative quantity having the same absolute value* [See Art 4, Cor 1 Note, Chap III]

Since $(-3)+5 = 2$, we have $2-(-3) = 5$ [by definition] $= 2+3$,

similarly, since $(-6)+(-4) = -10$,
we have $(-10)-(-6) = -4 = (-10)+6$

Thus, generally, since $(-b)+(a+b) = a$, we have $a-(-b)=a+b$, *i e, to subtract a negative quantity is the same as to add a positive quantity having the same absolute value.*

Note One quantity a is said to be greater than another b when $a-b$ is a positive quantity Thus -4 is greater then -5 for $(-4)-(-5) = -4+5 = 1$ Similarly, $-5 > -7$, $-10 > -20$ and so on Hence in the series 5 4 3, 2, 1, 0, $-1, -2, -3, -4, -5, -6, -7, -8$, &c each number is less than the one before it.

3. Illustration. Suppose AD is a Railway line run-

$A \qquad O \qquad B \qquad\qquad C \quad D$

ning from west to east, and A, O, B, C, D, are stations on it such that $AO = OB = 20$ miles, $BC = 30$ miles and $CD = 10$ miles. Suppose a man travels from O to C in two days.

Then evidently, (the distance travelled on the first day) + (the distance travelled on the second day) = 50 miles ; and hence, by definition, 50 miles − (the distance travelled on the first day) = the distance travelled on the second day.

Now, (i) if on the first day the man travels from O to B, *i e*, travels 20 miles towards the *east* of O, then on the second day he has to travel from B to C, a distance of 30 miles more towards the east, thus we have (50 miles) − (20 miles) = 30 miles.

(ii) If on the first day the man travels from O to A, *i e*, travels a distance of 20 miles towards the *west*, then on the second day he must travel from A to C, a distance of 70 miles towards the east, thus we have (50 miles)−(−20 miles) = 70 miles

(iii) Again, if on the first day the man travels from O to D, *i.e.*, a distance of 60 miles towards the east, then on the second day he must travel from D to C *i e*, a distance of 10 miles *towards the west* ; thus we have (50 miles)−(60 miles) = − 10 miles

Hence, taking a mile as the unit of distance we get the *following results* :—

$$\left.\begin{aligned} 50-\quad 20 &= \quad 30 \\ 50-(-20) &= \quad 70 \\ 50-\quad 60 &= -10 \end{aligned}\right\}$$

Example 1. Find the value of $a-b+c$, when $a = 5$, $b = -2$, $c = -3$

$$\begin{aligned} a-b+c &= 5-(-2)+(-3) \\ &= 5+2-3 = 4. \end{aligned}$$

Example 2 Find the value of $-a-(-b)+c$, when $a = -2$, $b = -3$, $c = -4$

$$\begin{aligned} \text{The given expression } &= -a+b+c \\ &= -(-2)+(-3)+(-4) \\ &= 2-3-4 \\ &= -5 \end{aligned}$$

Exercise (10).

If $a = 3$, $b = -5$, $c = -6$, $d = -8$, find the value of :—

1. $-a+b-c+d$ 2. $a-b+c+d$ 3. $c-d-(-b)-a$

4. $c-(-d)+b-a$. 5. $-(-a)+b-(-c)-d$

If $m = -47$, $n = 50$, $x = -154$, $y = -234$, find the value of —

6. $n-m-(-x)+y$ 7. $-(-m)+y-(-n)-x$.

8. $-(-x)+m-y-(-n)$. 9. $-(-y)-m-x-(-n)$

10. $-(-n)-y-(-x)-m$

4. To prove that $a-(b+c) = a-b-c$,

and $a-(b-c) = a-b+c$.

Since $(b+c)+(a-b-c) = a$,

∴ by definition, $a-(b+c) = a-b-c$.

Again, since $(b-c)+(a-b+c) = a$,

$a-(b-c) = a-b+c$ *

Cor. Thus we arrive at the following rule for subtracting one algebraical expression from another :—*Change the sign of every term of the subtrahend from + to − or from − to +, as the case may be, and then write down those terms in succession after the minuend.* Thus the result of subtracting $2a+3b-5c$ from $a-2b+c = a-2b+c-2a-3b+5c = -a-5b+6c$

Example 1. Subtract $-3a+2b-5c$ from $2a+b-8c$

The required result $= 2a+b-8c+3a-2b+5c$

$= (2a+3a)+(b-2b)+(-8c+5c)$

$= 5a+(-b)+(-3c)$

$= 5a+(-b)+(-3c)$

$= 5a-b-3c$

---

* When a, b, c are all positive quantities and a is greater than b, and b is greater than c, the following proof is generally given of this result in most treatises on Algebra :—

If we subtract b from a, we get $a-b$, but we thus subtract too much from a, for we have to subtract not b but a quantity which is less than b by c. Hence we must add c to the result [illegible] $= a-b+c$

Example 2. Subtract $2a^2+3ab-5b^2$ from $-3a^2+2ab-4b^2$.

The required result $=-3a^2+2ab-4b^2-2a^2-3ab+5b^2$

$=(-3a^2-2a^2)+(2ab-3ab)+(-4b^2+5b^2)$

$=-5a^2-ab+b^2$.

Exercise (11).

Subtract :—

1. $a-b+c$ from $2a+3b-c$.
2. $2a-5b+4c$ from $-a-2b+8c$.
3. $-x+y-z$ from $2x+3y-4z$.
4. $-3m^2+2mn+5$ from $-5m^2-mn+4$
5. $x^2-2y^2+8z^2$ from $3x^2-y^2+2z^2$.
6. $3ax-4xy+5y^2$ from $2ax+xy-6y^2$.
7. $-3a^2+2ab-7b^2$ from $a^2-5ab-8b^2$.
8. $-2bc+6c^2-8xy$ from $5bc-c^2+2xy$.
9. $2x^3-4x^2+7x+5$ from x^3-3x^2+6x+7
10. $x^4+2x^3y-3x^2y^2+6xy^3-y^3$ from $2x^4-2x^3y-3x^2y^2+4xy^3-y^3$
11. $3m^4-7m^2n+8mn^2-13n^4$ from $2m^4-13m^2n+15mn^2-37n^4$.
12. $8p^4-7p^3q+10p^2q^2-13pq^3+5q^4$ from $5p^4-12p^3q+7p^2q^2-23pq^3+3q^4$.
13. $-7x^5+6x^4y-8x^3y^2-13x^2y^3+9y^4$ from $3x^5-5x^4y+2x^3y^2-7x^2y^3+6y^4$.
14. $3m^3nx-10n^3xm+14x^3mn-20m^2n^2x-27n^2x^2m$ from $5m^3nx-17n^3xm+26x^3mn-13m^2n^2x-19n^2x^2m$.
15. $37x^6-28x^5y+43x^4y^2-54x^3y^3-67x^2y^4+84xy^5-93y^6$ from $48x^6-31x^5y-7x^4y^2-39x^3y^3-41x^2y^4+65xy^5-53y^6$.

5. The Ordinary Rule for subtracting one compound Expression from another. Put the subtrahend below the minuend in such a way that the different sets of like terms may stand in vertical columns and draw a line below the subtrahend ; then supposing the sign of every term of the subtrahend to be changed, write down the sum of each vertical column underneath it.

Example 1. Subtract $-2x^2+3xy-y^2$ from $x^2-2xy+3y^2$.

The minuend $= x^2-2xy+3y^2$

The subtrahend $= -2x^2+3xy-y^2$

$\therefore$ The remainder $= 3x^2-5xy+4y^2$

Note It must be noticed that the signs of the terms of the subtrahend are not actually altered in the process, but they are *supposed* to be altered and the operation of combining each pair of like terms is performed mentally

Example 2. Subtract $a^2-3ab+5x^2-y^2$ from $3x^2+2y^2-7a^2$.

The minuend $= 3x^2+2y^2-7a^2$

The sub trahend $= 5x^2-y^2+a^2-3ab$

$\therefore$ The remainder $= -2x^2+3y^2-8a^2+3ab$

Exercise (12).

Subtract —

1. $3a-4b+5c$ from $2a+3b-5c$
2. $-3m+5n-7r$ from $7m-2n+5r$
3. $-x^2+xy-y^2$ from $2x^2-3xy+4y^2$.
4. x^3-2x^2+5x+6 from $2x^3+3x^2-3x+5$.
5. $a^2-5ab+2b^2+7bc$ from $2a^2-3ab+3bc$
6. $-5x^3+6x^2y-4xy^2+7y^3$ from $3x^3-2x^2y+8xy^2-5y^3$.
7. $-2+3x-5x^2+7x^3$ from $4-6x+7x^2-9x^3$.
8. x^4-7x^2+9 from $3x^4-2x^3-8x^2+7$.
9. $-5x^5+2x^4-x^3+6x^2+9x+8$ from $x^5-6x^4-4x^2+5$
10. $2x^3-3x^2y+7xy^2-8y^3+a^3-3ab$ from $x^3-4x^2y-8xy^2-5ab$.
11. $3xyz^2-2x^4y-7x^2yz-2x^3y^2$ from $12x^4y-5x^3y^2+8xyz^2-9x^2yz$
12. $2bc-3c^2+ac+5a^2-4b^2-7ab$ from $2a^2-3b^2+5c^2-6ab+7bc-8ac$.
13. $-5x-3x^2+12+2xy-7y+4y^2$ from $2x^2-3xy+5y^2-6x-8y+9$.

14. $-5a^2+2a^3+4b^2-7ab^2-2a^2b-7b^3+5ab$ from $a^3-3a^2b+5ab^2-9b^3+3a^2-2ab+b^2$.

15. $-2yzbc^2+4yz^2bc-2ax^4-9y^2zbc+3a^2x^3$ from $3ax^4-5a^2x^3+6yzbc^2-7y^2zbc+8yz^2bc$

16. $19x^3z^5y-15x^3y^5z+27+11xyz^4-12x^2y^2z^2-19xy^3z^5$ from $25-16x^3y^5z-17xy^3z^5+21x^3z^5y-6x^2y^2z^2+8xyz^4$.

17. $43x^3y^4z^2-23x^3y^2z^4+25x^4y^3z^2-66x^2y^4z^3+26x^2y^3z^4+35x^4y^2z^3$ from $29x^4y^3z^2-37x^3y^4z^2+54x^2y^3z^4-45x^3y^2z^4-67x^4y^2z^3+89x^2y^4z^3$

18. $-29x^4y^3z^5+75x^5y^4z^3+13x^3y^5z^4+53x^3y^4z^5-94x^5y^3z^4-86x^4y^5z^3$ from $41x^3y^4z^5-87x^3y^5z^4-28x^4y^5z^3+63x^4y^3z^5-55x^5y^3z^4+37x^5y^4z^3$.

19. What must be added to $3x^2-5xy+6y^2+7yz$ in order that the sum may be $-x^2-y^2-yz$?

20. What must be added to $-5x^3+13x^2y^2-a^2bx+5bxy^2+7xyab$ in order that the sum may be $x^3+x^2y^2+a^2bx-2bxy^2-2xyab$?

21. What must be added to $5x^4-6x^3y+7x^2y^2-8xy^3-19y^4$ in order that the sum may be $3x^4+5x^2y^2-12y^4$?

22. What must be added to $-5x^5-3x^4y+6x^3y^2+17x^2y^3+13xy^4-21y^5$ in order that the sum may be $-7x^5-4x^3y^2+13x^2y^3+29y^5$?

23. What must be subtracted from $2a^2+5ab-6b^2$ in order that the remainder may be a^2+2b^2 ?

24. What must be subtracted from $5x^2-6xy+4y^2-8x-10y+15$ in order that the remainder may be $x^2+2xy+3y^2+4x+5y+6$?

25. What must be subtracted from $3a^3-4a^2b+5ab^2-8b^3$ in order that the remainder may be $a^3-2ab^2+7b^3$?

26. What must be subtracted from $-8x^3y+4x^2y^2-11xy^3+12x^2-13y+27$ in order that the remainder may be $4x^3y-3x^2y^2-11xy^3+20x^2-30y+56$?

27. From what expression must $3a^2-7ab-8bc-9b^2$ be subtracted in order that the remainder may be $2a^2+3ab+3bc+2b^2$?

28. From what expression must $-3x^3+5y^2-7xy+8x-9$ be subtracted in order that the remainder may be $x^3-8y^2+2xy-11x+7$?

29. From what expression must $-7a^3-8b^2c-18ac^2+3b^3$ be subtracted in order that the remainder may be $4a^3-3b^2c+7ac^2-8b^3$?

30. From what expression must $21x^3-37xy^2+42y^3-18x^2+19xy-39$ be subtracted in order that the remainder may be $-25x^3+15xy^2-87y^3+7x^2-43xy+24$?

6. Removal of Brackets. The laws for the removal of brackets are —

(1) If any number of terms be enclosed within a pair of brackets preceded by the sign $+$, the brackets may be struck out as of no value

(2) If any number of terms be enclosed within a pair of brackets preceded by the sign $-$, the brackets may be removed provided that the sign of every term within the brackets be changed, namely, $+$ to $-$ and $-$ to $+$.

The reason is obvious, for, any expression, included within brackets preceded by the sign $+$, has to be added to, whilst one, enclosed within brackets preceded by the sign $-$, has to be subtracted from what goes before.

Thus $a-b+(c-d+e) = a-b+c-d+e$,

whilst, $a-b-(c-d+e) = a-b-c+d-e$

Conversely,

(i) Any number of terms in an expression may be enclosed within a pair of brackets, with the sign $+$ prefixed;

(ii) Any number of terms in an expression may be enclosed within a pair of brackets with the sign $-$ prefixed, if the sign of every term put within the brackets be altered.

Thus $a-b+c-d+e-f = a-b-(-c+d-e+f)$

Note We often find brackets within brackets as in the expression $2a-[3b-\{4c-(5d-6e)\}]$, here it is meant that the expression within the braces $\{\ \}$ is to be subtracted from $3b$ and the result thus obtained is to be subtracted from $2a$, whilst the expression within the braces is to be found by subtracting the expression within the parentheses $(\ )$ from $4c$

When an expression of this kind is to be cleared of brackets, it is best for a beginner to remove first the innermost pair, then the innermost of those that remain, and so on, and lastly the outermost pair

Example 1 Simplify $a-\{b-(c-d)\}$.

$$a-\{b-(c-d)\} = a-\{b-c+d\}$$
$$= a-b+c-d.$$

Example 2. Simplify $a-[b-\{c-(d-e)\}-f]$.

$$\begin{aligned} a-[b-\{c-(d-e)\}-f] &= a-[b-\{c-d+e\}-f] \\ &= a-[b-c+d-e-f] \\ &= a-b+c-d+e+f \end{aligned}$$

Example 3. Simplify $a+[-b-\{c-(d-\overline{e-f})-g\}-h]$.

$$\begin{aligned} &a+[-b-\{c-(d-\overline{e-f})-g\}-h] \\ &\quad= a+[-b-\{c-(d-e+f)-g\}-h] \\ &\quad= a+[-b-\{c-d+e-f-g\}-h] \\ &\quad= a+[-b-c+d-e+f+g-h] \\ &\quad= a-b-c+d-e+f+g-h \end{aligned}$$

Example 4. Simplify $2a-[3a+\{4b-(2a-b)+5a\}-7b]$.

$$\begin{aligned} \text{The given expression} &= 2a-[3a+\{4b-2a+b+5a\}-7b] \\ &= 2a-[3a+\{5b+3a\}-7b] \\ &= 2a-[3a+5b+3a-7b] \\ &= 2a-[6a-2b] \\ &= 2a-6a+2b = -4a+2b \end{aligned}$$

Example 5. Simplify $a-[-b-\{c-(d-\overline{e-f})\}]$, first removing [ ], then { }, then ( ), and last of all the vinculum.

$$\begin{aligned} &a-[-b-\{c-(d-\overline{e-f})\}] \\ &\quad= a+b+\{c-(d-\overline{e-f})\} \\ &\quad= a+b+c-(d-\overline{e-f}) \\ &\quad= a+b+c-d+\overline{e-f} \\ &\quad= a+b+c-d+e-f \end{aligned}$$

Note The expression within [ ] consists of two terms, namely, $-b$ and $-\{c-(d-\overline{e-f})\}$, hence, when this pair of brackets, which is preceded by the sign $-$, is removed we get $b+\{c-(d-\overline{e-f})\}$ A similar reasoning applies to the removal of the other brackets It must be noticed carefully that *only one pair of brackets is to be removed at a time*

Example 6. Simplify

$$[a-\{b-(c-d)\}]-[2a-\{3b+(2c-4d)\}].$$

We have $a-\{b-(c-d)\} = a-\{b-c+d\}$
$= a-b+c-d$,

and $2a-\{3b+(2c-4d)\} = 2a-\{3b+2c-4d\}$.

$$= 2a-3b-2c+4d.$$

Hence the given expression

$$= [a-b+c-d]-[2a-3b-2c+4d]$$
$$= a-b+c-d-2a+3b+2c-4d$$
$$= -a+2b+3c-5d.$$

Example 7. Of the expression $a+b-c+d-e-f$ enclose the first three terms within a pair of brackets and the last three in another, each preceded by the sign $-$, and then put the last two terms of each of these bracketed expressions within an inner pair of brackets preceded by the sign $-$

According to the given directions,

$$a+b-c+d-e-f = -\{-a-b+c\}-\{-d+e+f\}$$
$$= -\{-a-(b-c)\}-\{-d-(-e-f)\}.$$

Exercise (13).

Simplify —

1. $2a-3b-(4a-6b)+(-2a+5b)$.
2. $x+(-y+4x)-(-2x+3y)$.
3. $-(5x-y)+(-3x+y)-(2y-6x)$.
4. $3a-\{6a-(2b-a)\}$
5. $-a-\{2b-(6a+4b)\}$.
6. $2a-\{5b-\overline{7b-2a}\}$
7. $3-\{5-(6-\overline{7-9})\}$.
8. $-2-[-3-\{-4-(-5-6)\}]$
9. $-a-[-3b-\{-2a-(-a-4b)\}]$
10. $a-[2b-\{3c-(a-\overline{2b-3c})\}]$
11. $3x-[5y-\{10z-(5x-\overline{10y-3z})\}]$
12. $-a-[-b-\{-c-(-\overline{a-b-c})\}]$.

Simplify the following expressions removing the brackets in the reverse order, *i e*, the outermost first and the innermost last —

13. $2x-[5y-\{9z-(10y-4x)\}]$
14. $-5a-[3b-\{6a-(5b-7a)\}]$
15. $-7m-[3n-\{8m-(4n-10m)\}]$.

16. $-2a-[-4b-\{-6c-(-8a-\overline{-10b-12c})\}]$.

17. $-3x-[-5y-\{-7z-(-9x-\overline{-11y-13z})\}]$.

18. $-2x-[-4y-\{-6z-(-3x-\overline{-5y-7z})\}]$.

19. $-x-[-3y+\{-5z-(-2x+\overline{-4y-6z})\}]$.

20. $-2a+[-5b-\{-8c+(-3a-\overline{-6b+9c})\}]$.

21. $-x+[-5y-\{-9z+(-3x-\overline{-7y+11z})\}]$.

Simplify —

22. $\{2a-(3b-5c)\}-[a-\{2b-(c-4a)\}-7c]$.

23. $[x-\{y-(z-x)\}-(y-z)]-[z-\{x-(y-z)\}]$

24. $[2a-(b-c)-\{3b-(2a-c)\}-\{-2a+(c-4b)\}]$
$-[-3b-(2a-4c)+\{6c-(2b-3a)\}-\{-5c+(6a-7b)\}]$.

In the expression $a-b-c+d-m+n-x+y-z$ —

25. Include the 2nd, 3rd and 4th terms in a pair of brackets preceded by the sign −, and the 5th, 6th and 7th in a pair of brackets preceded by the sign +.

26. Include all the terms after the 1st, in a pair of brackets preceded by the sign −, and of the expression thus enclosed put the last four terms within a pair of brackets preceded by the sign +.

27. Enclose the first five terms within a pair of brackets preceded by no sign and the last four within a pair of brackets preceded by the sign −, and then put the last three terms of each of these bracketed expressions within a pair of brackets preceded by the sign −.

28. Enclose every three terms from the first in a pair of brackets preceded by the sign −, and then put the last two terms of each of these bracketed expressions in a pair of brackets preceded by the sign −

Miscellaneous Exercises (I).

I.

1. What number will represent an interval of 5 hours (i) if the unit of time be half an hour, (ii) if the unit of time be 10 hours ?

2. If x stands for 17 and y for 25 what does $x\sim y$ denote ?

3. Define "Co-efficient" Distinguish between a *numerical* co-efficient and a *literal* co-efficient

What are the co-efficients of x^3 in $15x^3$, $2ax^3$, $7ab^2x^3$ and $16m^2pqx^3$?

4. Distinguish between $\sqrt{ab}$ and $\sqrt{a}b$ Find the value of $\sqrt{ab} \sim \sqrt{a}b$, when $a = 9$, $b = 4$.

5. If a distance of half a mile to the north of a place be represented by 40, what will represent a distance of 11 yards to the south of it?

6. State the result when a negative quantity is added to a positive quantity Hence deduce that $+(-b) = -b$

7. Define *subtraction*. Hence deduce that $4-6=-2$ and that $5-(-3) = 8$

8. Arrange the following numbers in descending order of magnitude $-2, 5, -3, 7, -8, -1, 9, -4, -12$

II.

1. If $a = 4$, $b = 5$, find the value of —

(i) $ab-a\times b$, (ii) $74-7a$,

(iii) $45-ab$, (iv) $85-8b$.

2. What does a^n mean? Distinguish between a^n and n^a Find the value of $a^4-4a^3b+6a^2b^2-4ab^3+b^4$, when $a = 7$, $b = 5$

3. What is the relation between a and each of the following $-\sqrt[3]{a}$, $\sqrt[5]{a}$, $\sqrt[8]{a}$, and $\sqrt[7]{a}$?

Find the value of $\sqrt{a^2-3d}\times\sqrt[3]{b^3-c^3-2e}$, when $a = 8$, $b = 7$, $c = 6$, $d = 5$ and $e = 1$.

4. What is meant by the *absolute value* of a positive or a negative quantity? Illustrate this by an example

5. Add together $3x^2y$, $-8x^2y$, $-19x^2y$ and $17x^2y$, and find the numerical value of the sum, when $x = 4$, $y = 5$

6. Write down the sum of $16x^4$, $-8xy^3$, $24x^2y^2$, y^4 and $-32x^3y$, and find its numerical value, when $x = 4$, $y = 5$.

7. Subtract $4a-13b-25c$ from $17b-12c-19a$.

8. Simplify $3x-[4y+\{2z-(x-5y+3z)\}]-(3x-7y)$.

III.

1. Express algebraically the following statements —

(i) The result of multiplying the sum of a and b by c is the same as the result of dividing x by the product of y and z.

(ii) The square of the sum of x and y is the same as the result of adding together the square of x, the square of y, and twice the product of x and y

(iii) If the cube root of the result of subtracting n from m be divided by the product of the cubes of m and n, we get a quantity which is less than the sum of the square roots of x and y

(iv) Since a is greater than b, therefore three times a is greater than three times b

2 A, B, C, D, E, F, G, are a number of successive points on a straight line such that the distances AB, BC, CD DE, EF, FG are respectively 3, 4, 6, 8, 5 and 7 inches If DC be represented by 3, what numbers will represent DB, DE, DF, DA and DG respectively?

3 State the result when one negative quantity is added to another Find the sum of $-a^3$, $-3a^2b$, $-3ab^2$, $-b^3$, when $a = 6$, $b = 4$.

4. Shew by a numerical example that when any number of quantities are added together, the result is the same in whatever order the quantities may be taken.

5 If $a = 16$, $b = 10$, $c = 5$, $d = 1$, find the value of $(a-b)(5\sqrt{a-b})+\sqrt{(a-b)(c+d)}$

6 If $a = \frac{1}{2}$, $b = \frac{2}{3}$, prove that

$$\frac{a^5+b^5}{a+b} = a^4-a^3b+a^2b^2-ab^3+b^4.$$

7. Add together $3a^2+4bc-x^2+10$, $2x^2-5a^2-15+6bc$ and $21-9bc-4a^2-10x^2$.

8 Simplify $a-[5b-\{a-(3c-3b)+2c-(a-2b-c)\}]$

IV

1. If $a = 9$, find the value of —

(i) $\sqrt{49}-\sqrt{4a}$; (ii) $\sqrt{49-\sqrt{4a}}$.

2 Shew by a numerical example that when any number of quantites are added together, they can be divided into groups and the result expressed as the sum of these groups

3 If $a = 2$, $b = 3$, $c = 4$, find the value of

$$\frac{a-b+c}{a+b-c}+\frac{b-c+a}{b+c-a}+\frac{c-a+b}{c+a-b}$$

4 Define an *Algebraical Expression* Distinguish between a *simple* expression and a *compound* expression

Is $42abx^2$ a simple or a compound expression? Give the names with illustrations of the different classes of compound expressions

5 If $x = 2$, $y = 3$, $a = 6$, $b = 5$, find the value of

$$\sqrt[3]{b(x+y)^2}+\sqrt[3]{(x+a)(b-2x)}+\sqrt[3]{x(b-y)^2}.$$

6 A certain sum is divided between A, B and C, B receives a pounds more than A, and C receives b pounds more than B, if A receives x pounds, find an expression for the whole sum divided

7 Add together $a^2-3ab-\frac{14}{31}b^2$, $2b^2-\frac{2}{3}b^3+c^2$, $ab-\frac{1}{3}b^2+b^3$ and $2ab-\frac{1}{3}b^3$

8. Reduce to its simplest form

$$\{2x^2-(y^2-xy)\}-\{y^2-(4x^2-y^2)\}+\{2y^2-(3xy-x^2)\}.$$

V

1 What is meant by the *dimensions* and *degree* of a product? What is a *Homogeneous Expression*? Write down two trinomial homogeneous expressions, one of six dimensions and the other of seven

2. If you were asked to find the value of the expression $a\times b-c-d\times e+f-gh$, how would you proceed?

3 Define *factor*. What are the *simple factors* of $2ab(a+b)$?

4 State the proposition from the converse of which is deduced the following rule —*To add together two or more algebraical expressions, write down the terms in succession with their proper signs*

5. If $a = 4$ and $x = 2$, find the numerical value of

$$\frac{2ax^2}{(a-x)^2}-\frac{6\sqrt[3]{ax}}{a^3\sqrt{2a+4x}}-\frac{29x^2}{64a}.$$

6. Find the value of $(x^3-7x^2+6x+5)+(-8x+2x^3+4+5x^2)+(-11-4x^3+2x-7x^2)+(9x^2+2+5x^3-4x)$, when $x=5$.

7. Prove that $a-(b-c)=a-b+c$. How is this generally proved, when a, b, c are all positive quantities and a is greater than b and b is greater than c?

8 Simplify $2x-[(3x-9y)-\{(2x-3y)-(x+5y)\}]$.

VI

1. Define the *power* of a number and the *index* of the power, and illustrate them by a numerical example

2. If $a=16$, $b=10$, $x=5$, $y=1$, find the numerical value of $(a-y)\sqrt{24bx+x^2} + \sqrt{(a-x)(b+y)}$

3. Shew that $a^3+b^3+c^3-3abc$

$$=(a+b+c)(a^2+b^2+c^2-ab-ac-bc),$$

(i) when $a=3$, $b=4$, $c=5$,

(ii) when $a=\frac{3}{2}$, $b=\frac{5}{2}$, $c=\frac{7}{2}$.

4. State the propositions from which the following result may be deduced:—

$$a-b+c-d+e-f=(a+c+e)+(-b-d-f).$$

5. Illustrate clearly by an example that

$$40-(-15)=55$$

6 Find the numerical value of the sum of $7x^3-25\sqrt{yz}+z^4$, $19\sqrt{yz}-3z^4-12x^3$ and $2z^4+5x^3+7\sqrt{yz}$, when $x=17$, $y=16$, $z=15$

7. State the operations indicated by the expression

$$5a-[4b-\{3c-(2d-7e)\}].$$

8 Find the value of

$$[(a^3+b^3+c^3+d^3)\{a+b-(c-a)\}+a^2b-c^2d]\times\{a^2-(b^2+c^2)+d^2\},$$ when $a=4$, $b=3$, $c=2$, $d=1$.

VII

1 Distinguish between — (i) $a-bc$ and $a-b\times c$; (ii) a^4 and $4a$, (iii) $3\sqrt{a}$ and $\sqrt[3]{a}$, (iv) $\sqrt{a+b}$ and $\sqrt{a}+b$, (v) $\sqrt{ab}$ and $\sqrt{a}\,b$.

2. If $a = 1$, $b = 2$, $c = 3$, $d = 0$, find the value of —

(i) $\dfrac{a^2b+b^2c+c^2d+d^2a}{(a+b)(c+d)-\{(a-d)+(c-b)\}}$,

(ii) $\sqrt[3]{b-a}+\sqrt[3]{4(c-a)}-\sqrt[4]{3(8a+5b+3c-2d)}$.

3 Show that the expressions $(a+b+c)^3+a^3+b^3+c^3$, $(a+b)^3+(b+c)^3+(c+a)^3+6abc$ and $2a^3+3b^2(a+c)+2b^3+3c^2(a+b)+2c^3+3a^2(b+c)+6abc$ are equal to one another, (i) when $a = 2$, $b = 3$, $c = 4$,
(ii) when $a = 7$, $b = 4$, $c = 1$

4. Simplify —

(i) $1-[1-\{1-(1-\overline{1+x})\}]$.

(ii) $3b-(b-2c)-\{a+c-(3a-b-2c)\}-(2a-3b+4c)$,

5 Express algebraically the following statements —

(i) That the product of the sum of two numbers multiplied by their difference is equal to the difference of the squares of the numbers

(ii) That the square of the sum of two numbers exceeds the sum of their squares by twice their product.

6. Find the value of

$17a-5b-[7a-3b-\{4(a-b)-(2a+3b)\}]$, when $a = 39$, $b = 52$.

7. If $V = 5a+4b-6c$, $X = -3a-9b+7c$,
$Y = 20a+7b-5c$, $Z = 13a-5b+9c$,
calculate the value of $V-(X+Y)+Z$

(Madras University Matriculation Paper, 1883).

8 From the sum of $a-\frac{1}{3}b+\frac{1}{4}c-\frac{1}{5}d$, $-\frac{1}{2}c+\frac{1}{3}a-\frac{1}{4}b+d$, $\frac{1}{4}d-\frac{1}{5}b+c-a$, $\frac{1}{6}a-\frac{3}{5}d+b-\frac{5}{8}c$ and $8a-6b+3c-4d$ subtract $\frac{13}{2}a-\frac{171}{30}b+\frac{9}{4}c-\frac{13}{4}d$

CHAPTER V.

MULTIPLICATION.

1. Definition. One number is said to be multiplied by another, when we do to the former what is done to unity to obtain the latter

Thus, since $4 = 1+1+1+1$, we must have

$$4 \times x \text{ or } 4x = x+x+x+x$$

Similarly,
$$\left.\begin{array}{lll} 4\times 5 & = 5+5+5+5 & = 20 \\ 3\times 6 & = 6+6+6 & = 18 \\ 5\times 3 & = 3+3+3+3+3 & = 15 \end{array}\right\} \quad \ldots\ldots \text{I}$$

$$\left.\begin{array}{ll} 3\times(-5)=(-5)+(-5)+(-5) & =-15 \\ 4\times(-3)=(-3)+(-3)+(-3)+(-3) & =-12 \\ 5\times(-4)=(-4)+(-4)+(-4)+(-4)+(-4) & =-20 \end{array}\right\} \quad \text{II}$$

Again, since $-4 = -1-1-1-1$, we must have

$$(-4)\times x = -x-x-x-x$$

Similarly,
$$\left.\begin{array}{ll} (-4)\times 5 = -5-5-5-5 & = -20 \\ (-3)\times 6 = -6-6-6 & = -18 \\ (-5)\times 3 = -3-3-3-3-3 & = -15 \end{array}\right\} \quad \ldots\text{III}$$

also,

$$\left.\begin{array}{ll} (-3)\times(-5) = -(-5)-(-5)-(-5) & \\ \qquad = 5+5+5 & = 15 \\ (-4)\times(-3) = -(-3)-(-3)-(-3)-(-3) & \\ \qquad = 3+3+3+3 & = 12 \\ (-5)\times(-4) = -(-4)-(-4)-(-4)-(-4)-(-4) & \\ \qquad = 4+4+4+4+4 & = 20 \end{array}\right\} \quad \ldots\text{IV}$$

The number multiplied is called the *multiplicand*, and the number by which it is multiplied is called the *multiplier*. the result is called the *product*

Exercise (14).

From the definition of multiplication deduce the result —

1. When 6 is multiplied by 5
2. When 7 is multiplied by 4
3. When 12 is multiplied by 3
4. When − 8 is multiplied by 4
5. When − 9 is multiplied by 5
6. When −13 is multiplied by 6
7. When 8 is multiplied by −3.
8. When 14 is multiplied by −5
9. When 15 is multiplied by −3.
10. When − 9 is multiplied by −4
11. When −12 is multiplied by −5
12. When −16 is multiplied by −4

2. The Law of Signs. From the last article it is clear that if a and b are two whole numbers, we have

$$\left.\begin{aligned}(+a)\times(+b)&=+(ab)\\(+a)\times(-b)&=-(ab)\\(-a)\times(+b)&=-(ab)\\(-a)\times(-b)&=+(ab)\end{aligned}\right\}$$

Thus, the product of two whole numbers is positive or negative according as the multiplicand and the multiplier have like or unlike signs

The same thing can be found when the numbers are fractional For instance, since $-\frac{2}{3}=-\frac{1}{3}-\frac{1}{3}$, *i e*, since $-\frac{2}{3}$ is obtained by subtracting a third part of unity twice, to multiply any number x by $-\frac{2}{3}$ we must subtract a third part of x twice

Hence $\quad (-\frac{2}{3})\times x \quad = -\frac{x}{3}-\frac{x}{3}=-\frac{2x}{3}$

Similarly, $\quad (-\frac{2}{3})\times\frac{4}{5} \quad = -\frac{4}{15}-\frac{4}{15} \quad = -\frac{8}{15}$.

$$(-\tfrac{2}{3})\times(-\tfrac{4}{5}) = -(-\tfrac{4}{15})-(-\tfrac{4}{15})$$
$$= \tfrac{4}{15}+\tfrac{4}{15} = \tfrac{8}{15}:$$

Hence, we can enunciate the *Law of signs* in a more general way, thus —The sign of the product of any two quantities is positive or negative according as the multiplicand and the multiplier have like or unlike signs Or, more briefly, thus :—Like signs produce + and unlike signs −

Cor. Since $(-x)\times(-x) = x^2$ and also $(+x)\times(+x) = x^2$, we have $\sqrt{x^2} = \pm x$. Thus every algebraical quantity has got two square roots which are equal in absolute value but opposite in sign

Example. Simplify $(a^2b-cd\,(c^2-d^2)$, when $a = -2$, $b = -3$, $c = -4$, $d = -5$

Since $a^2b = (-2)^2\times(-3) = 4\times(-3) = -12$

and $cd = (-4)\times 5 = -20$,

$\therefore\ a^2b-cd = -12-(-20) = -12+20 = 8$. (A)

Also, since $c^2 = (-4)^2 = 16$

and $d^2 = (5)^2 = 25$,

$\therefore\ c^2-d^2 = 16-25 = -9$ .. .. (B)

Hence, from (A) and (B), we have

$$(a^2b-cd)(c^2-d^2) = 8\times(-9) = -72$$

Exercise (15).

Find the value of —

1. $ab-cd$, when $a = -2$, $b = -3$, $c = -8$, $d=6$.
2. $(a^2-b^2)c$, when $a = -5$, $b = -6$, $c = -9$.
3. $abc+b^2c-c^2d$, when $a = 3$, $b = -5$, $c = -7$, $d = -4$.
4. $(-a)b^2-cd^2+b(-c)^2$, when $a = 5$, $b = -7$, $c = 4$, $d = -3$
5. $\{a^2+(-b)^2-(-c)^2\}(bc-ad)$, when $a = -3$, $b = 4$, $c = 6$, $d = -9$.
6. $a^2(b-c)+b^2(c-a)+c^2(a-b)$, when $a = -2$, $b = -5$, $c = -7$.
7. $x^3(y-z)+y^3(z-x)+z^3(x-y)$, when $x = -3$, $y = 8$, $z = -5$
8. $p^3(q^2-r^2)+q^3(r^2-p^2)+r^3(p^2-q^2)$, when $p = -3$, $q = -5$, $r = -7$.

9. $a^3+b^3+c^3-3abc$, when $a=-12$, $b=-13$, $c=-15$.

10. Show that $(a+b)^5 = a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$, when $a=-5$, $b=3$.

3 To prove that $a \times b = b \times a$, *i.e.*, b multiplied by a gives the same result as a multiplied by b.

(1) First let a and b be any two positive integers.

Place b units in a horizontal row and write down a such rows in such a manner that units in similar positions in the different rows may be in the same vertical column, thus —

1	1	1	1	1	.	b times
1	1	1	1	1	. .	b times
1	1	1	1	1	. . .	b times
.			.	.		
.	.		.		. . .	

to a lines.

This being done, evidently it may also be said that we have written down b columns, each containing a units

Now let us count up the total number of units thus written down.

Since we have got a rows each containing b units, the total number of units = (the number in the 1st row)+(the number in the 2nd row)+(the number in the 3rd row)+ +(the number in the a^{th} row) = $b+b+b+\ldots$to a terms $= a\times b$ . .. . (1)

Also, since we have got b columns each containing a units, the total number of units = (the number in the 1st column)+(the number in the 2nd column)+(the number in the 3rd column)+ +(the number in the b^{th} column) $= a+a+a+$ to b terms $= b\times a$ . .. (2)

Hence, from (1) and (2), we have $a\times b = b\times a$*

i e, b taken a times = a taken b times

---

* Since $ab=ba$, it does not matter much whether we read ab as a times b or b times a (*i e*, as b multiplied by a or a multiplied by b), but *until the proposition of the present article had been proved* it seems expedient to stick to one and the same mode of interpreting it. If a beginner is taught to read $7a$ as "7 times a" whilst 7×4 as "4 times 7", he is but unconsciously led to think that such expressions as ba and ab mean the same, and that consequently no amount of reasoning is necessary to establish the above proposition As a safeguard against this evil I have hitherto throughout taken $a\times b$ to mean "a times b", or "b multiplied by a"

(ii) Next let a and b be two positive fractions ; suppose $a=\dfrac{m}{n}$ and $b=\dfrac{p}{q}$, where m, n, p, q are positive integers.

$$\text{Then } a\times b=\frac{m}{n}\times\frac{p}{q}=m\times\left\{\left(\frac{p}{q}\right)\div n\right\}$$

$$=m\times\frac{p}{nq}=\frac{mp}{nq} \qquad \dots \quad \dots \quad \dots \qquad \text{(I)}$$

$$\text{and } b\times a=\frac{p}{q}\times\frac{m}{n}=p\times\left\{\left(\frac{m}{n}\right)\div q\right\}$$

$$=p\times\frac{m}{qn}=\frac{pm}{qn} \qquad \dots \quad \dots \quad \dots \qquad \text{(II)}$$

But m and p are positive integers, therefore $mp=pm$, and similarly $nq=qn$.

Hence, from (I) and (II), we have $a\times b=b\times a$. †

Thus it is established that for *all positive values* of a and b we must have $a\times b=b\times a$. .. .. (A)

Cor. 1. From Art 2, we have $x\times(-y)=-(xy)$ and $(-y)\times x=-(yx)$, but $xy=yx$, $\therefore\ x\times(-y)=(-y)\times x$. (B)

Cor. 2. From Art. 2, $(-x)\times(-y)=+xy$, and $(-y)\times(-x)=+yx$; but $xy=yx$, $\therefore\ (-x)\times(-y)=(-y)\times(-x)$ .. (C)

Hence, from (A), (B) and (C) we conclude that for *all values* of a and b, $a\times b=b\times a$.

---

† We can illustrate $a\times b=b\times a$ when a and b are fractions as follows —

Let us prove that $\frac{2}{3}\times\frac{4}{5}=\frac{4}{5}\times\frac{2}{3}$

$\frac{2}{3}\times\frac{4}{5}$ means that we have to divide $\frac{4}{5}$ of any thing into 3 equal parts and take 2 of those parts, whilst $\frac{4}{5}\times\frac{2}{3}$ means that we have to divide $\frac{2}{3}$ of a thing into 5 equal parts and take 4 of those parts

A ______________ B

Take a line AB 15 inches long, then $\frac{4}{5}$ of the line will be 12 inches and evidently $\frac{2}{3}$ of 12 inches $=8$ inches, thus $\frac{2}{3}\times\frac{4}{5}$ of the line $=8$ inches.

Again, since $\frac{2}{3}$ of the line is 10 inches, and $\frac{4}{5}$ of 10 inches $=8$ inches,

$\frac{4}{5}\times\frac{2}{3}$ of the line also $=8$ inches

Hence, we have $\frac{2}{3}\times\frac{4}{5}=\frac{4}{5}\times\frac{2}{3}$ Similarly, any other case may be illustrated.

4

Exercise (16).

Prove that :—

1. $4\times5=5\times4$ 2. $6\times3=3\times6$ 3. $7\times5=5\times7$.
4. $4\times8=8\times4$. 5. $9\times5=5\times9$.

4. To prove that $(ab)\times c=a\times(bc)$ or, $= b\times(ac)$, *i e*, to multiply c by the product of a and b is the same as to multiply c first by either of them and then that result by the other.

Place b brackets in a horizontal row each containing c units and write down a such rows in such a manner that the brackets in similar positions in the different rows may be in the same vertical column, thus —

$[c]$ $[c]$ $[c]$ $[c]$ b times
$[c]$ $[c]$ $[c]$ $[c]$. b times
$[c]$ $[c]$ $[c]$ $[c]$. b times
. . . .
. . . .
to a rows.

This being done, it may also be said that we have written down b columns each containing a brackets

As we have got altogether $a\times b$ brackets and as each bracket contains c units, the total number of units $=$ $(ab)\times c$ .. .. (α)

Again, since we have got b brackets in a row each containing c units, the number of units in a row $= bc$, and as there are a rows altogether, therefore the total number of units $= a\times(bc)$ ... . (β)

Again, since we have got a brackets in a column each containing c units, the number of units in a column $= ac$, and as there are b columns altogether, therefore the total number of units $= b\times(ac)$ .. .. (γ)

Hence, from (α), (β) and (γ) we have

$$(ab)\times c = a\times(bc) = b\times(ac).$$

Cor. From the results of the last article and this, we deduce that $abc = bca = cab$ For, by the present article $abc = a\times(bc)$, and by the last article $a\times(bc)=(bc)\times a=bca$;

hence we have $abc = bca$, and similarly $bca = cab$. Thus we are led to conclude that *the value of a product is the same in whatever order the factors may be taken.**

Note 1 Although the factors of a product can be taken in any order, it is always found convenient to place first the factor expressed in figures and to put after it the factors expressed in letters in the alphabetical order of those letters Thus $c^3 \times d \times 7 \times b \times a^4$ is written $7a^4bc^3d$

Note 2 We are now in a position to modify a little the definition of *Co-efficient* given in Art 10, Chap I In an algebraical product one or more of the factors may be called the co-efficient of the remaining factors

For instance, in $7abcd$ we may call $7ac$ as the co-efficient of bd, for $7abcd$ can be written as $7acbd$ and therefore by the definition alluded to, $7ac$ is the co-efficient of bd

5. To prove that $a^m \times a^n = a^{m+n}$ where m and n are any two positive integers.

N B From Art 3 we know that the quantity on either side of $\times$ may be regarded as the multiplier and that on the other as the multiplicand Hence, we need not any longer observe the restriction we have hitherto placed upon the meaning of $a \times b$

[See foot note, page 47]

$$\begin{aligned}
\text{Since} \quad a^2 &= aa \\
\text{and} \quad a^3 &= aaa \\
\therefore \quad a^2 \times a^3 &= (aa) \times (aaa) \\
&= a \times a \times a \times a \times a \qquad \text{[Art 4]} \\
&= a^5 = a^{2+3}.
\end{aligned}$$

$$\begin{aligned}
\text{Again, since} \quad a^4 &= aaaa \\
\text{and} \quad a^6 &= aaaaaa, \\
\therefore \quad a^4 \times a^6 &= (aaaa) \times (aaaaaa) \\
&= a \times a \times a \times a \times a \times a \times a \times a \times a \times a \; \text{[Art 4.]} \\
&= a^{10} = a^{4+6}.
\end{aligned}$$

$$\begin{aligned}
\text{Generally, since} \quad a^m &= aaaa \ldots \text{ to } m \text{ factors} \\
\text{and} \quad a^n &= aaaa \ldots \text{ to } n \text{ factors}, \\
\therefore \quad a^m \times a^n &= (aaaa \ldots \text{ to } m \text{ factors}) \times \\
&\qquad (aaaa \ldots \text{ to } n \text{ factors}) \\
&= aaaaaaaaa \ldots \text{ to } (m+n) \text{ factors} \\
&= a^{m+n}
\end{aligned}$$

---

* The validity of the conclusion has been established only for three factors A general proof, however, has not been attempted as being too tedious for the class of students for whom the book is meant

Cor. 1. $a^m \times a^n \times a^p = a^{m+n+p}$, where m, n and p are positive integers.

For $a^m \times a^n = a^{m+n}$, $\therefore a^m \times a^n \times a^p = a^{m+n} \times a^p = a^{(m+n)+p}$ $= a^{m+n+p}$.

Cor. 2. $\left(a^m\right)^n = a^{nm}$, where m and n are positive integers.

For $\left(a^m\right)^n = a^m \times a^m \times a^m \times \ldots$ to n factors

$$= a^{m+m+m+\text{ to } n \text{ terms}}$$

$$= a^{nm}$$

6. Applications of the principles established in the preceding articles —

Example 1. Show that $(-ab)^2 = a^2b^2$.

$(-ab)^2$	$= (-ab) \times (-ab)$	
	$= (ab) \times (ab)$	[Art. 2.]
	$= a \times b \times a \times b$	[Art. 4.]
	$= a \times a \times b \times b$	[Cor. Art. 4.]
	$= (aa) \times (bb)$	[Art. 4.]
	$= a^2b^2.$	[Art. 5.]

Example 2. Multiply $-5a^3b^2$ by $4a^5b^4$.

$(-5a^3b^2) \times (4a^5b^4)$	$= -\{(5a^3b^2) \times (4a^5b^4)\}$	[Art. 2.]
	$= -\{5 \times a^3 \times b^2 \times 4 \times a^5 \times b^4\}$	[Art. 4.]
	$= -\{5 \times 4 \times a^3 \times a^5 \times b^2 \times b^4\}$	[Cor. Art. 4.]
	$= -\{20 \times (a^3a^5) \times (b^2b^4)\}$	[Art. 4.]
	$= -20a^8b^6$	[Art. 5.]

Example 3. Simplify $(-2x^5y^4z) \times (4x^2y^7z^2) \times (-6xy^3z^4)$.

We have $(-2x^5y^4z) \times (4x^2y^7z^2)$

$$= -\{(2x^5y^4z) \times (4x^2y^7z^2)\}$$

$$= -\{2 \times x^5 \times y^4 \times z \times 4 \times x^2 \times y^7 \times z^2\}$$

$$= -\{2 \times 4 \times x^5 \times x^2 \times y^4 \times y^7 \times z \times z^2\}$$

$$= -\{8 \times (x^5x^2) \times (y^4y^7) \times (zz^2)\}$$

$$= -8x^7y^{11}z^3.$$

Hence, the given expression

$$= (-8x^7y^{11}z^3)\times(-6xy^3z^4)$$
$$= (8x^7y^{11}z^3)\times(6xy^3z^4)$$
$$= 8\times x^7\times y^{11}\times z^3\times 6\times x\times y^3\times z^4$$
$$= 8\times 6\times x^7\times x\times y^{11}\times y^3\times z^3\times z^4$$
$$= 48\times(x^7x)\times(y^{11}y^3)\times(z^3z^4)$$
$$= 48x^8y^{14}z^7.$$

Exercise (17).

Show that :—

1. $(-a)\times 3b = -3ab.$
2. $(2a)\times(-4b) = -8ab.$
3. $(-3a)\times(-6a^2) = 18a^3.$
4. $(-4b)\times(-5a) = 20ab.$
5. $(-7c)\times(-3ab) = 21abc.$
6. $10\times 35 = 25\times 14.$
7. $15\times 75 = 5^3\times 3^2.$
8. $(-a)^3 = -a^3.$
9. $(ab)^3 = a^3b^3.$
10. $(-ab)^3 = -a^3b^3.$
11. $(-a^3b^5)^2 = a^6b^{10}.$
12. $(-x)^5 = -x^5.$
13. $(-xy)^5 = -x^5y^5.$

Multiply :—

14. $6x^2y$ by $-7x^5y^4.$
15. $-3x^4y^3$ by $8x^7y^8.$
16. $-5x^{12}y^3$ by $-8x^5y^{13}.$
17. $-12x^3y^3z^2$ by $13x^7y^6z^4.$
18. $-14xy^5z^8$ by $-10x^5y^2z^{12}$

Simplify :—

19. $(-3ab)\times(4a^2b^2)\times(-2a^3b^3).$
20. $(-2a^4)\times(-7a^4b^7)\times(-5a^9b^5)$
21. $(-6x^5y^2z)\times(2z^4x^3y^5)\times(-4y^3z^2x^8)$
22. $(-3x^2y)\times(4zy^2x)\times(-x^3z^5y^4)\times(2zxy)$

7. Products of monomial expressions can be always found by the method illustrated in the last article ; it is necessary however when dealing with more complicated cases of multiplication, that such products should be found mentally. Hence the student must get thoroughly accustomed to this kind of mental work, for which an exercise is added below

Example 1. Write down the product of $3x^2$ and $-5xy$.

$$3x^2 \times (-5xy) = -15x^3y$$

Example 2. Write down the product of $-5a^2b$ and $-8ab^2$.

$$(-5a^2b) \times (-8ab^2) = 40a^3b^3.$$

Exercise (18).

Write down the product of —

1. $-2x^3$	and $5x^4$	2. $5a^3b$	and $-4ab^5$.
3. $-3m^2n^5$	and $-7n^3m^5$.	4. $3x^3y^5$	and $-6xy^2$.
5. $-a^3b^2$	and $-3a^4b^8$	6 $5mn^6$	and $-8m^7n$
7. $-10xyz^2$	and $-5xy^2z$.	8. $4x^3y^3z$	and $-6xyz^3$.
9. $-6x^2y^3z^4$	and $-8x^3y^2z$	10. $-5a^3b^5c^7$	and $-5a^2b^4c^6$.
11. $3x^2yz^4$	and $-8xy^2z$.	12. $-4abxy$	and $-8a^2xby^2$.
13. $-7a^2b^2z^3$	and $-5abz$.	14. $5a^4x^2y$	and $-12x^5y^4a^2$.
15. $-14xy^4$	and $-5x^4yz$.	16. $2abc^5$	and $-9a^7b^6c$.
17 $-7a^3x^5y$	and $-9x^3ya^6$.	18. $-8x^6y^2z^5$	and $-20y^5z^2x^8$.

19. $-13a^8b^{13}c^{15}$ and $-5bc^5a^2$.

20. $-7a^7x^8y^6z^2$ and $-16z^5x^2a^6y^3$

8. To prove that $a(b+c)$* $= ab+ac$.

Whatever b and c may be if a be a *positive integer*, we have

$$a(b+c) = (b+c)+(b+c)+(b+c)+ \ldots \text{ to } a \text{ terms}$$
$$= (b+b+b+ \ldots \text{ to } a \text{ terms})$$
$$+(c+c+c+ \ldots \text{ to } a \text{ terms})$$
$$= ab+ac \quad \ldots \quad (1)$$

Hence, conversely, $\frac{ab+ac}{a} = b+c = \frac{ab}{a}+\frac{ac}{a}$, that is, if p and q be any two quantities and r a *positive integer*, then

$$\frac{p+q}{r} = \frac{p}{r}+\frac{q}{r} \quad \ldots \quad (A)$$

---

* Every binomial expression can be put in the form $b+c$ For instance the expression $2x^2-3y^2$, which can also be written as $(2x^2)+(-3y^2)$ is of the form $b+c$, $2x^2$ being regarded as b and $-3y^2$ as c.

Next suppose a is a positive fraction, *i e*, suppose $a=\frac{m}{n}$ where m and n are positive integers

Then, $\frac{m}{n}(b+c) = m\times\frac{b+c}{n}$ [by the definition of multiplication]

$$= \frac{m(b+c)}{n}$$

$$= \frac{mb+mc}{n} \text{ [by (1)]}$$

$$= \frac{mb}{n}+\frac{mc}{n} \text{ [by (A)]}$$

$$= \frac{m}{n}b + \frac{m}{n}c \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

Hence, from (1) and (2), for all *positive values* of a we have $a(b+c) = ab+ac$ . .. ... (3)

Next suppose a is any negative quantity, *i e*, suppose $a=-x$ where x is any positive quantity.

Then $(-x)\ (b+c) = -[x(b+c)]$
$= -(xb+xc)$ [by (3)]
$= -xb-xc$
$= (-x).b+(-x)c$;

thus for any *negative* value of a also we have

$$a(b+c) = ab+ac \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

Hence from (3) and (4), for *all values* of a, b and c we have $a(b+c) = ab+ac$.

***Cor.* 1.** Conversely, $ab+ac = a(b+c)$.

Similarly, $xya^2+xyb^2 = xy(a^2+b^2)$

***Cor.* 2.** Since $b-c = b+(-c)$, we have

$$a(b-c) = a[b+(-c)] = ab+a(-c) = ab-ac.$$

Conversely, $ab-ac = a(b-c)$ Hence $2ax-2ay=2a(x-y)$.

***Cor.* 3.** $a(b+c+d) = a\{b+(c+d)\} = ab+a(c+d)$
$= ab+ac+ad.$

Similarly, $a(b+c+d+e+f+\ldots) = ab+ac+ad+ae+af+\ldots\ \ldots$

Thus, when any multinomial expression is multiplied by a monomial, the result is the sum of the products obtained by multiplying the different terms of the multinomial, by the monomial

Conversely, $ab+ac+ad+ae+\ \ldots = a(b+c+d+e+\ldots)$.

Example 1. Multiply $2ab-3b^2$ by $5ab$

$$\begin{aligned}5ab(2ab-3b^2) &= 5ab\{2ab+(-3b^2)\}\\ &= 5ab\times 2ab+5ab\times(-3b^2)\\ &= 10a^2b^2-15ab^3.\end{aligned}$$

Example 2. Multiply $x^4-3x^3+5x^2-6x+4$ by $-6x^2$.

$$\begin{aligned}&(-6x^2)(x^4-3x^3+5x^2-6x+4)\\ &= (-6x^2)\{x^4+(-3x^3)+5x^2+(-6x)+4\}\\ &= (-6x^2)\,x^4+(-6x^2)(-3x^3)+(-6x^2)\,5x^2\\ &\qquad +(-6x^2)(-6x)+(-6x^2).4\\ &= -6x^6+18x^5-30x^4+36x^3-24x^2.\end{aligned}$$

N B The beginner is particularly recommended to work out at first each such example in the method shown above, but after some practice he can safely do away with the intermediate steps and write down the result at once in the manner exemplified below

Example 3 Write down the product of

$-4a^4+5a^3b-6a^2b^2-8ab^3+9b^4$ and $-3a^2b^2$.

$$\begin{array}{l}-4a^4+5a^3b-6a^2b^2-8ab^3+9b^4\\ -3a^2b^2\\ \hline 12a^6b^2-15a^5b^3+18a^4b^4+24a^3b^5-27a^2b^6\end{array}$$

Example 4. Simplify $2x^2(3x-2)+2x(2x+3)-6(x-3)$.

We have
$$\begin{aligned}2x^2(3x-2) &= 6x^3-4x^2,\\ 2x(2x+3) &= 4x^2+6x,\\ 6(x-3) &= 6x-18\end{aligned}$$

Therefore, the given expression

$$\begin{aligned}&= (6x^3-4x^2)+(4x^2+6x)-(6x-18)\\ &= 6x^3-4x^2+4x^2+6x-6x+18 = 6x^3+18.\end{aligned}$$

Example 5. Simplify $3a(2a-5)-3a(a-6)$

Putting x for $2a-5$ and y for $a-6$.

we have $3a(2a-5)-3a(a-6) = 3ax-3ay$

$$= 3a(x-y)$$

$$= 3a\{(2a-5)-(a-6)\}$$

$$= 3a(a+1) = 3a^2+3a.$$

Exercise (19).

Multiply :—

1. $2a-3b$ by $-ab$.
2. $a-2b+3c$ by $-5a$.
3. $2a^2-3b^2-c^2$ by abc
4. $x^2y-2xy^2-y^3$ by $-3xy$.
5. $-a^2b+ab^2-3a^2b^2+5a^3$ by $-7b^2$.
6. $2ab-3bc+4ac-5abc$ by $-8a^2bc$.

Write down the product of :—

7. $3a^2x-4ax^2+5ax$ and $-2a^2$.
8. $-2m^3+3m^2n-5mn^2$ and $4mn$.
9. $5x^4-6x^3+7x^2-8x$ and $-4x^3$
10. $-2c^2d+3d^3c-5cd^2-4c^2d^2$ and $-6c^2d^4$
11. $8a^4-6a^3b+5a^2b^2-4ab^3$ and $-2a^3b^3$.
12. $-a^2bc+ab^2c-abc^2+a^2b^2-b^2c^2$ and $-a^3b^3c^3$.

Simplify :—

13. $7x^3(x-2)-2x^2(x-3)-8x^2(1-2x)$
14. $-5x^3(x+7)-7x^2(3-5x)+3x(7x-4)+17x$.
15. $9x^3(x^3-2y^2)+5y^2(3x^3+y^2)+3y^2(x^3-10y^2)$.
16. $x^3(x^3+2x^2+2x)-2x^2(x^3+2x^2+2x)+2x(x^3+2x^2+2x)$.
17. $a^6b^3(a^6b^3-2a^4b^2+2a^2b)+2a^4b^2(a^6b^3-2a^4b^2+2a^2b)$ $+2a^2b(a^6b^3-2a^4b^2+2a^2b)$.
18. $2a^9b^6(2a^9b^6+6a^6b^4+9a^3b^2)-6a^6b^4(2a^9b^6+6a^6b^4$ $+9a^3b^2)+9a^3b^2(2a^9b^6+6a^6b^4+9a^3b^2)$.
19. $a^2(2x-3y)+a^2(3x+4y)-a^2(5x-2y)$.
20. $2ab(3a^2-4b^2)-2ab(2a^2-3b^2)-2ab(2a^2-b^2)$.
21. $x^2(2ax+3bx+4cx)-x^2(ax+2bx+3cx)-x^2(bx+cx)$.

9. To prove that $(a+b)(c+d)=ac+ad+bc+bd$.

Putting x for $c+d$, we have

$$
\begin{aligned}
(a+b)(c+d) &= (a+b)x \\
&= x(a+b) \\
&= xa+xb \qquad \text{[Art 8]} \\
&= ax+bx \\
&= a(c+d)+b(c+d) \\
&= ac+ad+bc+bd \qquad \text{[Art 8]}
\end{aligned}
$$

Cor. 1 In like manner it may be shown that

$$
\begin{aligned}
(a+b+c+d+\ldots)(m+n+p+q+\ldots) \\
= a(m+n+p+q+\ldots)+b(m+n+p+q+\ldots) \\
+c(m+n+p+q+\ldots)+d(m+n+p+q+\ldots)+\&c
\end{aligned}
$$

Thus to multiply one multinomial expression by another we have to multiply every term of the one by every term of the other and take the algebraic sum of these partial products.

Cor. 2. Since $a-b=a+(-b)$ and $c-d=c+(-d)$, therefore,

$$
\begin{aligned}
(a-b)(c-d)=\{a+(-b)\}\{c+(-d)\}=ac+a(-d)+(-b)c \\
+(-b)(-d)=ac-ad-bc+bd
\end{aligned}
$$

Example 1. Multiply $2a+3b$ by $4a+5b$

$$
\begin{aligned}
(4a+5b)(2a+3b) &= (4a)(2a)+(4a)(3b)+(5b)(2a)+(5b)(3b) \\
&= 8a^2+12ab+10ab+15b^2 \\
&= 8a^2+22ab+15b^2
\end{aligned}
$$

Example 2 Multiply $3x-7y$ by $2x-5y$.

$$
\begin{aligned}
(2x-5y)(3x-7y) &= (2x)(3x)+(2x)(-7y)+(-5y)(3x) \\
&\qquad +(-5y)(-7y) \\
&= 6x^2-14xy-15xy+35y^2 \\
&= 6x^2-29xy+35y^2.
\end{aligned}
$$

Exercise (20).

Multiply —

1. $2a+3b$ by $a+b$.
2. $2m-3n$ by $m-n$
3. $a+b+c$ by $a+b+c$
4. $a-b+c$ by $a-b+c$.

5. $a-b-c$ by $a-b-c$ 6. $a-2b-3c$ by $2a-b-c$.
7. $2x-3y-4z$ by $x-y-z$. 8. $-5x+2a-3b$ by $-x-a+b$.
9. $x^2+y^2+z^2$ by $x-y-z$. 10. $xy+yz+zx$ by $xy-yz-zx$.

10. Arrangement of an expression according to descending or ascending powers of a letter.

When the different terms of an expression contain different powers of any letter, if we arrange the terms in such a way that the term containing the highest power of that letter is put first on the left, the term containing the next highest power is put next, and so on, and the term which either contains the lowest power of that letter or does not contain that letter at all is put last, then we are said to arrange the expression according to *descending* powers of the letter considered. If the order of the terms be reversed, the arrangement is said to be according to *ascending* powers of the letter. Thus the expression $a^5x^3+3a^4xy-5a^3x^6y^2+4a^2x^4y^3-2ax^2y^4+x^5y^5$ as it stands may be considered as arranged either according to descending powers of a, or according to ascending powers of y, but if it is arranged as $-5a^3x^6y^2+x^5y^5+4a^2x^4y^3+a^5x^3-2ax^2y^4+3a^4xy$, it is arranged according to descending powers of x.

N. B. When one expression is to be multiplied by another it is found *convenient* to arrange both the multiplicand and the multiplier according to descending or ascending powers of some letter common to them, and proceed as exemplified below.

Example 1. Multiply a^2-b^2-ab by $ab-b^2+a^2$.

$$
\begin{array}{lrrrrr}
\text{Multiplicand} & = a^2 & -ab & -b^2 & & \\
\text{Multiplier} & = a^2 & +ab & -b^2 & & \\
\hline
\text{Product by } a^2 & = a^4 & -a^3b & -a^2b^2 & & \\
\text{Product by } +ab & = & +a^3b & -a^2b^2 & -ab^3 & \\
\text{Product by } -b^2 & = & & -a^2b^2 & +ab^3 & +b^4 \\
\hline
\therefore \textit{ Complete product} & = a^4 & & -3a^2b^2 & & +b^4.
\end{array}
$$

Note. The process shown above may be described as follows:—

The multiplier has been placed under the multiplicand after having arranged them both according to descending powers of a, and a line has been drawn below the multiplier. The successive products of the multiplicand by the different terms of the multiplier

beginning from the left have then been placed in different horizontal rows, in such a manner that each set of like terms may be in the same vertical column A line having been now drawn below the lowest of the rows, the complete product has been found by writing down the sum of each vertical column immediately below it

Example 2. Multiply $2a^2-3x^2-5ax$ by $-3x^2+2a^2+5ax$.

Arranging the multiplicand and the multiplier according to ascending powers of x, we have—

$$\begin{array}{lllllll}
\text{Multiplicand} & = 2a^2 & -5ax & -3x^2 & & \\
\text{Multiplier} & = 2a^2 & +5ax & -3x^2 & & \\
\hline
 & 4a^4 & -10a^3x & -6a^2x^2 & & \\
 & & +10a^3x & -25a^2x^2 & -15ax^3 & \\
 & & & -6a^2x^2 & +15ax^3 & +9x^4 \\
\hline
\text{Product} & = 4a^4 & & -37a^2x^2 & & +9x^4.
\end{array}$$

Example 3. Multiply $2a^3b-5ab^3-a^4+3a^2b^2$ by $2a^4-3a^3b+4ab^3-5a^2b^2$

Arranging the multiplicand and the multiplier according to descending powers of a, we have—

$$\begin{array}{lllllll}
\text{Multiplicand} = & -a^4 & +2a^3b & +3a^2b^2 & -5ab^3 & & & \\
\text{Multiplier} \quad = & 2a^4 & -3a^3b & +5a^2b^2 & +4ab^3 & & & \\
\hline
-2a^8 & +4a^7b & +6a^6b^2 & -10a^5b^3 & & & & \\
 & +3a^7b & -6a^6b^2 & -9a^5b^3 & +15a^4b^4 & & & \\
 & & +5a^6b^2 & -10a^5b^3 & -15a^4b^4 & +25a^3b^5 & & \\
 & & & -4a^5b^3 & +8a^4b^4 & +12a^3b^5 & -20a^2b^6 & \\
\hline
\end{array}$$

Product

$$=-2a^8+7a^7b+5a^6b^2-33a^5b^3+8a^4b^4+37a^3b^5-20a^2b^6.$$

Note In this example the multiplicand and the multiplier are each homogeneous and of the 4th degree, whilst the product also is homogeneous and of the 8th degree Similarly it may be seen that whenever the expression to be multiplied together are homogeneous, the product also is homogeneous and the degree of the product is equal to the sum of the degrees of the expressions This law is of great importance in testing the accuracy of a multiplication when the multiplicand and the multiplier are both homogeneous, for in this case if the product obtained does not turn out to be homogeneous, we are sure there has been an error somewhere.

Example 4. Multiply together a^2-ab+b^2, a^2+ab+b^2 and $a^4-a^2b^2+b^4$

$$
\text{(i)}\quad
\begin{array}{lllll}
a^2 & -ab & +b^2 & & \\
a^2 & +ab & +b^2 & & \\
\hline
a^4 & -a^3b & +a^2b^2 & & \\
 & +a^3b & -a^2b^2 & +ab^3 & \\
 & & +a^2b^2 & -ab^3 & +b^4 \\
\hline
a^4 & & +a^2b^2 & & +b^4,
\end{array}
$$

$$
\text{(ii)}\quad
\begin{array}{lllll}
a^4 & +a^2b^2 & +b^4 & & \\
a^4 & -a^2b^2 & +b^4 & & \\
\hline
a^8 & +a^6b^2 & +a^4b^4 & & \\
 & -a^6b^2 & -a^4b^4 & -a^2b^6 & \\
 & & +a^4b^4 & +a^2b^6 & +b^8 \\
\hline
a^8 & & +a^4b^4 & & +b^8
\end{array}
$$

Thus the required product $= a^8+a^4b^4+b^8$.

Example 5. Multiply mx^2-nx-p by x^2+px-1.

$$
\begin{array}{llllll}
\text{Multiplicand} & = & mx^2 & -nx & -p & \\
\text{Multiplier} & = & x^2 & +px & -1 & \\
\hline
 & & mx^4 & -nx^3 & -px^2 & \\
 & & & +pmx^3 & -pnx^2 & -p^2x \\
 & & & & -mx^2 & +nx+p \\
\hline
\end{array}
$$

$$\text{Product} = mx^4-(n-pm)x^3-(p+pn+m)x^2+(n-p^2)x+p.$$

Note The sums of the vertical columns are put down as above by the help of the corollaries to Art 8 and the laws for the insertion of brackets

Exercise (21).

Multiply :—

1. x^2-x+1 by x^2+x+1. 2. a^2-ab+b^2 by $a+b$.
3. a^2+ab+b^2 by $a-b$. 4. $a^2-2ab+b^2$ by $a^2+2ab+b^2$.
5. x^4+x^2+1 by x^4-x^2+1.

6. $y^3-x^2y^2+x^3$ by $x^3+x^2y^2+y^3$
7. $m^4-m^2n^2+n^4$ by m^2+n^2
8. $p^2q^2+p^4+q^4$ by $-q^2+p^2$
9. $25b^2+30ab+9a^2$ by $3a-5b$.
10. $a^3+5ab^2-6a^2b$ by $5b^2+a^2+6ab$
11. $2a-3b+4c$ by $2a+3b-4c$
12. x^3-3x^2+3x-1 by x^2+3x+1
13. $2ax^3+a^4+3a^2x^2+x^4+2a^3x$ by a^2+x^2-2ax
14. $a^3+3a^2b+b^3+3ab^2$ by $3ab^2-b^3+a^3-3a^2b$
15. $x^2-11+x^4-4x+2x^3$ by $3+x^2-2x$
16. $1+2x+x^4+2x^3+3x^2$ by $1+x^2-2x$
17. $b^4+a^2b^2+a^3b+a^4+ab^3$ by $a^2b^2-a^3b+b^4-ab^3+a^4$.
18. $x^2-xy-xz+y^2-yz+z^2$ by $x+y+z$
19. $a^2+b^2+c^2-bc-ca-ab$ by $a+b+c$.
20. $5a^2b+4b^3+2a^3-3ab^2$ by $2ab^2-3a^2b+a^3-5b^3$

Multiply together —

21. $a+b$, $a-b$ and a^2+b^2.
22. $a^4+a^2b^2+b^4$, $a+b$ and $a-b$
23. $x^8+x^4y^4+y^8$, x^2+y^2, $x+y$ and $x-y$
24. $x^2+3xy+5y^2$, $x^2-3xy+5y^2$ and $x^4-x^2y^2+y^4$.
25. $a^{12}+a^6b^6+b^{12}$, $a^4+a^2b^2+b^4$, $a+b$ and $a-b$

Multiply —

26. ax^2+bx-c by $px-q$ 27. mx^2-nx-r by $nx-r$
28. ax^2-bx+c by x^2-bx-c.
29. ax^3-bx^2+cx-d by bx^2-cx+d
30. $px^2-(q-r)x+s$ by mx^2-nx-s.
31. Multiply together $x+a$, $x+b$ and $x+c$
32. Multiply together $x-a$, $x-b$ and $x-c$

Assuming $a^m\times a^n=a^{m+n}$ to be true for *all* values of m and n, prove that —

33. $a^{\frac{1}{2}}\times a^{\frac{1}{2}}=a$. $\left[a^{\frac{1}{2}}\times a^{\frac{1}{2}}=a^{\frac{1}{2}+\frac{1}{2}}=a^1=a\right]$

34. $x^{\frac{1}{3}}\times x^{\frac{2}{3}}=x$

35. $a^{\frac{1}{3}} \times a^{\frac{1}{3}} \times a^{\frac{1}{3}} = a$ $\left[a^{\frac{1}{3}} \times a^{\frac{1}{3}} \times a^{\frac{1}{3}} = a^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = a^1 = a\right]$

36. $a^{\frac{1}{4}} = \sqrt[4]{a}$.

$\left[\left(a^{\frac{1}{4}}\right)^4 = a^{\frac{1}{4}} \times a^{\frac{1}{4}} \times a^{\frac{1}{4}} \times a^{\frac{1}{4}} = a^{\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}} = a^1 = a, \quad a^{\frac{1}{4}} = \sqrt[4]{a}\right]$

37. $x^{\frac{2}{3}} = \sqrt[3]{x^2}$.

38. $y^{\frac{4}{5}} = \sqrt[5]{y^4}$.

39. $z^{\frac{3}{4}} = \sqrt[4]{z^3}$

40. $b^{\frac{3}{2}} \times b^{\frac{5}{2}} = b^4$.

41. $c^{\frac{3}{5}} \times c^{\frac{4}{5}} \times c^{\frac{8}{5}} = c^3$

42. $y^2 \times y^{\frac{3}{2}} \times y^{\frac{7}{2}} = y^7$.

43. $x^{-2} \times x^5 = x^3$ $[x^{-2} \times x^5 = x^{-2+5} = x^3]$

44. $y^4 \times y^{-7} = y^{-3}$.

45. $z^{\frac{3}{2}} \times z^{-\frac{1}{2}} = z$.

46. $a^{-\frac{3}{2}} = \sqrt{a^{-3}}$.

$\left[\left(a^{-\frac{3}{2}}\right)^2 = a^{-\frac{3}{2}} \times a^{-\frac{3}{2}} = a^{-\frac{3}{2}-\frac{3}{2}} = a^{-3}, \quad a^{-\frac{3}{2}} = \sqrt{a^{-3}}\right]$

47. $b^{-\frac{5}{3}} = \sqrt[3]{b^{-5}}$

48. $c^{-\frac{2}{5}} = \sqrt[5]{c^{-2}}$.

49 $a^{-\frac{3}{4}} \times a^{-\frac{1}{4}} = a^{-1}$

50. $x^{-\frac{5}{3}} \times x^{-\frac{4}{3}} = x^{-3}$.

Write down the product of :—

51. $-3x^{\frac{1}{2}}$ and $2x^{\frac{3}{2}}$

52. $5y^{\frac{3}{2}}$ and $-\frac{2}{5}y^{\frac{5}{2}}$

53. $2x^{\frac{1}{2}}y^{\frac{1}{2}}$ and $3x^{\frac{3}{2}}y^{\frac{1}{2}}$.

54. $-5xy^{\frac{3}{4}}$ and $-3x^{\frac{2}{3}}y^{\frac{1}{2}}$.

55. $4a^{-2}b^3$ and $-\frac{3}{4}a^3b^{-5}$

56. $\frac{3}{5}a^{\frac{3}{5}}y^3$ and $-\frac{5}{3}a^{\frac{2}{5}}y^{-4}$.

57. $-4a^{\frac{1}{2}}b^{\frac{2}{3}}c^{\frac{3}{4}}$ and $-8a^{\frac{3}{2}}b^{\frac{4}{3}}c^{\frac{5}{4}}$

58. $-5x^{\frac{2}{3}}y^{\frac{3}{5}}z^{\frac{4}{3}}$ and $-8x^{\frac{1}{3}}y^{\frac{2}{5}}z^{-\frac{1}{3}}$.

59. $-6a^{\frac{5}{3}}b^{-\frac{3}{4}}c^{-\frac{2}{7}}$ and $5a^{-\frac{1}{3}}b^{\frac{7}{4}}c^{-\frac{5}{7}}$.

60. $-4a^{\frac{5}{3}}x^{\frac{5}{8}}y^{-\frac{4}{5}}$ and $-19a^{\frac{1}{2}}x^{-\frac{3}{2}}y^{-\frac{6}{5}}$.

Multiply —

61. $a^{\frac{1}{2}}+b^{\frac{1}{2}}$ by $a^{\frac{1}{2}}+b^{\frac{1}{2}}$.

62. $a^{\frac{1}{2}}-b^{\frac{1}{2}}$ by $a^{\frac{1}{2}}-b^{\frac{1}{2}}$.

63. $3x^{\frac{2}{3}}-4y^{\frac{1}{3}}$ by $3x^{\frac{2}{3}}+4y^{\frac{1}{3}}$

64. $a^{\frac{2}{3}}-a^{\frac{1}{3}}b^{\frac{1}{3}}+b^{\frac{2}{3}}$ by $a^{\frac{1}{3}}+b^{\frac{1}{3}}$.

65. $x^{\frac{2}{3}}+x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}}$ by $x^{\frac{1}{3}}-y^{\frac{1}{3}}$.

66. $a^{\frac{3}{2}}-a^{\frac{3}{4}}b^{\frac{3}{4}}+b^{\frac{3}{2}}$ by $a^{\frac{3}{2}}+a^{\frac{3}{4}}b^{\frac{3}{4}}+b^{\frac{3}{2}}$.

67. $2x^{\frac{4}{3}}-5x^{\frac{2}{3}}y^{\frac{2}{3}}-3y^{\frac{4}{3}}$ by $2x^{\frac{4}{3}}+5x^{\frac{2}{3}}y^{\frac{2}{3}}-3y^{\frac{4}{3}}$

68. $a^{\frac{5}{2}}+a^{2}b^{\frac{1}{3}}+a^{\frac{3}{2}}b^{\frac{2}{3}}+ab+a^{\frac{1}{2}}b^{\frac{4}{3}}+b^{\frac{5}{3}}$ by $a^{\frac{1}{2}}-b^{\frac{1}{3}}$.

69. $x^{\frac{3}{2}}-xy^{\frac{1}{2}}+x^{\frac{1}{2}}y-y^{\frac{3}{2}}$ by $x^{\frac{1}{2}}+y^{\frac{1}{2}}$.

70. $a^{\frac{3}{4}}+a^{\frac{1}{2}}b^{\frac{1}{2}}+a^{\frac{1}{4}}b+b^{\frac{3}{2}}$ by $a^{\frac{1}{4}}-b^{\frac{1}{2}}$.

71. $x^{\frac{2}{3}}+y^{\frac{2}{3}}+z^{\frac{2}{3}}-x^{\frac{1}{3}}y^{\frac{1}{3}}-y^{\frac{1}{3}}z^{\frac{1}{3}}-z^{\frac{1}{3}}x^{\frac{1}{3}}$ by $x^{\frac{1}{3}}+y^{\frac{1}{3}}+z^{\frac{1}{3}}$.

72. $a^{2n}-a^{n}x^{n}+x^{2n}$ by $a^{n}+x^{n}$.

73. $a^{-3}-4a^{-2}b+4a^{-1}b^{2}-b^{3}$ by $a^{-2}-2a^{-1}b+b^{2}$.

74. $x^{-3}+3x^{-\frac{3}{2}}y^{\frac{3}{2}}+2y^{3}$ by $x^{-3}-3x^{-\frac{3}{2}}y^{\frac{3}{2}}+2y^{3}$

75. $2a^{-5}+3a^{-\frac{5}{2}}b^{-\frac{3}{2}}-5b^{-3}$ by $2a^{-5}+3a^{-\frac{5}{2}}b^{-\frac{3}{2}}+5b^{-3}$.

76. $3x^{-\frac{5}{4}}-5x^{-\frac{5}{8}}y^{-\frac{3}{8}}+7y^{-\frac{3}{4}}$ by $3x^{-\frac{5}{4}}+5x^{-\frac{5}{8}}y^{-\frac{3}{8}}-7y^{-\frac{3}{4}}$

77. Arrange $8x^{-4}a^{6}-3x^{\frac{7}{2}}a+6x^{-1}a^{4}+2x^{5}-7x^{-\frac{5}{2}}a^{5}-5x^{\frac{1}{2}}a$ $+4x^{3}a^{2}$ according to descending powers of x.

78. Arrange $5a^{-\frac{2}{3}}b^{-\frac{4}{3}}+8a^{\frac{7}{4}}b^{-2}-3a^{-\frac{5}{6}}-5a^{\frac{3}{2}}b^{-4}-7a^{-\frac{5}{8}}b^{-\frac{2}{7}}$ $+6a^{\frac{5}{4}}b^{-\frac{3}{4}}-2a^{-\frac{3}{4}}b^{-\frac{2}{5}}$ according to ascending powers of a.

CHAPTER VI.

FORMULÆ AND THEIR APPLICATION.

1. Definition. Any general result expressed in symbols is called a *formula* In other words, a formula is the most general expression for any theorem respecting numerical quantities

Note As a complete knowledge of some of the principal formulæ is essential for performing certain algebraical operations with neatness and accuracy it has been thought fit to lay before the student, at this stage of his progress a somewhat copious treatment of the subject of formulæ The beginner is recommended, however, in reading this chapter for the first time, to familiarise himself with the application of such of them only as are comparatively simple and of more frequent use Hence for his guidance some of the later articles of the chapter are marked with asterisks, several of which may be conveniently omitted for the first time

2. Formula $(a+b)^2 = a^2+2ab+b^2$.

$$\begin{aligned}[(a+b)^2 &= (a+b)(a+b)\\ &= a(a+b)+b(a+b)\\ &= a^2+2ab+b^2\ ]\end{aligned}$$

That is *the square of the sum of any two quantities is equal to the sum of their squares plus twice their product.*

Cor. $a^2+b^2 = (a^2+2ab+b^2)-2ab$
$= (a+b)^2-2ab.$

Example 1. Find the square of $2x+3y$.

$$\begin{aligned}(2x+3y)^2 &= (2x)^2+2(2x)(3y)+(3y)^2\\ &= 4x^2+12xy+9y^2.\end{aligned}$$

Example 2. Find the square of $5x+4$.

$$\begin{aligned}(5x+4)^2 &= (5x)^2+2(5x)\ 4+4^2\\ &= 25x^2+40x+16.\end{aligned}$$

Example 3. Find the square of $4a^3+7b^4$.

$$\begin{aligned}(4a^3+7b^4)^2 &= (4a^3)^2+2(4a^3)(7b^4)+(7b^4)^2\\ &= 16a^6+56a^3b^4+49b^8.\end{aligned}$$

Example 4. Find the square of $a+b+c$.

$$\begin{aligned}(a+b+c)^2 &= \{a+(b+c)\}^2, \quad \text{(regarding } b+c \text{ as one term)}\\ &= a^2+2a(b+c)+(b+c)^2\\ &= a^2+2ab+2ac+b^2+2bc+c^2\\ &= a^2+b^2+c^2+2ab+2ac+2bc\end{aligned}$$

Example 5. Find the square of $a+b+c+d$

$$\begin{aligned}(a+b+c+d)^2 &= \{(a+b)+(c+d)\}^2, \quad \text{(regarding } a+b \text{ as one term and } c+d \text{ as another)}\\ &= (a+b)^2+2(a+b)(c+d)+(c+d)^2\\ &= (a^2+2ab+b^2)+2(ac+ad+bc+bd)\\ &\qquad +(c^2+2cd+d^2)\\ &= a^2+b^2+c^2+d^2+2ab+2ac+2ad+2bc+2bd\\ &\qquad +2cd.\end{aligned}$$

Example 6. Simplify $(a+b-c)^2+2(a+b-c)(a-b+c)+(a-b+c)^2$.

Putting x for $(a+b-c)$ and y for $(a-b+c)$, we have the given expression

$$\begin{aligned}&= x^2+2xy+y^2\\ &= (x+y)^2\\ &= \{(a+b-c)+(a-b+c)\}^2\\ &= (2a)^2 = 4a^2.\end{aligned}$$

Example 7. Find the value of $9x^2+30xy+25y^2$, when $x=15, y=-9$.

The given expression $= (3x)^2+2(3x)(5y)+(5y)^2$

$= (3x+5y)^2$

But $\quad 3x+5y = 3\times15+5\times(-9) = 45-45 = 0.$

$\therefore$ The given expression $= 0$.

Exercise (22).

Find the square of each of the following expressions —

1. $3x+4$. 2. $7x+8$. 3. $a+5b$. 4. $2a+7b$.
5. $3x+8y$. 6. $5m+8n$ 7. $ax+3by$. 8 $4ab+c^2$.
9. $2a^2+5b^2$. 10. $4m^3+n^2$. 11. $a+2b+3c$.
12. $ab+bc+ca$. 13. $2p+3q+4r$. 14. $x^2+y^2+z^2$.

15. $2x+3y+4z$ **16.** $x^2+y^3+z^4$.

17. $x+y+2a+3b$. **18.** $3a+4b+c+2d$.

19. $2a+x+4y+3z$. **20.** $4m+3n+3p+2q$.

Simplify —

21. $(x+y)^2+2(x+y)(x-y)+(x-y)^2$.

22. $(x-y+z)^2+(y+z-x)^2+2(x-y+z)(y+z-x)$.

23. $(2a-3b+4c)^2+(2a+3b-4c)^2$
$$+2(2a-3b+4c)(2a+3b-4c).$$

24. $(5a-7b)^2+2(5a-7b)(9b-4a)+(9b-4a)^2$.

25. $(2x-5y-3z)^2+(6y+3z-x)^2+2(2x-5y-3z)(6y+3z-x)$.

Find the value of :—

26. $4x^2+28x+49$, when $x=-8$.

27. $25a^2+40ab+16b^2$, when $a=-18$, $b=23$.

28. $81x^2+90xy+25y^2$, when $x=15$ and $y=-27$.

29. $16m^2+56mn+49n^2$, when $m=-13$ and $n=7$.

30. $64a^2+16ac+c^2$, when $a=6$ and $c=-49$.

31. $81x^2+18xz+z^2$, when $x=7$ and $z=-67$.

32. $36p^2+132pq+121q^2$, when $p=12$ and $q=-7$.

33. If $m+\dfrac{1}{m}=4$, show that $m^2+\left(\dfrac{1}{m}\right)^2=14$.

3. Formula $(a-b)^2=a^2-2ab+b^2$.

$$[(a-b)^2=(a-b)(a-b)$$
$$=a(a-b)-b(a-b)$$
$$=a^2-2ab+b^2\,]$$

That is, *the square of the difference of any two quantities is equal to the sum of their squares minus twice their product.*

Note This formula is virtually included in the formula of the last article For $(a-b)^2=\{a+(-b)\}^2=a^2+2a(-b)+(-b)^2=a^2-2ab+b^2$

***Cor.* 1.** $a^2+b^2=(a^2-2ab+b^2)+2ab=(a-b)^2+2ab$.

***Cor.* 2.** Since $(a+b)^2=a^2+2ab+b^2$
and $(a-b)^2=a^2-2ab+b^2$,
evidently we have
$$(a+b)^2=(a-b)^2+4ab \text{ and } (a-b)^2=(a+b)^2-4ab.$$

Example 1. Find the square of $3a-4b$

$$
\begin{aligned}
(3a-4b)^2 &= (3a)^2-2(3a)(4b)+(4b)^2 \\
&= 9a^2-24ab+16b^2
\end{aligned}
$$

Example 2. Find the square of $x-y-z$

$$
\begin{aligned}
(x-y-z)^2 &= \{x-(y+z)\}^2 \\
&= x^2-2x(y+z)+(y+z)^2 \\
&= x^2-2xy-2xz+y^2+2yz+z^2 \\
&= x^2+y^2+z^2-2xy-2xz+2yz
\end{aligned}
$$

Example 3. Find the square of $2x-3y-4z$

$$
\begin{aligned}
(2x-3y-4z)^2 &= \{2x-(3y+4z)\}^2 \\
&= (2x)^2-2(2x)(3y+4z)+(3y+4z)^2 \\
&= 4x^2-2(6xy+8xz) \\
&\qquad +\{(3y)^2+2(3y)(4z)+(4z)^2\} \\
&= 4x^2-12xy-16xz+9y^2+24yz+16z^2 \\
&= 4x^2+9y^2+16z^2-12xy-16xz+24yz.
\end{aligned}
$$

Example 4. Find the square of $a-b-c+d$.

$$
\begin{aligned}
(a-b-c+d)^2 &= \{(a-b)-(c-d)\}^2 \\
&= (a-b)^2-2(a-b)(c-d)+(c-d)^2 \\
&= (a^2-2ab+b^2)-2(ac-ad-bc+bd) \\
&\qquad +(c^2-2cd+d^2) \\
&= a^2-2ab+b^2-2ac+2ad+2bc-2bd \\
&\qquad +c^2-2cd+d^2 \\
&= a^2+b^2+c^2+d^2-2ab-2ac+2ad \\
&\qquad +2bc-2bd-2cd.
\end{aligned}
$$

Example 5. Simplify

$$(ax-by+cz)^2+(ax-by-cz)^2-2(ax-by+cz)(ax-by-cz).$$

Putting m for $(ax-by+cz)$ and n for $(ax-by-cz)$, we have the given expression

$$
\begin{aligned}
&= m^2+n^2-2mn \\
&= (m-n)^2 \\
&= \{(ax-by+cz)-(ax-by-cz)\}^2 \\
&= (2cz)^2 = 4c^2z^2.
\end{aligned}
$$

Example 6. Find the value of $9a^2-48ab+64b^2$, when $a=15$ and $b=6$.

$$\begin{aligned}\text{The given expression } &= (3a)^2-2(3a)(8b)+(8b)^2\\ &= (3a-8b)^2\\ &= (45-48)^2\\ &= (-3)^2\\ &= 9.\end{aligned}$$

Exercise (23).

Find the square of each of the following expressions —

1. $2x-7$ **2.** $8x-5$ **3.** $ax-by$.

4. $-5x-3y$. **5.** $3m-8n$. **6.** $-ab-cd$.

7. a^2b-c^2d. **8.** x^3-2yz. **9.** $-mnp-qr$.

10. $2a^3-5b^3$ **11.** $a-2b-3c$. **12.** $2x-3y-4z$

13 $3m-4n-5q$ **14.** $a^2-3b^2-5c^2$. **15.** $x-y-a-b$.

16. $a-2x-3b-4y$ **17.** $90-1$. **18.** $120-3$.

19. $500-2$ **20.** $1000-7$.

Simplify —

21. $(a+3b)^2-2(a+3b)(a-3b)+(a-3b)^2$

22. $(2a-4b+5c)^2+(2a+4b+5c)^2-2(2a-4b+5c)(2a+4b+5c)$.

23. $(3a+5b+7c)^2+(7c-4a+5b)^2-2(3a+5b+7c)(7c-4a+5b)$.

24. $(2x^2-y^2-5z^2)^2-2(2x^2-y^2-5z^2)(6z^2+2x^2-y^2)+(6z^2+2x^2-y^2)^2$.

25. $(ab-bc+ca)^2+(ab+4bc+2ca)^2-2(ab-bc+ca)(ab+4bc+2ca)$.

Find the value of —

26. $a^2b^2-12abc+36c^2$, when $a=4$, $b=7$ and $c=5$.

27. $x^2y^2-24xyz+144z^2$, when $x=7$, $y=9$ and $z=6$.

28. $25(x+y)^2+z^2-10z(x+y)$, when $x=47$, $y=-22$ and $z=129$.

29. $9c^2-42c(a+b)+49(a+b)^2$, when $a=-37$, $b=57$ and $c=45$.

30. $64(7p-5q)^2-96(7p-5q)r+36r^2$, when $p=28$, $q=32$ and $r=46$.

31. If $c-\dfrac{1}{c}=4$, show that $c^2+\left(\dfrac{1}{c}\right)^2=18$.

4. Formula $(a+b)(a-b)=a^2-b^2$.

$$\left[(a+b)(a-b)=a(a-b)+b(a-b)\right.$$
$$\left.=a^2-b^2\right]$$

That is, *the product of the sum and difference of any two quantities is equal to the difference of their squares*

Note Conversely, $a^2-b^2=(a+b)(a-b)$ Hence we can always find the factors of an expression which is of the form a^2-b^2

[When one expression is the product of two or more expressions each of the latter is called a factor of the former]

Example 1. Multiply $3x+5y$ by $3x-5y$.

$$(3x+5y)(3x-5y)=(3x)^2-(5y)^2$$
$$=9x^2-25y^2.$$

Example 2. Multiply $a+b-c$ by $a-b+c$

$$(a+b-c)(a-b+c)=\{a+(b-c)\}\{a-(b-c)\}$$
$$=a^2-(b-c)^2$$
$$=a^2-(b^2-2bc+c^2)$$
$$=a^2-b^2+2bc-c^2.$$

Example 3. Multiply x^2+xy+y^2 by x^2-xy+y^2.

$$(x^2+xy+y^2)(x^2-xy+y^2)=\{(x^2+y^2)+xy\}\{(x^2+y^2)-xy\}$$
$$=(x^2+y^2)^2-(xy)^2$$
$$=x^4+2x^2y^2+y^4-x^2y^2$$
$$=x^4+x^2y^2+y^4.$$

Example 4. Simplify $(a^2+ab+b^2)^2-(a^2-ab+b^2)^2$.

The given expression $=\{(a^2+ab+b^2)+(a^2-ab+b^2)\}$

$$\times\{(a^2+ab+b^2)-(a^2-ab+b^2)\}$$
$$=(2a^2+2b^2)\times 2ab$$
$$=2(a^2+b^2)\times 2ab$$
$$=4ab(a^2+b^2)$$

Example 5. Find the value of $(9726854)^2-(9726849)^2$.

The given expression

$$= (9726854+9726849)(9726854-9726849)$$
$$= 19453703 \times 5$$
$$= 97268515$$

Example 6. Resolve into factors $(a+b)^2-(c-d)^2$.

The given expression $= \{(a+b)+(c-d)\}\{(a+b)-(c-d)\}$
$= (a+b+c-d)(a+b-c+d)$.

Example 7. Resolve into factors $16a^4-81x^4$.

The given expression $= (4a^2)^2-(9x^2)^2$
$= (4a^2+9x^2)(4a^2-9x^2)$.

Again, $4a^2-9x^2 = (2a)^2-(3x)^2$
$= (2a+3x)(2a-3x)$.

Hence, the given expression

$$= (4a^2+9x^2)(2a+3x)(2a-3x).$$

Exercise (24).

Multiply together —

1. $ax+by$ and $ax-by$. **2.** $5x+8$ and $5x-8$.

3. $cx+d^2$ and $cx-d^2$. **4.** $ab+bc$ and $ab-bc$.

5. $7a+3b$ and $7a-3b$ **6.** $4a^2-5b^2$ and $4a^2+5b^2$.

7 m^2-3n and m^2+3n **8** $2ab-7bc$ and $2ab+7bc$

9. $x+1, x-1$ and x^2+1 **10** a^2+b^2, a^2-b^2 and a^4+b^4.

11. $a+b+c$ and $a+b-c$. **12** $a+b+c$ and $a-b-c$.

13. m^2+mn+n^2 and m^2-mn+n^2.

14. $ax+by-cz$ and $ax-by+cz$.

15. $a-2b+3c$ and $a+2b-3c$.

16. x^2-x+1 and x^2+x+1. **17.** x^4-x^2+1 and x^4+x^2+1.

18. $x^2+2xy+2y^2$ and $x^2-2xy+2y^2$.

19. $a^2+ab\sqrt{2}+b^2$ and $a^2-ab\sqrt{2}+b^2$.

20. x^2-2x+1, x^2+2x+1 and x^4+2x^2+1

Simplify —

21. $(a+b-c)^2-(a-b+c)^2$. **22.** $(a-2b+3c)^2-(a+2b-3c)^2$.

23. $(x^2+xy+y^2)^2-(x^2-xy+y^2)^2$.
24. $(x+y-a+b)^2-(x-y+a-b)^2$
25. $(2a+3b-5c+7d)^2-(2a-3b+5c-7d)^2$

Find the value of —

26. $2345\times2345-2343\times2343$ **27** $(53497)^2-(53487)^2$.
28. $498567\times498567-498562\times498562$

Resolve into factors —

29. $25x^2-36$. **30.** $9a^2-16c^2$ **31.** $16m^2-49n^2$
32. $4p^2-81q^2$ **33** $a^2x^2-64b^2$ **34.** $36x^4-121y^4$.
35. $49-64d^2$ **36.** $144c^2-25d^2$. **37.** $(a+b)^2-c^2$
38. $(a+2b)^2-25c^2$. **39.** $4x^2-(3a-4b)^2$.
40. $a^2-(2b-3c)^2$ **41.** a^4-81b^4 **42.** $(x-y)^2-(a-b)^2$.
43. $81x^4-625y^4$ **44.** $(2a-5x)^2-36x^2$.
45. $25x^2-(4x+1)^2$. **46.** $(4a+7b)^2-(3a-8b)^2$.
47. $(3x+5y)^2-(2x-7y)^2$ **48.** $(a+2b-3c)^2-(a+b-c)^2$.
49. $(2m+3n-5p)^2-(2n+3p)^2$.
50. $(3x-4y+7z)^2-(2x-3y+5z)^2$.

5. Formula $(a+b)^3 = a^3+3a^2b+3ab^2+b^3$,
or $= a^3+b^3+3ab(a+b)$

$$[(a+b)^3 = (a+b)(a+b)^2$$
$$= (a+b)(a^2+2ab+b^2)$$
$$= a(a^2+2ab+b^2)+b(a^2+2ab+b^2)$$
$$= a^3+3a^2b+3ab^2+b^3,$$
and this latter $= a^3+3ab(a+b)+b^3$
$$= a^3+b^3+3ab(a+b)]$$

Cor $a^3+b^3 = \{a^3+b^3+3ab(a+b)\}-3ab(a+b)$
$$= (a+b)^3-3ab(a+b)$$

Example 1. Find the cube of $3a+5b$.

$$(3a+5b)^3 = (3a)^3+3(3a)^2(5b)+3(3a)(5b)^2+(5b)^3$$
$$= 27a^3+3(9a^2)(5b)+3(3a)(25b^2)+125b^3$$
$$= 27a^3+135a^2b+225ab^2+125b^3.$$

Example 2 Simplify $(x-y)^3+(x+y)^3+3(x-y)^2(x+y)$
$+3(x+y)^2(x-y)$.

Putting a for $x-y$ and b for $x+y$, we have

the given expression $= a^3+b^3+3a^2b+3b^2a$

$= a^3+3a^2b+3ab^2+b^3$

and $\therefore$ $= (a+b)^3$

$= \{(x-y)+(x+y)\}^3$

$= (2x)^3 = 8x^3$

Example 3. If $a+b=5$ and $ab=6$, find the value of a^3+b^3.

We have $a^3+b^3 = (a+b)^3-3ab(a+b)$

and $\therefore$ by the given condition, $= 5^3-3\times6\times5$

$= 125-90 = 35$

Example 4. If $x+\dfrac{1}{x}=p$, show that $x^3+\left(\dfrac{1}{x}\right)^3 = p^3-3p$.

Since $a^3+b^3 = (a+b)^3-3ab(a+b)$,

$$\therefore \quad x^3+\frac{1}{x^3} = \left(x+\frac{1}{x}\right)^3-3\,x\,\frac{1}{x}\left(x+\frac{1}{x}\right)$$

$$= \left(x+\frac{1}{x}\right)^3-3\left(x+\frac{1}{x}\right).$$

Hence, the required value $= p^3-3p$

Example 5. Find the cube of $p+q+r$

$$(p+q+r)^3 = \{(p+q)+r\}^3$$
$$= (p+q)^3+3(p+q)^2r+3(p+q)r^2+r^3$$
$$= (p^3+3p^2q+3pq^2+q^3)+3(p^2+2pq+q^2)r$$
$$+3(p+q)r^2+r^3$$
$$= p^3+q^3+r^3+3p^2q+3pq^2+3p^2r+3pr^2$$
$$+3q^2r+3qr^2+6pqr.$$

Example 6. Find the value of $x^3+9x^2y+27xy^2+27y^3$, when $x=5$ and $y=-2$

The given expression $= x^3+3x^2(3y)+3x(3y)^2+(3y)^3$

$= (x+3y)^3 = (5-6)^3$

$= (-1)^3 = -1.$

Exercise (25).

Find the cube of —

1. $2x+1$ 2. $3x+y$ 3. $2x+3a$ 4. a^2+2b.
5. $ab+bc$. 6. $a+2x+b$. 7. $2m+3n+p$. 8. $xy+yz+zx$.

Simplify —

9. $(3m+5n)^3+3(3m+5n)^2(2m-5n)+3(3m+5n)(2m-5n)^2$
$+(2m-5n)^3$.

10. $(3x-8y)^3+(9y-2x)^3+3(x+y)(3x-8y)(9y-2x)$.

11. $(3a-7b)^3+(10b-3a)^3+9b(3a-7b)(10b-3a)$.

12. $(5x-2)^3+(3-4x)^3+3(x+1)(5x-2)(3-4x)$.

13. $(3-7x)^3+(8x-1)^3+3(8x-1)(3-7x)(x+2)$

14. $(a-b+c)^3+(a+b-c)^3+6a\{a^2-(b-c)^2\}$.

Find the value of a^3+b^3 —

15. When $a+b=6$ and $ab=7$.

16. When $a+b=7$ and $ab=8$.

17. If $a+\frac{1}{a}=3$, show that $a^3+\left(\frac{1}{a}\right)^3=18$.

18. If $z+\frac{1}{z}=4$, find the value of $z^3+\left(\frac{1}{z}\right)^3$.

Find the value of —

19. $64+48x+12x^2+x^3$, when $x=-8$.

20. $8m^3+36m^2n+54mn^2+27n^3+64$, when $m=6$ and $n=-5$.

21. $x^3+15x^2+75x+189$, when $x=-9$.

22. $x^3+18x^2+108x+351$, when $x=-11$.

23. If $x+y=5$, show that $x^3+y^3+15xy=125$.

24. If $a^2+b^2=c^2$, show that $a^6+b^6+3a^2b^2c^2=c^6$.

25. If $p+q=2$, show that $p^3+q^3+6pq=8$.

6. Formula $\boldsymbol{(a-b)^3=a^3-3a^2b+3ab^2-b^3}$
$\boldsymbol{\text{or}=a^3-b^3-3ab(a-b)}$.

$$
\begin{aligned}
[(a-b)^3 &= (a-b)(a-b)^2\\
&= (a-b)(a^2-2ab+b^2)\\
&= a(a^2-2ab+b^2)-b(a^2-2ab+b^2)\\
&= a^3-3a^2b+3ab^2-b^3,\\
\text{and this latter } &= a^3-3ab(a-b)-b^3\\
&= a^3-b^3-3ab(a-b)]
\end{aligned}
$$

Cor. $a^3-b^3=\{a^3-b^3-3ab(a-b)\}+3ab(a-b)$
$=(a-b)^3+3ab(a-b)$.

Example 1. Find the cube of $3x-4y$.

$$\begin{aligned}(3x-4y)^3 &= (3x)^3-3(3x)^2(4y)+3(3x)(4y)^2-(4y)^3\\ &= 27x^3-3(9x^2)(4y)+3(3x)(16y^2)-64y^3\\ &= 27x^3-108x^2y+144xy^2-64y^3.\end{aligned}$$

Example 2. Find the cube of $a-b-c$.

$$\begin{aligned}(a-b-c)^3 &= \{(a-b)-c\}^3\\ &= (a-b)^3-3(a-b)^2c+3(a-b)c^2-c^3\\ &= (a^3-3a^2b+3ab^2-b^3)-3(a^2-2ab+b^2)c\\ &\qquad\qquad +3(a-b)c^2-c^3\\ &= a^3-b^3-c^3-3a^2b+3ab^2-3a^2c+3ac^2\\ &\qquad\qquad -3b^2c-3bc^2+6abc.\end{aligned}$$

Example 3. Find the value of $27x^3-54x^2+36x-64$, when $x = 2\frac{1}{3}$.

$$\begin{aligned}\text{The given expression} &= (3x)^3-3(9x^2)\,2+3(3x).4-8-56\\ &= (3x-2)^3-56.\end{aligned}$$

$$\begin{aligned}\text{Hence, the required value} &= (7-2)^3-56\\ &= 125-56\\ &= 69.\end{aligned}$$

Exercise (26).

Find the cube of —

1. $1-2a$ **2.** $2-3x$. **3.** $3-4x$.

4. $5m-4n$. **5.** $2p-5q$. **6.** $2x-y-z$

7. $2m-3n-p$. **8.** $l^2-m^2-n^2$.

Simplify .—

9. $(a+2b)^3-3(a+2b)^2(a-2b)+3(a+2b)(a-2b)^2-(a-2b)^3$.

10. $(3x-8y)^3-(2x-7y)^3-3(3x-8y)(2x-7y)(x-y)$.

11. $(5x-8)^3-(3x-8)^3-6x(5x-8)(3x-8)$.

Find the value of .—

12. $m^3-12m^2n+48mn^2-64n^3$, when $m = 12$ and $n = 3$.

13. $27a^3-135a^2+225a-125$, when $a=4$

14. $8-9a+27a^2-27a^3$, when $a=3$.

15. $216-144x+108x^2-27x^3$, when $x=3$

16. If $a-\frac{1}{a}=3$, find the value of $a^3-\left(\frac{1}{a}\right)^3$.

17. If $c-\frac{1}{c}=5$, find the value of $c^3-\left(\frac{1}{c}\right)$

18. If $x-y=3$, show that $x^3-y^3-9xy=27$.

19. If $p-2q=4$, show that $p^3-8q^3-24pq=64$.

20. If $2a-3b=5$, show that $8a^3-27b^3-90ab=125$.

7. Formula $(a+b)(a^2-ab+b^2)=a^3+b^3$.

$$\begin{aligned}[(a+b)(a^2-ab+b^2) &= a(a^2-ab+b^2)+b(a^2-ab+b^2)\\ &= (a^3-a^2b+ab^2)+(a^2b-ab^2+b^3)\\ &= a^3+b^3\,]\end{aligned}$$

Note Conversely, $a^3+b^3=(a+b)(a^2-ab+b^2)$ Hence, we can always resolve an expression into factors when it is of the form a^3+b^3

Example 1. Multiply x^4-x^2+1 by x^2+1

Putting a for x^2 and b for 1, we have

$$x^4-x^2+1=(x^2)^2-x^2\,1+1^2=a^2-ab+b^2$$

Hence,
$$\begin{aligned}(x^2+1)(x^4-x^2+1) &= (a+b)(a^2-ab+b^2)\\ &= a^3+b^3\\ &= (x^2)^3+1^3=x^6+1\end{aligned}$$

Example 2 Multiply $9x^2-12x+16$ by $3x+4$

Putting a for $3x$ and b for 4, we have

$$\begin{aligned}9x^2-12x+16 &= (3x)^2-(3x)\,4+4^2\\ &= a^2-ab+b^2\end{aligned}$$

Hence,
$$\begin{aligned}(3x+4)(9x^2-12x+16) &= (a+b)(a^2-ab+b^2)\\ &= a^3+b^3\\ &= (3x)^3+4^3\\ &= 27x^3+64\end{aligned}$$

Example 3. Multiply $16a^2-20ab+25b^2$ by $4a+5b$.

Putting x for $4a$ and y for $5b$, we have

$$\begin{aligned}16a^2-20ab+25b^2 &= (4a)^2-(4a)(5b)+(5b)^2\\ &= x^2-xy+y^2\end{aligned}$$

Hence, $(4a+5b)(16a^2-20ab+25b^2)$

$$= (x+y)(x^2-xy+y^2)$$
$$= x^3+y^3$$
$$= (4a)^3+(5b)^3$$
$$= 64a^3+125b^3$$

Example 4. Resolve $a^3b^3+8c^3$ into factors.

$$a^3b^3+8c^3 = (ab)^3+(2c)^3$$
$$= (ab+2c)\{(ab)^2-(ab)(2c)+(2c)^2\}$$
$$= (ab+2c)(a^2b^2-2abc+4c^2).$$

Exercise (27).

Multiply :—

1. a^2-a+1 by $a+1$
2. $4x^2-2x+1$ by $2x+1$.
3. $25m^2-5m+1$ by $5m+1$.
4. $16x^2-28xy+49y^2$ by $4x+7y$
5. $9m^2-24mn+64n^2$ by $3m+8n$
6. $a^2b^2-4abc+16c^2$ by $ab+4c$
7. $a^2x^2-5abx+25b^2$ by $ax+5b$.
8. $25a^2-45ab+81b^2$ by $5a+9b$

Resolve into factors :—

9. x^3+8
10. $8a^3+1$.
11. m^3+27.
12. $27x^3+1$
13. z^3+64
14. $125m^3+1$.
15. $8a^3+343x^3$
16. $64a^3x^6+27y^3$.

8. Formula $(a-b)(a^2+ab+b^2) = a^3-b^3$

$$[(a-b)(a^2+ab+b^2) = a(a^2+ab+b^2)-b(a^2+ab+b^2)$$
$$= (a^3+a^2b+ab^2)-(a^2b+ab^2+b^3)$$
$$= a^3-b^3]$$

Note Conversely, $a^3-b^3 = (a-b)(a^2+ab+b^2)$ Hence we can always resolve into factors an expression which is of the form a^3-b^3

Example 1. Multiply $4a^2b^4+2ab^2+1$ by $2ab^2-1$.

$$(2ab^2-1)(4a^2b^4+2ab^2+1)$$
$$= (2ab^2-1)\{(2ab^2)^2+(2ab^2)\,1+1^2\}$$
$$= (2ab^2)^3-1^3$$
$$= 8a^3b^6-1.$$

Example 2. Resolve $64x^6-a^3y^6$ into factors.

$$64x^6-a^3y^6 = (4x^2)^3-(ay^2)^3$$
$$= (4x^2-ay^2)\{(4x^2)^2+(4x^2)(ay^2)+(ay^2)^2\}$$
$$= (4x^2-ay^2)(16x^4+4ax^2y^2+a^2y^4).$$

Exercise (28).

Multiply :—

1. $1+3a+9a^2$ by $1-3a$. 2. $16x^2+4x+1$ by $4x-1$.
3. $25m^2+15mn+9n^2$ by $5m-3n$
4. $x^2+2xyz+4y^2z^2$ by $x-2yz$
5. $9a^2+3abc+b^2c^2$ by $3a-bc$

Resolve into factors —

6. $125a^3-1$. 7. $343x^3-8y^6$ 8 $216k^3-125l^3$.
9. $1-512k^3$. 10. $729m^3-64a^3n^6$.

9. Formula $(x+a)(x+b) = x^2+(a+b)x+ab$.

$$[(x+a)(x+b) = x(x+b)+a(x+b)$$
$$= x^2+(a+b)x+ab]$$

Note It is easy to see that the above formula includes the following results —

$$\left.\begin{array}{ll}(1) & (x-a)(x-b) = x^2-(a+b)x+ab \\ (2) & (x-a)(x+b) = x^2+(b-a)x-ab \\ (3) & (x+a)(x-b) = x^2+(a-b)x-ab\end{array}\right\}$$

For instance, $(x-a)(x-b) = \{x+(-a)\}\{x+(-b)\}$

$$= x^2+\{(-a)+(-b)\}x+(-a)\times(-b)$$
$$= x^2-(a+b)x+ab$$

Similarly, the truth of the other results can be proved, which is left as an exercise for the student

Hence, we can express the formula more clearly as follows —

$(x+a)(x+b) = x^2+$(algebraic sum of a and b)$x+$(product of a and b)

Example 1. Write down the product of $x+3$ and $x+4$.

$$\left.\begin{array}{lr}\text{Since} & 3+4 = 7 \\ \text{and} & 3\times4 = 12\end{array}\right\}, \therefore \text{ the required product } = x^2+7x+12$$

Example 2. Write down the product of $x-7$ and $x+4$.

$$\left.\begin{array}{lr}\text{Since} & -7+4 = -3 \\ \text{and} & (-7)\times4 = -28\end{array}\right\}, \therefore \text{ the required product } = x^2-3x-28.$$

Example 3. Write down the product of $x+5$ and $x-9$.

Since $\left.\begin{array}{l} 5-9=-4 \\ 5\times(-9)=-45 \end{array}\right\}$, $\therefore$ the required product $=x^2-4x-45$.

Example 4. Write down the product of $x-2$ and $x+7$.

Since $\left.\begin{array}{l} -2+7=5 \\ (-2)\times 7=-14 \end{array}\right\}$, $\therefore$ the required product $=x^2+5x-14$.

Example 5. Write down the product of $x-5$ and $x-8$.

Since $\left.\begin{array}{l} -5-8=-13 \\ (-5)\times(-8)=40 \end{array}\right\}$, $\therefore$ the required product $=x^2-13x+40$.

Exercise (29).

Write down the product of —

1. $x+2$ and $x+8$.
2. $x+3$ and $x+5$.
3. $x+6$ and $x+11$.
4. $m+7$ and $m+9$.
5. $x-6$ and $x+2$.
6. $m-2$ and $m+8$
7. $a-3$ and $a-4$.
8. $x+5$ and $x-3$.
9. $x-4$ and $x+9$.
10. $x-5$ and $x-10$.
11. $x-12$ and $x+5$.
12. $k-13$ and $k+2$.
13. $a+5$ and $a+14$.
14. $m-14$ and $m+6$.
15. $x-5$ and $x-13$.
16. $x+7$ and $x+12$.
17. $a-3$ and $a-11$.
18. $x+4$ and $x-13$.
19. $m+5$ and $m-16$.
20. $x-8$ and $x-10$.
21. $a+6$ and $a-12$.
22. $m-7$ and $m+13$.
23. $x-10$ and $x-16$.
24. $x+5$ and $x-18$.
25. $x-16$ and $x+10$.

*10. Formula $(x+a)(x+b)(x+c)$

$$=x^3+(a+b+c)x^2+(ab+ac+bc)x+abc.$$

Note. The student can easily verify this It is also evident that the following results are included in it —

$$(x-a)(x-b)(x-c)=x^3-(a+b+c)x^2+(ab+ac+bc)x-abc\,,$$
$$(x+a)(x+b)(x-c)=x^3+(a+b-c)x^2+(ab-ac-bc)x-abc\,;$$
$$(x+a)(x-b)(x-c)=x^3+(a-b-c)x^2-(ab+ac-bc)x+abc$$

For instance, $(x-a)(x-b)(x-c) = \{x+(-a)\}\{x-(-b)\}\{x+(-c)\}$

$$= x^3+\{(-a)+(-b)+(-c)\}x^2$$
$$+\{(-a)(-b)+(-a)(-c)+(-b)(-c)\}x$$
$$+(-a)(-b)(-c)$$
$$= x^3-(a+b+c)x^2+(ab+ac+bc)x$$
$$-abc.$$

Similarly, the other two results can be established, which is left as an exercise for the student

Example 1. Write down the product of $x+2$, $x+4$ and $x+6$.

$$2+4+6 = 12,$$
$$2\times4+2\times6+4\times6 = 8+12+24 = 44,$$
$$2\times4\times6 = 48$$

Hence, the required product $= x^3+12x^2+44x+48$.

Example 2. Write down the product of $x-3$, $x-5$ and $x-7$

$$(-3)+(-5)+(-7) = -15,$$
$$(-3)(-5)+(-3)(-7)+(-5)(-7) = 15+21+35 = 71,$$
$$(-3)(-5)(-7) = -105.$$

Hence, the required product $= x^3-15x^2+71x-105$.

Example 3 Write down the product of $x-4$, $x+5$ and $x-3$.

$$(-4)+5+(-3) = -2,$$
$$(-4)\,5+(-4)(-3)+5\,(-3) = -20+12-15 = -23,$$
$$(-4)\times5\times(-3) = 60$$

Hence, the required product $= x^3-2x^2-23x+60$.

Example 4. Write down the product of $x+3$, $x+5$ and $x-8$.

$$3+5+(-8) = 0,$$
$$3\times5+3\times(-8)+5\times(-8) = 15-24-40 = -49,$$
$$3\times5\times(-8) = -120$$

Hence, the required product $= x^3-0\,x^2-49x-120$

Exercise (30).

Write down the product of —

1. $x+1$, $x+2$ and $x+3$.
2. $x+2$, $x+5$ and $x+7$.
3. $x+3$, $x-6$ and $x+2$.
4. $x+4$, $x+5$ and $x-10$.
5. $x-8$, $x+3$ and $x+1$
6. $x-5$, $x-2$ and $x+8$.
7. $x-3$, $x+7$ and $x-4$.
8. $x+6$, $x-5$ and $x-7$
9. $x-5$, $x-7$ and $x-11$.
10. $x-3$, $x-6$ and $x-9$
11. $x+4$, $x-5$ and $x-12$.
12. $x+5$, $x+9$ and $x+11$.
13. $x-6$, $x+8$ and $x-2$
14. $x-3$, $x-7$ and $x-13$
15. $x-3$, $x+12$ and $x+4$
16. $x-9$, $x-10$ and $x+12$
17. $x+9$, $x-5$ and $x-7$.
18. $x+8$, $x+12$ and $x+15$.
19. $x-14$, $x+8$ and $x+6$
20. $x-5$, $x-10$ and $x-16$.

11. Squares of multinomials. It has been respectively shown in examples 4 and 5 of Art 2 that

$$(a+b+c)^2 = a^2+b^2+c^2+2ab+2ac+2bc \text{ and } (a+b+c+d)^2$$
$$= a^2+b^2+c^2+d^2+2ab+2ac+2ad+2bc+2bd+2cd.$$

Thus in each of these cases we may observe that the square of the whole expression is obtained by taking the sum of the squares of the different terms and of twice the product of each term by every term which *follows* it The results are best remembered when put as follows —

$$(a+b+c)^2 = a^2+b^2+c^2+2a(b+c)+2bc\ ;$$
$$(a+b+c+d)^2 = a^2+b^2+c^2+d^2+2a(b+c+d)+2b(c+d)+2cd.$$

The same rule may be shown to hold in every other case , for instance, let us find the square of $a+b+c+d+e$.

We have $(a+b+c+d+e)^2 = \{(a+b+c)+(d+e)\}^2$

$$= (a+b+c)^2+2(a+b+c)(d+e)+(d+e)^2$$
$$= \{a^2+b^2+c^2+2a(b+c)+2bc\}+\{2a(d+e)+2b(d+e)$$
$$+2c(d+e)\}+(d^2+e^2+2de)$$
$$= a^2+b^2+c^2+d^2+e^2+2a(b+c+d+e)$$
$$+2b(c+d+e)+2c(d+e)+2de.$$

Hence we conclude that the square of any multinomial is equal to the sum of the squares of its different terms together with twice the product of each term by every term which *follows* it.

It is needless to add that the above rule will also hold good when the multinomial under consideration contains one or more negative terms, for the symbols used above are perfectly general in character and any of them may stand either for a positive or a negative quantity.

Note Since $(a+b+c)^2 = a^2+b^2+c^2+2(ab+ac+bc)$,

we have $2(ab+ac+bc) = \{a^2+b^2+c^2+2(ab+ac+bc)\}-(a^2+b^2+c^2)$
$$= (a+b+c)^2-(a^2+b^2+c^2)$$

Similarly, $a^2+b^2+c^2 = (a+b+c)^2-2(ab+ac+bc)$

Example 1. Write down the square of $x-y+z-v$.

$$(x-y+z-v)^2 = x^2+y^2+z^2+v^2+2x(-y+z-v)$$
$$+2(-y)(z-v)+2z(-v)$$
$$= x^2+y^2+z^2+v^2-2xy+2xz-2xv$$
$$-2yz+2yv-2zv.$$

Example 2. Write down the square of $-a+2b-3c-d$.

$$(-a+2b-3c-d)^2 = a^2+4b^2+9c^2+d^2+2(-a)(2b-3c-d)$$
$$+2(2b)(-3c-d)+2(-3c)(-d)$$
$$= a+4b^2+9c^2+d^2-4ab+6ac+2ad$$
$$-12bc-4bd+6cd.$$

Example 3. Find the value of $a^2+b^2+c^2+2ab-2ac-2bc$, when $a = 19$, $b = 18$ and $c = 32$.

The given expression $= a^2+b^2+c^2+2a(b-c)+2b(-c)$
$$= (a+b-c)^2.$$

Hence, the required value
$$= (19+18-32)^2$$
$$= (5)^2 = 25.$$

Example 4. If $x = b+c$, $y = c-a$, $z = a-b$, prove that $x^2+y^2+z^2-2xy-2xz+2yz=4b^2$. (C U Entrance Paper, 1883).

$$x^2+y^2+z^2-2xy-2xz+2yz$$
$$= x^2+y^2+z^2+2x(-y-z)+2(-y)(-z)$$
$$= (x-y-z)^2$$
$$= \{(b+c)-(c-a)-(a-b)\}^2$$
$$= (2b)^2 = 4b^2.$$

Exercise (31).

Write down the square of —

1. $x+y-z$ 2. $x-y+z$. 3. $-x+y+z$.
4. $-x-y+z$. 5. $x-y-z$. 6. $a-x+y-z$.
7. $a-x-y-z$. 8. $m+n+p+q+r$. 9. $p-q+r-x-y$.
10. $-a+b-c+x-y-z$ 11. $a-2x-3y-4z$.
12. $2a-b+2c-d$

Find the value of :—

13. $l^2+m^2+n^2-2lm+2ln-2mn$, when $l = 17$, $m = 23$, and $n = 13$.
14. $p^2+q^2+r^2+2pq-2pr-2qr$, when $p = 16$, $q = 12$, and $r = 25$.
15. $a^2+b^2+c^2-2ab-2ac+2bc$, when $a = 28$, $b = 13$, and $c = 15$.
16. $x^2+y^2+1+2xy-2x-2y$, when $x = 6$ and $y = 7$.
17. $x^2+y^2+2xy-2x-2y+36$, when $x = 23$ and $y = 18$.
18. $x^2+4y^2+1-4xy-2x+4y$, when $x = 26$ and $y = 12$.
19. $x^2+9y^2-6xy-2x+6y+64$, when $x = 49$ and $y = 16$.
20. $9x^2+y^2-6xy+6x-2y-24$, when $x = 14$ and $y = 38$.
21. If $a+b+c = 12$, and $a^2+b^2+c^2 = 50$, find the value of $ab+ac+bc$.
22. If $a+b+c = 13$, and $ab+ac+bc = 50$, find the value of $a^2+b^2+c^2$.

*12. Powers of Binomials, Involution.

By actual multiplication it may be seen that

$$\left.\begin{aligned}(a+b)^2 &= a^2+2ab+b^2\\(a-b)^2 &= a^2-2ab+b^2\end{aligned}\right\}$$

$$\left.\begin{aligned}(a+b)^3 &= a^3+3a^2b+3ab^2+b^3\\(a-b)^3 &= a^3-3a^2b+3ab^2-b^3\end{aligned}\right\}$$

$$\left.\begin{aligned}(a+b)^4 &= a^4+4a^3b+6a^2b^2+4ab^3+b^4\\(a-b)^4 &= a^4-4a^3b+6a^2b^2-4ab^3+b^4\end{aligned}\right\}$$

$$\left.\begin{aligned}(a+b)^5 &= a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5\\(a-b)^5 &= a^5-5a^4b+10a^3b^2-10a^2b^3+5ab^4-b^5\end{aligned}\right\}$$

$$\left.\begin{aligned}(a+b)^6 &= a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6\\(a-b)^6 &= a^6-6a^5b+15a^4b^2-20a^3b^3+15a^2b^4-6ab^5+b^6\end{aligned}\right\}$$

Note On examining the above cases we observe —

(1) The total number of terms in the resulting expression is one more than the index of the binomial Thus in the *fifth* power the number of terms is *six*, in the *sixth* power the number of terms is *seven*, and so on

(2) Any power of $a-b$ differs from the same power of $a+b$ *only* in this that the signs of the terms of the former are *alternately* $+$ and $-$, whilst those of the latter are *all* $+$

(3) The first term is a raised to a power equal to that of the binomial and the last term is b raised to the same power Thus in the *fourth* power, the first term is a^4 and the last b^4, in the *fifth* power, the first term is a^5 and the last b^5, and so on As to the other terms the power of a in any term is one less, whilst the power of b is one greater than that in the preceding term

(4) The co efficient of the second term is the same as the index of the power to which the binomial is raised, and if the co efficient of any term be multiplied by the index of a in that term and divided by the number *indicating the position* of that term, the result gives the co efficient of the next term. Thus, if we multiply the co efficient of the *second* term by the index of a in it and divide the product by *two*, we get the co efficient of the 3rd term, again, if the co efficient of the *third* term be multiplied by the index of a in it and the product divided by *three*, we obtain the co efficient of the 4th term, and so on

(5) The co-efficients of the terms equidistant from the beginning and the end are the same, in other words, the co-efficient of the term which has any number of terms *before* it, is equal to that of the term which has the same number of terms *after* it

The laws observed above, a proof of the universal truth of which is beyond the scope of our limits, furnish us with a ready means of raising a binomial to any power without the process of actual multiplication. The following examples are intended to illustrate the application of those laws.

[The resulting expression in each case is called the ***expansion*** of the corresponding power of the binomial]

The operation of raising any expression to any power is called **Involution.**

Example 1. Raise $a+b$ to the *seventh* power.

The total number of terms in the expansion $= 8$.

$$\left.\begin{array}{lll}\text{The first term} & = a^7 & \\ \text{,, 2nd ,,} & = 7a^6b & \\ \text{,, 3rd ,,} & = \frac{7\times6}{2}a^5b^2 & = 21a^5b^2 \\ \text{,, 4th ,,} & = \frac{21\times5}{3}a^4b^3 & = 35a^4b^3\end{array}\right\}\text{[Laws (3) and (4)]}$$

Now, since the four terms from the end will have respectively the same co-efficients as the four terms from the beginning [law (5)], the next four terms of the expansion will respectively be $35a^3b^4$, $21a^2b^5$, $7ab^6$ and b^7

Hence we have $(a+b)^7 = a^7+7a^6b+21a^5b^2+35a^4b^3+35a^3b^4+21a^2b^5+7ab^6+b^7$.

Example 2. Expand $(x-y)^8$.

The total number of terms in the expansion = 9.

$$\left.\begin{array}{llll} \text{The first term} & = x^8 & & \\ \text{,, 2nd ,,} & = -8x^7y & & \\ \text{,, 3rd ,,} & = \frac{8\times 7}{2}x^6y^2 & = & 28x^6y^2 \\ \text{,, 4th ,,} & = -\frac{28\times 6}{3}x^5y^3 & = & -56x^5y^3 \\ \text{,, 5th ,,} & = \frac{56\times 5}{4}x^4y^4 & = & 70x^4y^4. \end{array}\right\}\text{[Laws (2) (3) and (4)]}$$

The co-efficients of the remaining four terms need not be calculated as the co-efficients of the first four terms only will now reappear in the reverse order [Law (5)]

Hence we have

$$(x-y)^8 = x^8-8x^7y+28x^6y^2-56x^5y^3+70x^4y^4-56x^3y^5+28x^2y^6-8xy^7+y^8$$

Example 3. Expand $(2x-3y)^7$.

The total number of terms in the expansion = 8.

As we have $2x$ for a and $3y$ for b, we must have

$$\left.\begin{array}{llll} \text{The 1st term} & = (2x)^7 & & \\ \text{,, 2nd ,,} & = -7(2x)^6(3y) & & \\ \text{,, 3rd ,,} & = \frac{7\times 6}{2}(2x)^5(3y)^2 & = & 21(2x)^5(3y)^2 \\ \text{,, 4th ,,} & = -\frac{21\times 5}{3}(2x)^4(3y)^3 & = & -35(2x)^4(3y)^3 \end{array}\right\}$$

We can now write down the remaining four terms which will respectively be $35(2x)^3(3y)^4, -21(2x)^2(3y)^5$, $7(2x)(3y)^6$ and $-(3y)^7$. Hence we have

$$\begin{aligned}(2x-3y)^7 &= (2x)^7-7(2x)^6(3y)+21(2x)^5(3y)^2-35(2x)^4(3y)^3 \\ &\quad +35(2x)^3(3y)^4-21(2x)^2(3y)^5+7(2x)(3y)^6-(3y)^7 \\ &= 128x^7-7(64x^6)(3y)+21(32x^5)(9y^2)-35(16x^4)(27y^3) \\ &\quad +35(8x^3)(81y^4)-21(4x^2)(243y^5)+7(2x)(729y^6)-2187y^7 \\ &= 128x^7-1344x^6y+6048x^5y^2-15120x^4y^3 \\ &\quad +22680x^3y^4-20412x^2y^5+10206xy^6-2187y^7.\end{aligned}$$

Example 4. Find the value of

$x^6+6x^5+15x^4+20x^3+15x^2+6x-8$, when $x = \sqrt[3]{3}-1$.

The given expression

$$= (x^6+6x^5+15x^4+20x^3+15x^2+6x+1)-9$$
$$= (x+1)^6-9$$
$$= (\sqrt[3]{3})^6-9$$
$$= 9-9 = 0.$$

Exercise (32).

Expand —

1. $(x+1)^5$. 2. $(x+1)^6$. 3. $(a+b)^8$. 4. $(a+b)^9$.
5. $(x-y)^7$ 6. $(m-n)^7$. 7. $(x+2)^4$. 8. $(x+2)^5$.
9. $(x+1)^8$. 10. $(x+3)^4$ 11. $(x-1)^5$ 12. $(2-x)^6$.
13. $(2x-1)^4$. 14. $(x-y)^9$, 15. $(3x-2)^5$ 16. $(1-a)^8$.
17. $(1-c)^7$ 18. $(1-3x)^6$. 19. $(1-2x)^7$. 20. $(2x-a)^8$.
21. $(x-a)^{10}$ 22. $(3x-2a)^5$

Simplify —

23. $(x+1)^5-(x-1)^5$. 24. $(x-1)^6+(x+1)^6$.
25. $(x+a)^7-(x-a)^7$.

Find the sum of the co-efficients in the expansion of .—

26 $(x+a)^4$. 27. $(x+a)^5$. 28. $(x+a)^6$.
29. $(x+a)^7$. 30. $(x+a)^8$.

Find the value of —

31. $x^5+5x^4+10x^3+10x^2+5x+32$, when $x = -2$.
32. $x^6-6x^5+15x^4-20x^3+15x^2-6x$, when $x = \sqrt[3]{2}+1$.
33. $16x^4-32x^3+24x^2-8x-80$, when $x = 2$
34. $x^4+12x^3+54x^2+108x+81$, when $x = -5$.
35. $x^4+8x^3+24x^2+32x-609$, when $x = -7$.

***13 Formula** $(a+b+c)(a^2+b^2+c^2-ab-ac-bc)$
$$= a^3+b^3+c^3-3abc$$

$[(a+b+c)(a^2+b^2+c^2-ab-ac-bc)$
$$= (a+b+c)\{(a^2+b^2-ab)-(ac+bc)+c^2\}$$
$$= (a+b+c)\{\overline{(a+b)^2}-3ab-c(a+b)+c^2\}$$

$$= (\overline{a+b}+c)\{\overline{(a+b)^2-c(a+b)+c^2}-3ab\}$$
$$= \overline{(a+b)^3+c^3}-3ab(\overline{a+b}+c)$$
$$= (a+b)^3-3ab(a+b)+c^3-3abc$$
$$= a^3+b^3+c^3-3abc\ ]$$

Cor. Conversely, $a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-ab-ac-bc)$. Hence we can always resolve an expression into factors whenever it is of the form $a^3+b^3+c^3-3abc$

Note Since $a^2+b^2+c^2-ab-bc-ca = \frac{1}{2}\{(a-b)^2+(b-c)^2+(c-a)^2\}$ we have $a^3+b^3+c^3-3abc=\frac{1}{2}(a+b+c)\{(a-b)^2+(b-c)^2+(c-a)^2\}$

Example 1. Multiply $x^2+y^2+z^2+xy+xz-yz$ by $x-y-z$.

Putting a for x, b for $-y$, and c for $-z$, we have

$$(x-y-z)(x^2+y^2+z^2+xy+xz-yz)$$
$$= (a+b+c)(a^2+b^2+c^2-ab-ac-bc)$$
$$= a^3+b^3+c^3-3abc$$
$$= x^3-y^3-z^3-3xyz.$$

Example 2. Resolve $m^3-n^3+1+3mn$ into factors.

Putting a for m, b for $-n$ and c for 1, we have

$$m^3-n^3+1+3mn = a^3+b^3+c^3-3abc$$
$$= (a+b+c)(a^2+b^2+c^2-ab-ac-bc)$$
$$= (m-n+1)(m^2+n^2+1+mn-m+n).$$

Example 3. Show that $(x-y)^3+(y-z)^3+(z-x)^3$
$$= 3(x-y)(y-z)(z-x).$$

Putting a for $x-y$, b for $y-z$ and c for $z-x$, we have

$$a+b+c = (x-y)+(y-z)+(z-x) = 0.$$

Hence, $\{(x-y)^3+(y-z)^3+(z-x)^3\}-3(x-y)(y-z)(z-x)$
$$= a^3+b^3+c^3-3abc$$
$$= (a+b+c)(a^2+b^2+c^2-ab-ac-bc)$$
$$= 0\times(a^2+b^2+c^2-ab-ac-bc)$$
$$= 0;$$
$$\therefore\ (x-y)^3+(y-z)^3+(z-x)^3 = 3\,(x-y)(y-z)(z-x).$$

Exercise (33).

Multiply —

1. $x^2+y^2+z^2-xy+xz+yz$ by $x+y-z$.
2. $p^2+4q^2+r^2+2pq+pr-2qr$ by $p-2q-r$

3. $4x^2+9y^2+z^2+6xy+2xz-3yz$ by $2x-3y-z$
4. $a^2+4b^2+2ab-3a+6b+9$ by $a-2b+3$
5. $9a^2+25b^2+15ab+12a-20b+16$ by $3a-5b-4$.

Resolve into factors —

6 $x^3-y^3-1-3xy$

7. $x^3-y^3+6xy+8$.

8 $x^3-8y^3-27z^3-18xyz$

Find the value of —

9. $x^3+y^3+18xy-216$, when $x+y=6$
10. $a^3-8b^3-24ab-64$, when $a-2b=4$.
11 $(s-a)^3+(s-b)^3+(s-c)^3-3(s-a)(s-b)(s-c)$, when $3s=a+b+c$.
12. Show that $(a-2b)^3+(2b-3c)^3+(3c-a)^3$
$$=3(a-2b)(2b-3c)(3c-a).$$
13 Show that $(x+y-2z)^3+(y+z-2x)^3+(z+x-2y)^3$
$$=3(x+y-2z)(y+z-2x)(z+x-2y).$$
14. Show that $(a+2b-3c)^3+(b+2c-3a)^3+(c+2a-3b)^3$
$$=3(a+2b-3c)(b+2c-3a)(c+2a-3b)$$
15 Show that $(2p-5q+3r)^3+(2q-5r+3p)^3+(2r-5p+3q)^3$
$$=3(2p-5q+3r)(2q-5r+3p)(2r-5p+3q).$$

14. Formula $(a-b)(a-c)(b-c)$
$$=a^2(b-c)+b^2(c-a)+c^2(a-b).$$

$$[(a-b)(a-c)(b-c)=\{a^2-a(b+c)+bc\}(b-c)$$
$$=a^2(b-c)-a(b^2-c^2)+bc(b-c)$$
$$=a^2(b-c)+b^2(c-a)+c^2(a-b)]$$

Cor. 1. Conversely, $a^2(b-c)+b^2(c-a)+c^2(a-b)$
$$=(a-b)(a-c)(b-c).$$

Hence, we know at once the factors of an expression which is of the form $a^2(b-c)+b^2(c-a)+c^2(a-b)$

Cor. 2. Since $a-c=-(c-a)$, we have
$$(a-b)(a-c)(b-c)=-(a-b)(b-c)(c-a).$$

Hence the above relation can also be put in the form
$$a^2(b-c)+b^2(c-a)+c^2(a-b)=-(a-b)(b-c)(c-a)$$

Cor. 3. Since $a^2(b-c)+b^2(c-a)+c^2(a-b)$ can be put in the form $ab(a-b)+bc(b-c)+ca(c-a)$, we have also

$$ab(a-b)+bc(b-c)+ca(c-a) = -(a-b)(b-c)(c-a).$$

Example 4. Simplify $(a+2b+3c)^2(a-2b+c)$

$$+(b+2c+3a)^2(b-2c+a)+(c+2a+3b)^2(c-2a+b)$$
$$+(a-2b+c)(b-2c+a)(c-2a+b).$$

Putting $\left.\begin{array}{r} x \text{ for } a+2b+3c, \\ y \text{ for } b+2c+3a, \\ z \text{ for } c+2a+3b, \end{array}\right\}$ we have $\left.\begin{array}{r} y-z = a-2b+c \\ z-x = b-2c+a \\ x-y = c-2a+b \end{array}\right\}$

Hence the given expression

$$= x^2(y-z)+y^2(z-x)+z^2(x-y)+(y-z)(z-x)(x-y)$$
$$= -(y-z)(z-x)(x-y)+(y-z)(z-x)(x-y) = 0$$

Exercise (34).

1. Show that $(x-2y+z)+(2x-y-z)(y-2z+x)$
$= (x-y)^2(y-2z+x)+(y-z)^2(z-2x+y)+(z-x)^2(x-2y+z)$.

2. Show that $(a+b)^2(b-a)+(b+c)^2(c-b)+(c+a)^2(a-c)$
$+(b-a)(c-b)(a-c) = 0$.

3. Resolve into factors
$2(a-b+c)^2(a-c)+2(b-c+a)^2(b-a)+2(c-a+b)^2(c-b)$.

4. Resolve into factors
$(x+y)^2(y-x)+(y+z)^2(z-y)+(z+x)^2(x-z)$.

5. Simplify $2(a-b-c)^2(b-c)+2(b-c-a)^2(c-a)$
$+2(c-a-b)^2(a-b)+8(a-b)(b-c)(c-a)$.

6. Simplify $(x-y)(y-z)(x-2y+z)$
$+(y-z)(z-x)(y-2z+x)+(z-x)(x-y)(z-2x+y)$
$+(x-2y+z)(y-2z+x)(z-2x+y)$.

* 15. Miscellaneous Examples.

Example 1. Prove that $4a^2b^2-(a^2+b^2-c^2)^2$
$= s(s-2a)(s-2b)(s-2c)$, where $s = a+b+c$.

$4a^2b^2-(a^2+b^2-c^2)^2$

$= (2ab)^2-(a^2+b^2-c^2)^2$

$= \{2ab+(a^2+b^2-c^2)\}\{2ab-(a^2+b^2-c^2)\}$ [See Art 4]

$= \{(a^2+b^2+2ab)-c^2\}\{c^2-(a^2+b^2-2ab)\}$

$= \{(a+b)^2-c^2\}\{c^2-(a-b)^2\}$

$= \{(\overline{a+b}+c)(\overline{a+b}-c)(c+\overline{a-b})(c-\overline{a-b})\}$

$= (a+b+c)(a+b-c)(c+a-b)(c-a+b)$

$= (a+b+c)(\overline{a+b+c}-2c)(\overline{a+b+c}-2b)(\overline{a+b+c}-2a)$

$= s(s-2c)(s-2b)(s-2a)$

Example 2. If $s = a+b+c$, show that

$$(s-a)(s-b)(s-c) = (ab+ac+bc)(a+b+c)-abc.$$

$(s-a)(s-b)(s-c) = s^3-(a+b+c)s^2+(ab+ac+bc)s-abc.$

[See Art 10]

$= s^3-s\,s^2+(ab+ac+bc)(a+b+c)-abc$

$= (ab+ac+bc)(a+b+c)-abc$

Example 3. Prove that $(x-y)^2+(y-z)^2+(z-x)^2$
$= 2(x-y)(x-z)+2(y-z)(y-x)+2(z-x)(z-y).$

Putting $\left.\begin{array}{l} a \text{ for } x-y \\ b \text{ for } y-z \\ c \text{ for } z-x \end{array}\right\}$ we have $a+b+c = 0.$

Hence, $\{(x-y)^2+(y-z)^2+(z-x)^2\}-\{2(x-y)(x-z)$
$+2(y-z)(y-x)+2(z-x)(z-y)\}$

$= (a^2+b^2+c^2)-\{2a(-c)+2b(-a)+2c(-b)\}$

$= a^2+b^2+c^2+2ac+2ab+2bc$

$= (a+b+c)^3 = 0,$

$\therefore \quad (x-y)^2+(y-z)^2+(z-x)^2$

$= 2(x-y)(x-z)+2(y-z)(y-x)+2(z-x)(z-y).$

Example 4. If $2s = a+b+c$, prove that

$$(s-a)^3+(s-b)^3+3(s-a)(s-b)c=c^3.$$

We have $c = 2s-(a+b) = (s-a)+(s-b).$

Hence, $(s-a)^3+(s-b)^3+3(s-a)(s-b)c$

$$=(s-a)^3+(s-b)^3+3(s-a)(s-b)\{(s-a)+(s-b)\}$$

$$=\{(s-a)+(s-b)\}^3 \qquad \text{[See Art. 5.]}$$

$$=c^3.$$

Example 5. Prove that

$$(x-a)(x-b)(a-b)+(x-b)(x-c)(b-c)+(x-c)(x-a)(c-a)$$
$$=(a-b)(a-c)(b-c)$$

Putting p for $x-a$, q for $x-b$, r for $x-c$, we have

$$\left.\begin{aligned} q-p &= a-b \\ r-q &= b-c \\ p-r &= c-a \end{aligned}\right\}.$$

Hence,

$$(x-a)(x-b)(a-b)+(x-b)(x-c)(b-c)+(x-c)(x-a)(c-a)$$
$$=pq(q-p)+qr(r-q)+rp(p-r)$$
$$=-(q-p)(r-q)(p-r) \qquad \text{[See Art. 14, Cor. 3.]}$$
$$=-(a-b)(b-c)(c-a)$$
$$=(a-b)(b-c)(a-c).$$

Example 6. Show that

$$(x+y+z)^3-x^3-y^3-z^3=3(x+y)(y+z)(z+x).$$

$$(x+y+z)^3=\{x+(y+z)\}^3$$
$$=x^3+3x^2(y+z)+3x(y+z)^2+(y+z)^3 \qquad \text{[See Art. 5.]}$$
$$=x^3+3x^2(y+z)+3x(y+z)^2+\{y^3+z^3+3yz(y+z)\}$$
$$=x^3+y^3+z^3+3(y+z)\{x^2+x(y+z)+yz\}$$
$$=x^3+y^3+z^3+3(y+z)\{(x+y)(x+z)\} \qquad \text{[See Art. 9.]}$$
$$=x^3+y^3+z^3+3(y+z)(x+y)(x+z).$$

Hence, $(x+y+z)^3-x^3-y^3-z^3$

$$=\{x^3+y^3+z^3+3(y+z)(x+y)(x+z)\}-x^3-y^3-z^3$$
$$=3(x+y)(y+z)(z+x).$$

Example 7. If $2s=a+b+c$, show that

$$2(s-a)(s-b)+2(s-b)(s-c)+2(s-c)(s-a)$$
$$=2s^2-a^2-b^2-c^2.$$

Since $2x+2y+2z=(x+y)+(y+z)+(z+x)$,

we must have $2(s-a)(s-b)+2(s-b)(s-c)+2(s-c)(s-a)$

$$=\{(s-a)(s-b)+(s-b)(s-c)\}+\{(s-b)(s-c)+(s-c)(s-a)\}$$
$$+\{(s-c)(s-a)+(s-a)(s-b)\}.$$

Now, $(s-a)(s-b)+(s-b)(s-c)$
$$=(s-b)\{(s-a)+(s-c)\}$$
$$=(s-b)(2s-a-c)$$
$$=(s-b)b,$$

similarly, $\quad (s-b)(s-c)+(s-c)(s-a)=(s-c)c$;

and $\quad (s-c)(s-a)+(s-a)(s-b)=(s-a)a$

Hence, the given expression $=(s-b)b+(s-c)c+(s-a)a$
$$=s(b+c+a)-b^2-c^2-a^2$$
$$=2s^2-a^2-b^2-c^2.$$

Example 8 If $s=a+b+c$, prove that
$$s(s-2b)(s-2c)+s(s-2c)(s-2a)+s(s-2a)(s-2b)$$
$$=(s-2a)(s-2b)(s-2c)+8abc$$

The sum of the first two terms of the given expression
$$=s(s-2c)\{(s-2b)+(s-2a)\}$$
$$=s(s-2c)\{2s-2(a+b)\}$$
$$=s(s-2c)\times 2c,$$

and the third term $=\overline{(s-2c}+2c)(s-2a)(s-2b)$
$$=(s-2c)(s-2a)(s-2b)+2c(s-2a)(s-2b).$$

Hence the given expression
$$=s(s-2c)2c+\{(s-2c)(s-2a)(s-2b)+2c(s-2a)(s-2b)\}$$
$$=(s-2a)(s-2b)(s-2c)+2c\{s(s-2c)+(s-2a)(s-2b)\}.$$

But $\quad s(s-2c)+(s-2a)(s-2b)$
$$=(s^2-2cs)+\{s^2-2s(a+b)+4ab\}$$
$$=2s^2-2s(a+b+c)+4ab$$
$$=2s^2-2s\,s+4ab=4ab$$

$\therefore$ The given expression $=(s-2a)(s-2b)(s-2c)+8abc.$

Exercise (35).

1. Show that $(a^2+ax-x^2)(a^2-ax+x^2)$
$$=a^4-a^2x^2+2ax^3-x^4.$$

2. Show that $(a^2-ax+x^2)(ax-a^2+x^2)$
$$=x^4-a^2x^2+2a^3x-a^4.$$

3. Show that $(a+b+c)(a-b-c)+(b+c-a)(a-b+c)$
$$= 2b(a-b-c).$$

4. Show that $2(a^2+b^2+c^2-ab-ac-bc)$
$$= (a-b)^2+(b-c)^2+(c-a)^2.$$

5. Show that $\frac{1}{2}(a+b+c)\{(a-b)^2+(b-c)^2+(c-a)^2\}$
$$= a^3+b^3+c^3-3abc.$$

6. Show that $(a+b)(a+c)(b+c)$
$$= a^2(b+c)+b^2(c+a)+c^2(a+b)+2abc$$

7. Show that $2(x^3-x)+3x(x+1) = x(x+1)(2x+1)$.

8. Show that $x^4+x+x(x+1)(2x+1)-2x(x+1)$
$$= x^2(x+1)^2.$$

9. Show that $(a^2+b^2)(c^2+d^2) = (ac-bd)^2+(ad+bc)^2$.

10. Show that $(a+b)^2-(c+d)^2+(a+c)^2-(b+d)^2$
$$= 2(a+b+c+d)(a-d).$$

11. Multiply $9x^2+4y^2+z^2-2yz-3xz-6xy$ by $3x+2y+z$.

12. Show that $(a+b+c-d)(d-a-b+c) = c^2-(a+b-d)^2$.

13. Show that the product of $(b+c)^2-a^2$ and
$a^2-b^2-c^2+2bc = 2a^2b^2+2a^2c^2+2b^2c^2-a^4-b^4-c^4$.

14. Multiply together $a+b+c$, $a+b-c$, $a-b+c$
and $b+c-a$.

15. Simplify $(a^2+b^2+c^2)^2$
$$+(a+b+c)(a+b-c)(a+c-b)(b+c-a).$$

16. Simplify $(a^2+b^2+c^2)^2$
$$-(a+b+c)(a+b-c)(a+c-b)(b+c-a).$$

17. If $x = a^2-bc$, $y = b^2-ca$, $z = c^2-ab$,
show that $ax+by+cz = (a+b+c)(x+y+z)$

18. If $x^2-yz = a^2$, $y^2-zx = b^2$, $z^2-xy = c^2$,
prove that $a^2x+b^2y+c^2z = (x+y+z)(a^2+b^2+c^2)$.

19. Simplify $(a+b+c)^2-(a+b-c)^2+(a+c-b)^2$
$$-(b+c-a)^2.$$

20. Simplify $(a^2+b^2+c^2)^2-(b^2+c^2-a^2)^2-(a^2-b^2+c^2)^2$
$$+(a^2+b^2-c^2)^2.$$

21. Show that $(b-c+d+a)(d+a-b+c)$
$$+(c-d+a+b)(b+c+d-a) = 4(ad+bc).$$

22. Show that $(b+c+a-d)(b+c-a+d)$
$$= 2(ad+bc)-(a^2-b^2-c^2+d^2).$$

23. Resolve $4(ab+bc)^2-(a^2-b^2-c^2+d^2)^2$ into four factors.

24. Multiply together x^2-2x+1, x^2+2x+1 and x^4+2x^2+1.

25. Multiply $a^2+2ab+b^2-c^2$ by $a^2-2ab+b^2+c^2$.

26. Show that
$$(x-y+z)^2+(y-z+x)^2+(z-x+y)^2$$
$$+2(x-y+z)(y-z+x)+2(y-z+x)(z-x+y)$$
$$+2(z-x+y)(x-y+z) = (x+y+z)^2.$$

27. Simplify
$$(a-b)(b-c)(c-a)$$
$$-[a^2(c-b)-\{c^2(a-b)-b^2(c+a)\}].$$

28 Simplify
$$(a-b)(b-c)(c-a)$$
$$-\{ab(b-a)+bc(c-b)-ca(c+a)\}.$$

29. Show that
$$(x-a)^2(b-c)+(x-b)^2(c-a)+(x-c)^2(a-b)$$
$$+(b-c)(c-a)(a-b) = 0.$$

30. Show that
$$(a^2+b^2+c^2)(x^2+y^2+z^2)-(ax+by+cz)^2$$
$$= (ay-bx)^2+(cx-az)^2+(bz-cy)^2.$$

31. If $s=a+b+c$, prove that
$$(as+bc)(bs+ca)(cs+ab)$$
$$= (b+c)^2(c+a)^2(a+b)^2.$$

(Allahabad University Entrance Paper, 1890).

32. Show that
$$(a+b)(a+c)(b+c)-2abc$$
$$= a^2(b+c)+b^2(c+a)+c^2(a+b).$$

33. Show that
$$(a+c)^3-(b+c)^3-3(a+c)(b+c)(a-b)$$
$$= (a-b)^3.$$

34. Simplify
$$(x-2y+3z)^3+(x+2y-3z)^3$$
$$+6x(x-2y+3z)(x+2y-3x).$$

35 Show that
$$4(a+b+c)^2 = (a+b)^2+(b+c)^2+(c+a)^2$$
$$+2(a+b)(b+c)+2(b+c)(c+a)+2(c+a)(a+b).$$

36. Show that
$$8(a+b+c) = (a+b)^3+(b+2c+a)^3$$
$$+6(a+b)(b+2c+a)(a+b+c).$$

37. Show that
$$27(a+b+c)^3 = (a+3b+2c)^3+(2a+c)^3$$
$$+9(a+3b+2c)(2a+c)(a+b+c).$$

38. Show that
$$8(a+b+c)^3-(a+b)^3-(b+c)^3-(c+a)^3$$
$$= 3(2a+b+c)(a+2b+c)(a+b+2c).$$

39. Show that
$$27(a+b+c)^3-(a+2b)^3-(b+2c)^3-(c+2a)^3$$
$$= 3(a+3b+2c)(b+3c+2a)(c+3a+2b).$$

40. If $2s = a+b+c$, show that $2(s-a)(s-b)(s-c)$
$$+a(s-b)(s-c)+b(s-c)(s-a)+c(s-a)(s-b) = abc$$

41. If $s = a+b+c$, show that $s(s-a)(s-b)+s(s-a)(s-c)$
$$+s(s+a)(s-c)+c(s+a)(s+b) = (s+a)(s+b)(s+c).$$

42. If $2s = a+b+c$, show that $(s-a)^3+(s-b)^3+(s-c)^3$
$$-3(s-a)(s-b)(s-c) = \tfrac{1}{2}(a^3+b^3+c^3-3abc).$$

43. If $s = a+b+c$, show that $(s-3a)^2+(s-3b)^2+(s-3c)^2$
$$= 3\{(a-b)^2+(b-c)^2+(c-a)^2\}.$$

44. Given $a+b+c = 13$ and $a^2+b^2+c^2 = 69$; } find
45. Given $a+b+c = 12$ and $ab+bc+ca = 47$; }
the value of $a^3+b^3+c^3-3abc$.

46. Given $x+y+z = 14$, $x^2+y^2+z^2 = 74$,
and $x^3+y^3+z^3 = 434$, find the value of xyz.

47. Given $x+y+z = 13$, $xy+yz+zx = 52$,
and $xyz = 60$, find the value of $x^3+y^3+z^3$.

16. Recapitulation of the Formulæ. The different formulæ treated of in the foregoing articles are grouped below to facilitate any reference to them. It is desired, however, that the student should commit them so fully to memory that the necessity even for occasional references may be altogether done away with.

1. $(a+b)^2 = a^2+2ab+b^2.$

2. $(a-b)^2 = a^2-2ab+b^2$

3. $(a+b)(a-b) = a^2-b^2.$

4. $(a+b)^3 = a^3+3a^2b+3ab^2+b^3 = a^3+b^3+3ab(a+b)$

5. $(a+b+c)^3 = a^3+b^3+c^3+3a^2(b+c)+3b^2(a+c)+3c^2(a+b)+6abc.$

6. $(a-b)^3 = a^3-3a^2b+3ab^2-b^3 = a^3-b^3-3ab(a-b)$

7. $a^3+b^3 = (a+b)^3-3ab(a+b) = (a+b)(a^2-ab+b^2)$

8. $a^3-b^3 \quad = (a-b)^3+3ab(a-b) \quad = (a-b)(a^2+ab+b^2)$

9. $(x+a)(x+b) \quad = x^2+(a+b)x+ab.$

10. $(x-a)(x+b) \quad = x^2+(b-a)x-ab$

11. $(x-a)(x-b) \quad = x^2-(a+b)x+ab.$

12. $(x+a)(x+b)(x+c) = x^3+(a+b+c)x^2 +(ab+ac+bc)x+abc.$

13 $(x-a)(x-b)(x-c) = x^3-(a+b+c)x^2+(ab+ac+bc)x -abc.$

14. $(a+b+c)^2 \quad = a^2+b^2+c^2+2ab+2ac+2bc.$

15. $(a+b+c+d)^2 \quad = a^2+b^2+c^2+d^2+2a(b+c+d) +2b(c+d)+2cd.$

16. $(a+b)^4 \quad = a^4+4a^3b+6a^2b^2+4ab^3+b^4.$

17 $(a+b)^5 \quad = a^5+5a^4b+10a^3b^2+10a^2b^3 +5ab^4+b^5.$

18. $a^3+b^3+c^3-3abc = (a+b+c)(a^2+b^2+c^2-ab-ac-bc).$

19. $a^2(b-c)+b^2(c-a)+c^2(a-b) = (a-b)(a-c)(b-c) \quad = -(a-b)(b-c)(c-a)$

20. $ab(a-b)+bc(b-c)+ca(c-a) = (a-b)(a-c)(b-c) \quad = -(a-b)(b-c)(c-a)$

CHAPTER VII

DIVISION.

1. Definition. One quantity a is said to be divided by another quantity b when a third quantity c is found such that $c\times b = a$. In other words, $a\div b = c$ when $a = b\times c$

Thus, when $x = y\times z$, we have $x\div y = z$, and $x\div z=y$.

When one quantity is divided by another, the former is called the *dividend* and the latter the *divisor*, the result is called the *quotient*.

Note The quotient of a by b is often expressed as $\frac{a}{b}$.

2. Fundamental Propositions.

(i) To prove that $a \div b \times b = a$

If we denote $a \div b$ by x, we must have, by definition,

$$x \times b = a$$

Hence, $a \div b \times b = x \times b = a$.

(ii) To prove that $a \div b \div c = a \div bc$

We have
$$\begin{aligned}(a \div b \div c) \times bc &= \{(a \div b) \div c\} \times c \times b \\ &= [\{(a \div b) \div c\} \times c] \times b \\ &= (a \div b) \times b \quad \text{[by the last result]} \\ &= a.\end{aligned}$$

Hence, by definition, $a \div b \div c = a \div bc$.

That is, *to divide any quantity successively by two others is the same as to divide it at once by their product*

Cor. Hence, $a \div b \div c = a \div c \div b$, for each of them $= a \div (bc)$.

(iii) To prove that $a \div b = a \times \frac{1}{b}$.

We have $\frac{1}{b} \times b = 1 \div b \times b = 1$. [by (i)]

Hence, $a \times \frac{1}{b} \times b = a \times \left(\frac{1}{b} \times b\right)$ [Art 4, Chap V]

$$= a \times 1 = a\,;$$

i. e., $\left(a \times \frac{1}{b}\right) \times b = a$

Therefore, by definition, $a \div b = a \times \frac{1}{b}$.

Thus, *to divide one quantity by another is the same as to multiply the former by the reciprocal of the latter.*

Cor $a \div b \times c = a \times c \div b$

For $a \div b \times c = a \times \frac{1}{b} \times c$

$$= a\times c\times\frac{1}{b},\qquad \text{[Cor Art. 4, Chap V]}$$

and this latter $= a\times c\div b$

3. Law of Signs.

Since $a\times(-b) = -ab$,

$$\therefore \text{ by definition, } \left.\begin{aligned}(-ab)\div a &= -b\\ \text{and}\quad (-ab)\div(-b) &= a\end{aligned}\right\}\quad \ldots\ \text{I.}$$

Again, since $(-a)\times(-b) = ab$,

$$\left.\begin{aligned}ab\div(-a) &= -b\\ \text{and}\quad ab\div(-b) &= -a\end{aligned}\right\}\quad \ldots\ \text{II.}$$

It is evident also that

$$\left.\begin{aligned}ab\div a &= b\\ \text{and}\quad ab\div b &= a\end{aligned}\right\}\quad \ldots\ \text{III}$$

Hence, from I, II, and III, we have the following law of signs in division —*When the dividend and the divisor have the same signs, the quotient is positive, and when they have different signs, the quotient is negative* In other words, *like signs produce + and unlike signs —*

4. Division of one monomial expression by another

Let us examine a few particular cases —

(i) Since $3a^2b\times5a^3b^2c = 15a^5b^3c$, we must have

$$(15a^5b^3c)\div(5a^3b^2c) = 3a^2b$$

$$\left.\begin{aligned}\text{Thus, if the dividend} &= 15a^5b^3c\\ &= 3\times5\times a^3\times a^2\times b^2\times b\times c,\\ \text{and the divisor} &= 5a^3b^2c,\\ \text{we have the quotient} &= 3a^2b\end{aligned}\right\}\text{I.}$$

(ii) Since $(-2a^{10}b^2cd)\times(-3a^5c^2) = 6a^{15}b^2c^3d$,
we must have $6a^{15}b^2c^3d\div(-2a^{10}b^2cd) = -3a^5c^2$

$$\left.\begin{aligned}\text{Thus, if the dividend} &= 6a^{15}b^2c^3d\\ &= 2\times3\times a^{10}\times a^5\times b^2\times c\times c^2\times d,\\ \text{and the divisor} &= -2a^{10}b^2cd,\\ \text{we have the quotient} &= -3a^5c^2.\end{aligned}\right\}\text{II.}$$

(iii) Since $(-5a^8b^5c^2d)\times(4b^3c^4) = -20a^8b^8c^6d$.
we must have $(-20a^8b^8c^6d)\div(-5a^8b^5c^2d) = 4b^3c^4$.

Thus, if the dividend $= -20a^8b^8c^6d$

$$= -5 \times 4 \times a^8 \times b^5 \times b^3 \times c^2 \times c^4 \times d,$$

and the divisor $= -5a^8b^5c^2d$,

we have the quotient $= 4b^3c^4$. III.

Hence, from I, II, and III, we are led to deduce the following rule for dividing one monomial expression by another :—

Take away from the dividend all those factors which make up the divisor and to the remaining factors prefix the sign + or no sign, if the two expressions have the same sign, and the sign —, if they have different signs.

Note. We have $a^{12} \div a^7 = (a^5 \times a^7) \div a^7 = a^5$ $[= a^{12-7}]$. Similarly, $a^{20} \div a^9 = a^{11}$, $a^{21} \div a^{14} = a^7$, and so on. Hence generally, $a^m \div a^n = a^{m-n}$, where m and n are positive integers and $m > n$.

Example 1. Divide $18m^3n^2p$ by $-6m^2n^2p$.

The dividend $= 18m^3n^2p$

$= 6 \times 3 \times m^2 \times m \times n^2 \times p$.

The divisor $= -6m^2n^2p$.

∴ The quotient $= -3m$.

Example 2. Divide $-24a^7b^3c$ by $-6a^4bc$.

The dividend $= -24a^7b^3c$

$= -6 \times 4 \times a^4 \times a^3 \times b \times b^2 \times c$.

The divisor $= -6a^4bc$.

∴ The quotient $= 4a^3b^2$.

Exercise (36).

Divide :—

1. $8a^3$ by $-2a$.
2. $-15a^6$ by $3a^3$.
3. $-20x^7a^3$ by $-5x^3a$
4. $27m^5n^3$ by $9m^4n$.
5. $-28p^4q^2$ by $7p^2q$.
6. $-30a^{12}b^8$ by $-10a^8b^5$.
7. $-70x^{16}y^9z$ by $-14x^{10}y^5$
8. $64a^{12}b^7c^5$ by $-8a^9b^7c^3$.
9. $-81m^{13}n^{14}p^5$ by $27m^8n^8p^4$.
10. $-69a^7b^4c^9$ by $-23a^5b^4c^7$
11. $25x^{20}y^3z^8$ by $-5x^{16}yz^8$.
12. $-42a^{23}x^{25}y^9z^3$ by $-14a^{17}x^{18}y^5z$.

13. a^{101} by a^{57} **14.** $28x^{205}$ by $-4x^{157}$.
15. $56m^{307}$ by $-8m^{289}$. **16.** $-91a^{138}b^{209}$ by $13a^{97}b^{81}$

5. Division of a Multinomial by a Monomial.

From Cor 3, Art 8, Chap V, we have

$a(b+c+d+e+f+\ldots) = ab+ac+ad+ae+af+\ldots$

Hence, $(ab+ac+ad+ae+\ldots)\div a = b+c+d+e+\ldots$

$= (ab\div a)+(ac\div a)+(ad\div a)+(ae\div a)+\ldots$

Thus, *to divide a multinomial expression by a monomial we have to divide each term of the dividend by the divisor and take the sum of those partial quotients for the complete quotient*

Example 1. Divide $4a^3x^2-6a^2x^3+10ax^4$ by $-2ax$.

The required quotient $= \dfrac{4a^3x^2-6a^2x^3+10ax^4}{-2ax}$

$$= \frac{4a^3x^2}{-2ax}+\frac{-6a^2x^3}{-2ax}+\frac{10ax^4}{-2ax}$$

$$= -2a^2x+3ax^2-5x^3.$$

Example 2. Divide $-9x^5-4x^4a-2x^3a^2$ by $3x^3$.

The required quotient $= \dfrac{9x^5-4x^4a-2x^3a^2}{3x^3}$

$$= \frac{9x^5}{3x^3}+\frac{-4x^4a}{3x^3}+\frac{-2x^3a^2}{3x^3}$$

$$= 3x^2-\tfrac{4}{3}xa-\tfrac{2}{3}a^2.$$

Note After a little practice the student can safely do away with the middle step in each case and write down the quotient at once.

Exercise (37).

Divide :—

1. $a^3b^2-2a^2b^3$ by a^2b^2. **2.** $3x^4-6x^3a$ by $-3x^2$.
3. $4x^3a^3-8x^4a^2$ by $4x^2a^2$. **4.** $-9a^5+12a^3b^2$ by $-3a^3$
5. $14a^6b^4-21a^7b^3$ by $-7a^5b^2$.
6. $2ax^3-4a^2x^2+6a^3x$ by $2ax$.
7. $-3a^3x^4+6a^2x^5-9a^4x^3$ by $-3a^2x^3$.
8. $12x^5-8x^3a^2+20ax^4$ by $-4x^3$.

9. $10m^5n^4-15m^7n^2-20m^3n^6$ by $5m^3n^2$.

10. $8p^4q^2-5p^3q^3-3p^2q^4$ by $-8p^2q^2$.

11. $-14x^8y^5+21x^{10}y^3-28x^7y^6$ by $7x^7y^3$.

12. $15a^4x^8-30a^7x^5-45a^6x^6$ by $20a^4x^5$.

13. $-60x^4a^5-75x^3a^6+80x^5a^4$ by $-20x^3a^4$.

14. $25m^3n^2p-35m^2n^3p-40mnp^4$ by $5mnp$

15. $-a^2x^3y^2+a^4x^2y-ax^3y^3+a^5xy$ by $-axy$.

6. Division of one multinomial expression by another.

Let us consider a particular example.

We have

$$\begin{aligned}&(2a^2+3ab+4b^2)(a+3b)\\&=2a^2(a+3b)+3ab(a+3b)+4b^2(a+3b)\\&=2a^3+9a^2b+13ab^2+12b^3.\end{aligned}$$

Hence,

$$(2a^3+9a^2b+13ab^2+12b^3)\div(a+3b)=2a^2+3ab+4b^2.$$

Now, let us review this result and see in what way, given the dividend and the divisor, we can discover the quotient The points noticed are —

(i) The dividend and the divisor *both* stand arranged according to descending powers of a common letter, namely, a,

(ii) The *first* term of the quotient, namely, $2a^2 = 2a^3\div a$, *i.e.*, = (the 1st term of the dividend)÷(the 1st. term of the divisor).

(iii) If we subtract $2a^2(a+3b)$ from the dividend, the remainder is $3a^2b+13ab^2+12b^3$, and the *second* term of the quotient, namely, $3ab = 3a^2b\div a$, *i.e.*, = (the 1st term of this remainder)÷(the 1st. term of the divisor).

(iv) If we subtract $3ab(a+3b)$ from the above remainder, the new remainder is $4ab^2+12b^3$ and the *third* term of the quotient, namely, $4b^2 = 4ab^2\div a$, *i.e.*, = (the 1st term of this remainder)÷(the 1st. term of the divisor).

(v) If we subtract $4b^2(a+3b)$ from the preceding remainder, nothing remains and the division is complete.

The process noted above can be shown as follows :—

$$
\begin{array}{l|l|l}
a+3b & 2a^3+9a^2b+13ab^2+12b^3 & 2a^2+3ab+4b^2 \\
 & 2a^3+6a^2b & \\
\hline
 & \quad 3a^2b+13ab^2+12b^3 & \\
 & \quad 3a^2b+\ \ 9ab^2 & \\
\hline
 & \qquad 4ab^2+12b^3 & \\
 & \qquad 4ab^2+12b^3 & \\
\hline
\end{array}
$$

Hence, we deduce the following rule —

Arrange both the dividend and the divisor according to the descending powers of some common letter and place them in a line as in the process of Division in Arithmetic

Divide the first term of the dividend by the first term of the divisor and write down the result as the first term of the quotient. Multiply the divisor by the quantity thus found and subtract the product from the dividend.

Regard the remainder as a new dividend and see if it is arranged according to the descending powers of the common letter. Divide its first term by the first term of the divisor and write down the result as the next term of the quotient Multiply the divisor by this term and subtract the product from the new dividend.

Then go on similarly with the successive remainders until there is no remainder

Note That the rule above stated gives us a correct result is evident For, the different quantities, that are one by one subtracted from the dividend, being the partial products of the divisor by successive terms of the quotient, their sum is equal to the product of the divisor by the *whole* quotient , and as this sum is clearly equal to the dividend, the dividend is equal to the product of the divisor by the quotient, and *this is what it should be*

Example 1. Divide $x^4-4x^2+12x-9$ by x^2-2x+3

Both the dividend and the divisor as they are, are arranged according to descending powers of x Hence, we may proceed at once as follows .—

$$
\begin{array}{r|llll|l}
x^2-2x+3 & x^4 & & -4x^2 & +12x-9 & x^2+2x-3 \\
 & x^4 & -2x^3 & +3x^2 & & \\
\hline
 & & 2x^3 & -7x^2 & +12x-9 & \\
 & & 2x^3 & -4x^2 & +\ \ 6x & \\
\hline
 & & & -3x^2 & +\ \ 6x-9 & \\
 & & & -3x^2 & +\ \ 6x-9 & \\
\hline
\end{array}
$$

Thus, the required quotient $= x^2+2x-3$.

Note. In the dividend it must be noticed that the term containing x^3 is wanting, and hence the second term, which contains x^2, has been put a little apart from the first as if leaving unoccupied the place of the absent term. This point should be attended to, although not strictly required, *for the purpose of having like terms placed under one another*: for instance, in the above example, if the second term of the dividend stood close to the first, $-2x^3$ would come under $-4x^2$, and $3x^2$ under $12x$, and this might confuse the beginner or otherwise lessen the neatness of the process.

Example 2. Divide $16x^4+36x^2+81$ by $4x^2+6x+9$.

$$
\begin{array}{r|llll|l}
4x^2+6x+9 & 16x^4 & & +36x^2 & +81 & 4x^2-6x+9 \\
 & 16x^4 & +24x^3 & +36x^2 & & \\
\hline
 & & -24x^3 & & -81 & \\
 & & -24x^3 & -36x^2 & -54x & \\
\hline
 & & & 36x^2 & +54x+81 & \\
 & & & 36x^2 & +54x+81 & \\
\hline
\end{array}
$$

Thus, the required quotient $= 4x^2-6x+9$.

Note. It may so happen that the dividend is not exactly divisible by the divisor. For instance, if in the present example the dividend were $16x^4+36x^2+6x+86$, the second remainder would be $36x^2+60x+86$, and hence the final remainder $6x+5$. As $6x+5$ cannot be divided by $4x^2+6x+9$, the division in this case would be incomplete and the result might be expressed as in Arithmetic, thus:—

$$\frac{16x^4+36x^2+6x+86}{4x^2+6x+9} = 4x^2-6x+9+\frac{6x+5}{4x^2+6x+9}.$$

The portion of the dividend which is thus left as a residue not divisible by the divisor is spoken of as *the remainder* in division. Hence, if D denote the dividend, d the divisor, Q the quotient, and R the *remainder*, we have the following invariable relation between these symbols:—

$$D = d \times Q + R$$

Example 3. Divide $x^6-4x^4-2x^3+3x^2+8x-12$ by x^2-4.

N B It is *not essential* to arrange the dividend and the divisor according to *descending* powers of some letter common to them, the arrangements may as well be according to *ascending* powers of that letter The only thing indispensable is that *both the expressions* should be arranged in the *same* order, be it descending or ascending For instance, let us work out the present example by arranging the expressions in the *ascending* order of the powers of x

$$
\begin{array}{r|l|l}
-4+x^2 & -12+8x+3x^2-2x^3-4x^4+x^6 & 3-2x+x^4 \\
 & -12 \qquad\;\; +3x^2 & \\ \hline
 & 8x \qquad -2x^3-4x^4+x^6 & \\
 & 8x \qquad -2x^3 & \\ \hline
 & -4x^4+x^6 & \\
 & -4x^4+x^6 & \\ \hline
\end{array}
$$

Thus, the required quotient $= 3-2x+x^4$.

Example 4. Divide $a^2b^2+2abc^2-a^2c^2-b^2c^2$

by $ab+ac-bc$.

The dividend, when arranged according to descending powers of a, becomes

$$(b^2-c^2)a^2+2bc^2a-b^2c^2.$$

The divisor, when so arranged, becomes $(b+c)a-bc$.

Thus, the dividend has become a trinomial and the divisor a binomial

$$
\begin{array}{r|l|l}
(b+c)a-bc & (b^2-c^2)a^2+2bc^2\,a-b^2c^2 & (b-c)a+bc \\
 & (b^2-c^2)a^2-(b^2c-bc^2)a & \\ \hline
 & (b^2c+bc^2)a-b^2c^2 & \\
 & (b^2c+bc^2)a-b^2c^2 & \\ \hline
\end{array}
$$

Thus, the required quotient $= ab-ac+bc$.

Example 5. Divide $a^3+b^3-c^3+3abc$ by $a+b-c$

The dividend and the divisor, arranged according to descending powers of a, respectively become

$$a^3+3bc\,a+(b^3-c^3) \text{ and } a+(b-c)$$

Thus, the dividend has become a trinomial and the divisor a binomial.

$$
\begin{array}{r|l|l}
a+(b-c) & a^3 \qquad\qquad\qquad + 3bc\,a + (b^3-c^3) & a^2-(b-c)a \\
 & a^3+(b-c)a^2 & +(b^2+bc+c^2) \\
\hline
 & -(b-c)a^2 + 3bc\,a + (b^3-c^3) & \\
 & -(b-c)a^2 - (b-c)^2\,a & \\
\hline
 & (b^2+bc+c^2)a + (b^3-c^3) & \\
 & (b^2+bc+c^2)a + (b^3-c^3) & \\
\hline
\end{array}
$$

Thus, the required quotient $= a^2+b^2+c^2-ab+ac+bc$.

Example 6. Divide $(b-c)\,a^3+(c-a)b^3+(a-b)c^3$ by $a^2-ab-ac+bc$.

Let us arrange the dividend and the divisor according to descending powers of a

The dividend $= (b-c)a^3-b^3a+c^3a+b^3c-bc^3$

$= (b-c)a^3-(b^3-c^3)a+bc(b^2-c^2)$.

The divisor $= a^2-(b+c)a+bc$.

Thus, the dividend has become a trinomial and the divisor also a trinomial.

$$
\begin{array}{r|l|l}
a^2-(b+c)a+bc & (b-c)a^3-(b^3-c^3)a \quad + bc(b^2-c^2) & (b-c)a \\
 & (b-c)a^3-(b^2-c^2)a^2+bc(b-c)a & +(b^2-c^2) \\
\hline
 & (b^2-c^2)a^2-(b^3+b^2c-bc^2-c^3)a+bc(b^2-c^2) & \\
 & (b^2-c^2)a^2-(b^3+b^2c-bc^2-c^3)a+bc(b^2-c^2) & \\
\hline
\end{array}
$$

Thus, the required quotient $= ab-ac+b^2-c^2$.

Note. It must be noted that the expressions which are enclosed within brackets as co-efficients of different powers of a are all arranged according to descending powers of b Such arrangements add to the neatness of the process and lessen the chance of confusion

Exercise (38).

Divide —

1. x^2-5x+6 by $x-3$
2. $2x^2-11x+5$ by $2x-1$
3. $6x^2-x-12$ by $2x-3$
4. $15x^2-x-28$ by $3x+4$.
5. $2a^2-7ab+6b^2$ by $a-2b$.

6. $x^4+x^2y^2+y^4$ by x^2+xy+y^2.
7. $4x^2-9a^2$ by $2x+3a$ 8 x^3+a^3 by $x+a$
9. $a^3-a^2b-7ab^2+3b^3$ by $a-3b$
10. $2m^3-9m^2n+13mn^2-6n^3$ by $2m-3n$
11. $a^4-3a^3b+3ab^3-b^4$ by a^2-b^2
12. $2x^4-3x^3y-3xy^3-2y^4$ by x^2+y^2.
13. $2a^4-36a^2x^2-16ax^3$ by $2a^2+8ax$
14. $3+2x+4x^2+5x^3-4x^4+2x^5$ by $1+2x^2$
15. $x^4-4x^2+12x-9$ by x^2+2x-3.
16. $4a^4-9a^2b^2+24ab^3-16b^4$ by $2a^2-3ab+4b^2$
17. $a^4+4a^2x^2+16x^4$ by $a^2+2ax+4x^2$.
18 a^4+4b^4 by $a^2+2ab+2b^2$
19. $2x^5-7x^4-2x^3+18x^2-3x-8$ by x^3-2x^2+1.
20. x^4-81 by $x-3$ 21. a^5-32 by $a-2$.
22. $3-9x+2x^2+5x^3-7x^4+2x^5$ by $1-3x+x^2$.
23. $82x^2+40-45x^3+18x^4-67x$ by $6x^2+8-7x$
24. $64-x^6$ by $2-x$ 25. $1+x^6-2x^3$ by x^2+1-
26. $18ab^3+2a^2b^2+6a^4-a^3b+4b^4$ by $4ab+b^2+3a^2$
27. $a^3b-15b^4-8a^2b^2+a^4+19ab^3$ by a^2+3b^2-2ab.
28. x^6-a^6 by $x^3-2x^2a+2xa^2-a^3$.
29. $8a^2b^3+3b^5+a^5-9a^3b^2-2ab^4-a^4b$ by $2ab-3b^2+$
30 $y^6+x^6-2x^3y^3$ by x^2+y^2-2xy
31. $x^6-2a^3x^3+a^6$ by $x^2-2ax+a^2$
32. $2x^3y^3+y^6+x^6$ by $2xy+x^2+y^2$.
33. $x^3+(a+b+c)x^2+(ab+ac+bc)x+abc$ by $x+c$
34. $x^3+(b-c-a)x^2+(ca-ab-bc)x+abc$
by $x^2+(b-a)x-ab$.
35. $a^3+a^2b+a^2c-abc-b^2c-bc^2$ by a^2-bc
36 $a^2(b+c)-b^2(a+c)+c^2(a+b)+abc$ by $a-b+c$
37. $a^2(b+c)+b^2(a-c)+c^2(a-b)+abc$ by $a+b+c$
38. $x^3-2ax^2+(a^2-ab-b^2)x+a^2b+ab^2$ by $x-a-b$
39. $a^3+b^3+c^3-3abc$ by $a+b+c$.
40. $x^3+y^3-1+3xy$ by $x+y-1$
41. $x^3-8y^3-27z^3-18xyz$ by $x-2y-3z$

42. $x^3-y^3+z^3+3xyz$ by $x-y+z$.

43. $8x^3-27y^3-z^3-18xyz$

by $4x^2+9y^2+z^2+6xy+2xz-3yz$.

44. $a^2(b-c)+b^2(c-a)+c^2(a-b)$ by $a-b$.

45. $(x^2-bx+cx)a-bc(x+a)+(x-b+c)x^2$ by $(x+a)(x-b)$.

46. $c(ab-x^2)+(a-b)(x-c)x+x(x^2-ab)$ by $(x-b)(x-c)$.

47. $a^3(b-c)+b^3(c-a)+c^3(a-b)$ by $ab+bc-ac-b^2$.

48. $a^3(b^2-c^2)+b^3(c^2-a^2)+c^3(a^2-b^2)$

by $a^2b-bc^2-ac^2-a^2c$.

49. $xy^3+2y^3z-xy^2z+xyz^2-x^3y-2yz^3+x^3z-xz^3$

by $y+z-x$.

50. $b(x^3+a^3)+ax(x^2-a^2)+a^3(x+a)$ by $(a+b)(x+a)$

51. $(a-b)^2c^2+(a-b)c^3-(c^2-a^2)b^2+(c-a)b^3$ by

$(a-b)c^2-(c-a)b^2$.

(Calcutta University Entrance Paper, 1883)

[Arrange the given expressions according to descending powers of c]

52. $(ax+by)^3+(ax-by)^3-(ay-bx)^3+(ay+bx)^3$ by

$(a+b)^2x^2-3ab(x^2-y^2)$.

(Calcutta University Entrance Paper, 1888)

[Simplify the dividend and then arrange the two expressions according to descending powers of x]

53. $x(1+y^2)(1+z^2)+y(1+z^2)(1+x^2)+z(1+x^2)(1+y^2)+$

$4xyz$ by $1+xy+yz+zx$.

(Calcutta University Entrance Paper, 1878)

(Arrange the expressions according to descending powers of x)

54. $(4x^3-3a^2x)^2+(4y^3-3a^2y)^2-a^6$ by $x^2+y^2-a^2$

(Bombay University Matriculation Paper, 1884)

Assuming the formula $a^m\div a^n = a^{m-n}$ to be true for *all* values of m and n, show that :—

55. $a^0=1$. [$a^0 = a^{m-m} = a^m\div a^m = 1$]

56. $a^{-n}=\dfrac{1}{a^n}$. [$a^{-n} = a^{0-n} = a^0\div a^n = 1\div a^n$.]

57. $x^{\frac{5}{2}}\div x^{\frac{3}{2}} = x$.

58. $x^{-\frac{3}{4}}\div x^{-\frac{7}{4}} = x$.

Divide —

59. $a^2b^{\frac{2}{3}}$ by $a^{-1}b^{\frac{1}{3}}$ 60. $a^{-2}b^{\frac{1}{2}}c^{\frac{5}{2}}$ by $a^{-3}b^{\frac{3}{2}}c^2$.

61. $15xyz$ by $-5x^{\frac{2}{3}}y^{\frac{3}{5}}z^{\frac{4}{3}}$. 62. $9x^{\frac{4}{3}}-16y^{\frac{2}{3}}$ by $3x^{\frac{2}{3}}+4y^{\frac{1}{3}}$.

63. $a+b$ by $a^{\frac{1}{3}}+b^{\frac{1}{3}}$. 64. $a^3+a^{\frac{3}{2}}b^{\frac{3}{2}}+b^3$ by $a^{\frac{3}{2}}+a^{\frac{3}{4}}b^{\frac{3}{4}}+b^{\frac{3}{2}}$.

65. $4x^{\frac{8}{3}}-37x^{\frac{4}{3}}y^{\frac{4}{3}}+9y^{\frac{8}{3}}$ by $2x^{\frac{4}{3}}+5x^{\frac{2}{3}}y^{\frac{2}{3}}-3y^{\frac{4}{3}}$.

66. $a-b^2$ by $a^{\frac{1}{4}}-b^{\frac{1}{2}}$

67. $4a^{-10}+12a^{-\frac{15}{2}}b^{-\frac{3}{2}}+9a^{-5}b^{-3}-25b^{-6}$

by $2a^{-5}+3a^{-\frac{5}{2}}b^{-\frac{3}{2}}-5b^{-3}$.

68 $9x^{-\frac{5}{2}}-25x^{-\frac{5}{4}}y^{-\frac{3}{4}}+70x^{-\frac{5}{8}}y^{-\frac{9}{8}}-49y^{-\frac{3}{2}}$

by $3x^{-\frac{5}{4}}+5x^{-\frac{5}{8}}y^{-\frac{3}{8}}-7y^{-\frac{3}{4}}$.

69. a^3-b^2 by $a^{\frac{1}{2}}-b^{\frac{1}{3}}$.

70. $x+y+z-3x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}}$ by $x^{\frac{1}{3}}+y^{\frac{1}{3}}+z^{\frac{1}{3}}$

7. *A few important results

The student already knows that

$$x^2-a^2 = (x-a)(x+a)$$

and $$x^3-a^3 = (x-a)(x^2+xa+a^2)$$

Hence, x^4-a^4 [which $= x^3(x-a)+a(x^3-a^3)$]

$$= (x-a)\{x^3+a(x^2+xa+a^2)\}$$
$$= (x-a)(x^3+x^2a+xa^2+a^3).$$

Hence, x^5-a^5 [which $= x^4(x-a)+a(x^4-a^4)$]

$$= (x-a)\{x^4+a(x^3+x^2a+xa^2+a^3)\}$$
$$= (x-a)(x^4+x^3a+x^2a^2+xa^3+a^4).$$

Similarly, it may be shown that $x-a$ is a factor of x^6-a^6, of x^7-a^7, of x^8-a^8, and so on, hence, generally, $x-a$ is a factor of x^n-a^n where n is any whole number.

We conclude, therefore, that for *all* positive integral values of n, x^n-a^n *is divisible* by $x-a$

Again, since $x^n+a^n = (x^n-a^n)+2a^n$, of which x^n-a^n is divisible by $x-a$ and $2a^n$ is not, $\therefore$ x^n+a^n is *not* divisible by $x-a$

Thus, when n is a positive integer,

$$\left.\begin{array}{ll} & x-a \textit{ always} \text{ divides } x^n-a^n, \\ \text{but} & \textit{never} \text{ divides } x^n+a^n. \end{array}\right\} \quad .. \quad (A)$$

Cor. 1. $x+a$ divides x^n-a^n *only* when n is an *even* integer.

$$\left.\begin{array}{l} \text{For, when } n \text{ is even, } (-a)^n = \ \ a^n, \dagger\text{and } \therefore x^n-a^n=x^n-(-a)^n, \\ \quad\text{when } n \text{ is odd, } (-a)^n=-a^n, \dagger\text{and } \therefore x^n-a^n=x^n+(-a)^n, \end{array}\right\}$$

$$\text{also, } x+a = x-(-a)$$

Now, from (A), we know that $x-(-a)$ divides $x^n-(-a)^n$, but not $x^n+(-a)^n$. Hence, $x+a$ divides x^n-a^n when n is even but not when n is odd, *i e*, $x+a$ divides x^n-a^n *only* when n is an *even* integer.

Cor. 2. $x+a$ divides x^n+a^n *only* when n is an *odd* integer.

$$\left.\begin{array}{l} \text{For, when } n \text{ is odd, } (-a)^n=-a^n, \text{ and } \therefore x^n+a^n=x^n-(-a)^n, \\ \quad\text{when } n \text{ is even, } (-a)^n= \ \ a^n, \text{ and } \therefore x^n+a^n=x^n+(-a)^n; \end{array}\right\}$$

$$\text{also, } x+a=x-(-a)$$

Now, from (A), we know that $x-(-a)$ divides $x^n-(-a)^n$, but not $x^n+(-a)^n$. Hence, $x+a$ divides x^n+a^n when n is odd, but not when n is even, *i e.*, $x+a$ divides x^n+a^n *only* when n is an *odd* integer

Thus, we have obtained the following results,—

$$\left.\begin{array}{l} x-a \text{ divides } x^n-a^n \textit{ always}, \\ \qquad\qquad\quad x^n+a^n \textit{ never} \end{array}\right\}$$

$$\left.\begin{array}{l} x+a \text{ divides } x^n-a^n \textit{ only} \text{ when } n \text{ is } \textit{even}, \\ \qquad\qquad\quad x^n+a^n \textit{ only} \text{ when } n \text{ is } \textit{odd}. \end{array}\right\}$$

Exercise (39).

Verify by actual division that the following expressions are divisible by $x+a$ —

1. x^3+a^3. 2. x^4-a^4 3. x^5+a^5.
4. x^6-a^6. 5. x^7+a^7 6. x^8-a^8.

---

†This follows from repeated applications of the laws of signs in multiplication, thus $(-a)^2=a^2$ hence, $(-a)^3=(-a)\times(-a)^2 =(-a)\times a^2=-a^3$, hence, $(-a)^4=(-a)\times(-a)^3=(-a)(-a^3)=a^4$; hence, $(-a)^5=(-a)(-a^4)=(-a)\times a^4=-a^5$, and so on That is, any power of $-a$ is positive or negative according as the index of the power is an even or an odd integer

Verify by actual division that the following expressions are *not* divisible by $x+a$ —

7. x^3-a^3. 8. x^4+a^4. 9. x^5-a^5

10. x^6+a^6 11. x^7-a^7. 12. x^8+a^8.

Write down the quotient —

13. x^4-1 by $x-1$ 14. x^4-y^4 by $x+y$

15. x^5-1 by $x-1$ 16. x^5+y^5 by $x+y$

17. x^6-1 by $x-1$ 18. x^6-y^6 by $x+y$

19. x^7-1 by $x-1$. 20 x^7+y^7 by $x+y$

CHAPTER VIII

FACTORS.

1. Definitions. When an expression is the product of two or more others, each of these latter is called a *factor* of the former

An expression is said to be *resolved into factors* when those expressions of which it is the product are found

Note In this chapter we shall confine our attentions to *rational* and *integral* expressions only (*i e*, expressions free from radical signs and in which no letter occurs in the denominator of any term) and by the factors of an expression will be meant the *rational* and *integral* expressions of which it is the product

[A few simple cases of resolution into factors have already been incidentally treated in the chapter on *formulæ and their application*. These cases, however, will not be altogether passed over in the following articles as the present chapter is intended for a more systematic treatment of the subject]

2. Simple cases. Any expression *all the terms of which have got a common factor* may, on inspection, be at once resolved into two factors, one of which is simple and the other compound, thus —

(1) $a^2x+ax^2 \qquad = ax(a+x).$

(2) $2a^3b^2-3a^2b^3 \qquad = a^2b^2(2a-3b)$

(3) $24x^4a^3-40x^3a^4+56x^2a^5 = 8x^2a^3(3x^2-5xa+7a^2)$.

Exercise (40).

Resolve into factors —

1. $ab+ac$ 2. $a^2b^3+a^3b^2$. 3. $x^3y^4-2x^4y^3$.
4. $2x^2yz+4xy^2z-6xyz^2$. 5. $4a^5b-6a^4b^2-8a^3b^3$.
6. $ax^2y-5a^2x^3y^2+3ax^3$
7. $3x^4y^3z^2-12x^2y^4z^3+21x^3y^2z^4$.
8. $28a^8b^5-42a^5b^8$ 9. $72x^{10}y^8+108x^8y^{10}$.
10. $39a^5b^7c^7-65b^5c^7a^7-91c^5a^7b^7$.

3. Expressions of the form a^2-b^2.

The method of resolving into factors an expression of this form has already been treated in Art. 4 Chap VI A few more examples are added here for the exercise of the student.

Exercise (41).

Resolve into factors —

1. $9a^2-16b^2$. 2. $4a^3-25ax^2$. 3. $36x^4-1$
4. $16x^4-1$. 5. $16x^5-9x$ 6. $16x^5-81x$.
7. $1-16a^4$. 8. x^2-81x^6. 9. $36-x^4a^2$.
10. $64a^4-49x^6$. 11. $121-m^6$
12 $49x^6a^{10}-81$. 13. $a^2b^2-25c^2d^2$.
14. $81x^{12}-64a^{10}$. 15. $p^2q^4-100p^2$.
16. $144x^7-25x^3a^4$ 17. $192a^9-243a^5x^4$.
18. $98a^3x^5-128ax$. 19. $324x^{17}a^9-484x^5a^3$.
20. $245m^{23}n^{13}-605m^{15}n^7$ 21. $(a+3b)^2-25c^2$.
22. $a^2-(3b-5c)^2$. 23. $(x+y)^2-(x-y)^2$.
24. $(3a+2x)^2-(2a+x)^2$. 25. $4(a-b)^2-9(c-d)^2$.
26. $49x^2-(5y-3z)^2$ 27. $(8x+5)^2-(2x-7)^2$.
28. $(a+b-c)^2-(a-b+c)^2$.
29. $(2a-3b+4c)^2-(a+4b-5c)^2$.
30. $64(a+3x-4y)^2-9(2a-x+3y)^2$.
31. $(4x^2-5a^2)^2-(5x^2-4a^2)^2$.
32. $(5a^2-3a+7)^2-(5a^2-3a-7)^2$.

4. Expressions which by mere inspection can be put into the form a^2-b^2 The following examples are intended for illustration

Example 1. Resolve into factors $a^4+a^2b^2+b^4$.

$$\begin{aligned} a^4+a^2b^2+b^4 &= (a^4+2a^2b^2+b^4)-a^2b^2 \\ &= (a^2+b^2)^2-(ab)^2 \\ &= \{(a^2+b^2)+ab\}\{(a^2+b^2)-ab\} \\ &= (a^2+ab+b^2)(a^2-ab+b^2). \end{aligned}$$

Example 2. Resolve into factors x^4+4.

$$\begin{aligned} x^4+4 &= (x^4+4x^2+4)-4x^2 \\ &= (x^2+2)^2-(2x)^2 \\ &= \{(x^2+2)+2x\}\{(x^2+2)-2x\} \\ &= (x^2+2x+2)(x^2-2x+2). \end{aligned}$$

Example 3. Resolve into factors x^4-6x^2+1.

$$\begin{aligned} x^4-6x^2+1 &= (x^4-2x^2+1)-4x^2 \\ &= (x^2-1)^2-(2x)^2 \\ &= \{(x^2-1)+2x\}\{(x^2-1)-2x\} \\ &= (x^2+2x-1)(x^2-2x-1). \end{aligned}$$

Example 4. Resolve into factors $a^2-b^2+2bc-c^2$.

$$\begin{aligned} a^2-b^2+2bc-c^2 &= a^2-(b^2-2bc+c^2) \\ &= a^2-(b-c)^2 \\ &= \{a+(b-c)\}\{a-(b-c)\} \\ &= (a+b-c)(a-b+c). \end{aligned}$$

Example 5. Resolve into factors

$$2(ab+cd)-a^2-b^2+c^2+d^2.$$

$$\begin{aligned} \text{The given expression} &= (c^2+2cd+d^2)-(a^2-2ab+b^2) \\ &= (c+d)^2-(a-b)^2 \\ &= \{(c+d)+(a-b)\}\{(c+d)-(a-b)\} \\ &= (c+d+a-b)(c+d-a+ \end{aligned}$$

Exercise (42).

Resolve into factors —

1. x^4+x^2+1 2. x^8+x^4+1. 3. $a^4+a^2x^2+x^4$.
4. $a^8+a^4x^4+x^8$. (Calcutta University Entrance Paper, 1887)

5. x^4+64
6. $4x^4+81$.
7. $9x^4+36$.
8. a^4+2a^2+9.
9. x^4-7x^2+9.
10. $4x^4+8x^2+9$.
11. $4x^4-16x^2+9$
12. $4x^4+3x^2+9$.
13. $4a^4-37a^2+9$.
14. $4a^4+625$
15. $9x^4+23x^2+16$.
16. $9a^4-25a^2+16$.
17. $9x^4-33x^2+16$
18. $9a^4-a^2+16$.
19. $16x^4+4x^2a^2+25a^4$.
20. $9a^4-19a^2x^2+25x^4$.
21. x^4+8x^2+144
22. $a^4-35a^2b^2+25b^4$.
23. $36a^4-16a^2b^2+b^4$.
24. $49m^4+16n^4-60m^2n^2$.
25. $64a^4+81x^4$.
26. $4x^4+(7a)^4$.
27. $x^2-y^2+2yz-z^2$.
28. $4a^2-b^2-9c^2+6bc$.
29. $9x^2-4y^2+12yz-9z^2$.
30. $a^2-4b^2-25c^2+20bc$
31. $30xz+16y^2-9x^2-25z^2$.
32. $a^2+4b^2-9c^2-4d^2-4ab+12cd$.
33. $(x^2-2xy)-(z^2-2yz)$.
34. $4x^2-1+9a^2-25b^2+12xa-10b$.
35. $9x^2-4y^2-49z^2-30x+28yz+25$
36. $16a^2-16c^2-9b^2-24a+24bc+9$.
37. $49y^2+20z+x^2-14xy-25z^2-4$.
38. $16x^2+42by-9y^2+40xa-49b^2+25a^2$.
39. $49x^2-1+16y^2-64z^2+16z-56xy$.
40. $a^2-b^2-c^2+d^2-2(ad-bc)$

5. Expressions of the form a^3+b^3 or a^3-b^3.

The resolution of such expressions into factors has already been considered in Articles 7 and 8, Chap. VI. A few cases, however, of a little more complicated character may, with advantage, be added here.

Example 1. Resolve into factors a^9+x^9.

Since $a^3+b^3 = (a+b)(a^2-ab+b^2)$,

we have
$$\begin{aligned} a^9+x^9 &= (a^3)^3+(x^3)^3 \\ &= (a^3+x^3)\{(a^3)^2-(a^3)(x^3)+(x^3)^2\} \\ &= (a^3+x^3)(a^6-a^3x^3+x^6) \\ &= (a+x)(a^2-ax+x^2)(a^6-a^3x^3+x^6). \end{aligned}$$

Example 2. Resolve into factors a^9-x^9

Since $a^3-b^3=(a-b)(a^2+ab+b^2)$,

we have
$$\begin{aligned} a^9-x^9 &= (a^3)^3-(x^3)^3 \\ &= (a^3-x^3)\{(a^3)^2+(a^3)(x^3)+(x^3)^2\} \\ &= (a^3-x^3)(a^6+a^3x^3+x^6) \\ &= (a-x)(a^2+ax+x^2)(a^6+a^3x^3+x^6). \end{aligned}$$

Example 3. Resolve into factors $64x^7-xa^6$

$$\begin{aligned} &64x^7-xa^6 \\ &= x(64x^6-a^6) \\ &= x\{(8x^3)^2-(a^3)^2\} \\ &= x(8x^3+a^3)(8x^3-a^3) \\ &= x\{(2x)^3+a^3\}\{(2x)^3-a^3\} \\ &= x\{(2x+a)(4x^2-2xa+a^2)\}\{(2x-a)(4x^2+2xa+a^2)\} \\ &= x(2x+a)(2x-a)(4x^2-2xa+a^2)(4x^2+2xa+a^2) \end{aligned}$$

Otherwise —

$$\begin{aligned} 64x^7-xa^6 &= x(64x^6-a^6) \\ &= x\{(4x^2)^3-(a^2)^3\} \\ &= x(4x^2-a^2)(16x^4+4x^2a^2+a^4) \\ &= x(2x+a)(2x-a)\{(16x^4+8x^2a^2+a^4)-4x^2a^2\} \\ &= x(2x+a)(2x-a)\{(4x^2+a^2)^2-(2xa)^2\} \\ &= x(2x+a)(2x-a)(\overline{4x^2+a^2}+2xa)(\overline{4x^2+a^2}-2xa) \\ &= x(2x+a)(2x-a)(4x^2+2xa+a^2)(4x^2-2xa+a^2) \end{aligned}$$

Note Although the resolution can be effected in either of the two ways shown above, it is generally found convenient to adopt the first method

Exercise (43).

Resolve into factors —

1. a^3-8b^3. 2. a^4-27ax^3 3. $512x^9+1$.
4 a^9-512b^9 5. $27a^6+125x^6$ 6 m^6-n^6.
7. $343x^3+512y^3$ (Calcutta University Entrance Paper, 1882).
8. $64x^{12}-1$. 9. a^6-64x^{12} 10. $125x^9-216a^9$.

11. $64a^{13}b+343ab^{13}$.

12. $729x^{20}y^2-64x^2y^{20}$.

13. $(a^2+b^2)^3+8a^3b^3$.

14. $(2x^2-3y^2)^3+y^6$.

15. $(2a^3-b^3)^3-b^9$.

6. Expressions of the form x^2+px+q resolved into factors by inspection.

From the relation $x^2+(a+b)x+ab=(x+a)(x+b)$, it is clear that to resolve an expression of the form x^2+px+q into two factors we have to find two quantities a and b such that $a+b=p$ and $ab=q$ This can be done by inspection whenever a and b are rational and integral The student can very well refer himself to the examples worked out in Art. 9 Chap VI, for a clearer comprehension of such cases.

Example 1. Resolve into factors $x^2+17x+30$

We have to find two numbers whose sum $=17$, and product $=30$.

Pairs of numbers whose product is 30 are —(i) 1 and 30, (ii) 2 and 15, (iii) 3 and 10, (iv) 5 and 6 Out of these four pairs then we must pick out that of which the sum is 17; the second pair, therefore, is the one sought.

Thus, 2 and 15 are the numbers required.

Hence, $x^2+17x+30=(x+2)(x+15)$

Example 2. Resolve into factors $x^2-11x+24$.

We must find two numbers whose product $=+24$, and sum $=-11$ Clearly then the two numbers must be both negative

The pairs of negative numbers whose product is 24 are —(i) -1 and -24, (ii) -2 and -12, (iii) -3 and -8, (iv) -4 and -6 Out of these four pairs we must pick out that of which the sum is -11, the third pair, therefore, is the one sought

Thus, the required numbers are -3 and -8

Hence, $x^2-11x+24=(x-3)(x-8)$

Example 3. Resolve into factors $x^2+6x-40$.

We must find two numbers whose product $=-40$, and sum $=+6$.

The pairs of numbers whose product is -40 are —(i) 1 and -40, (ii) -1 and 40, (iii) 2 and -20, (iv) -2 and 20, (v) 4 and -10, (vi) -4 and 10, (vii) 5 and -8, (viii) -5 and 8 Out of these 8 pairs we must pick out that of which the sum is $+6$, the sixth pair, therefore, is the one sought

Thus, the required numbers are -4 and 10

Hence, $x^2+6x-40 = (x-4)(x+10)$

Note From the fact that the sum of the two numbers is positive it is clear that the positive number must be *numerically* greater than the negative Hence we might at once reject the first, third, fifth and seventh of the above pairs

Example 4 Resolve into factors $x^2-5x-36$

We have to find two numbers whose product $= -36$, and sum $= -5$ Clearly then the numbers must have different signs and the negative number must be numerically greater than the positive

Hence, the only admissible pairs of numbers whose product is -36 are —(i) 1 and -36. (ii) 2 and -18, (iii) 3 and -12, (iv) 4 and -9 Out of these four pairs we must pick out that of which the sum is -5 , the last pair, therefore, is the one sought. Thus, the required numbers are 4 and -9

Hence, $x^2-5x-36 = (x+4)(x-9)$

Example 5. Resolve into factors $a^2+7ab+12b^2$

The factors will evidently be $a+pb$ and $a+qb$ where p and q are such that $p+q = 7$, and $pq = 12$.

Arguing as before it is easy to see that 3 and 4 are the numbers whose sum is 7, and product 12

Hence, $a^2+7ab+12b^2 = (a+3b)(a+4b)$

Example 6. Resolve into factors $m^2-12mn+20n^2$

We have to find two numbers whose sum $= -12$, and product $= 20$

Arguing in the usual way we find that -10 and -2 are the required numbers.

Hence, $m^2-12mn+20n^2 = (m-10n)(m-2n)$

Example 7. Resolve into factors a^4-a^2-12.

Putting x for a^2, the given expression becomes x^2-x-12 and it is easy to see that $x^2-x-12 = (x-4)(x+3)$

Hence, $a^4-a^2-12 = (a^2-4)(a^2+3)$
$= (a+2)(a-2)(a^2+3)$.

Example 8. Resolve into factors

$$(x^2+2x)^2-3(x^2+2x)-18.$$

Putting a for x^2+2x, the given expression becomes $a^2-3a-18$, and it is easy to see that

$$a^2-3a-18 = (a-6)(a+3)$$

Hence, the given expression $= \{(x^2+2x)-6\}\{(x^2+2x)+3\}$
$= (x^2+2x-6)(x^2+2x+3)$.

Example 9. Resolve into factors

$$(5a+b)^2+(5a+b)(a+2b)-20(a+2b)^2$$

Putting x for $5a+b$ and y for $a+2b$, the given expression becomes $x^2+xy-20y^2$.

Now it can be easily seen that

$$x^2+xy-20y^2 = (x+5y)(x-4y)$$

Hence, the given expression

$$= \{(5a+b)+5(a+2b)\}\{(5a+b)-4(a+2b)\}$$
$$= (10a+11b)(a-7b)$$

Example 10. Resolve into factors $8x^2+2x-3$

$$8x^2+2x-3 = \tfrac{1}{8}(8\times 8x^2+2\times 8x-3\times 8)$$
$$= \tfrac{1}{8}(a^2+2a-24) \qquad [\text{Putting } a \text{ for } 8x\,]$$

Now it can be easily seen that $a^2+2a-24=(a+6)(a-4)$.

Hence, the given expression $= \frac{1}{8}(a+6)(a-4)$
$= \frac{1}{8}(8x+6)(8x-4)$
$= \frac{1}{8}\{2(4x+3)\times 4(2x-1)\}$
$= (4x+3)(2x-1)$

Example 11. Resolve into factors $12x^2+7x-10$

$$12x^2+7x-10 = \tfrac{1}{12}(12\times 12x^2+7\times 12x-10\times 12)$$
$$= \tfrac{1}{12}(a^2+7a-120) \qquad [\text{Putting } a \text{ for } 12x\,]$$

Now it can be easily seen that

$$a^2+7a-120 = (x+15)(a-8)$$

Hence, the given expression $= \frac{1}{12}(12x+15)(12x-8)$
$= \frac{1}{12}\{3(4x+5)\times 4(3x-2)\}$
$= (4x+5)(3x-2)$.

Exercise (44).

Resolve into factors —

1. x^2+4x+3	2. x^2+5x+6	3. $x^2+7x+12$.
4. $x^2+9x+20$.	5. $x^2+9x+18$	6. $x^2+11x+28$
7. $x^2-10x+24$.	8. $x^2-8x+15$	9. $x^2-11x+30$.
10. $x^2-12x+32$	11. $x^2-14x+24$	12. $x^2-22x+40$.
13. $x^2+7x-30$	14. $x^2+2x-48$	15. $x^2+16x-36$
16. $x^2+9x-36$	17. $x^2+11x-42$	18. $x^2+14x-72$.
19. $x^2-8x-40$.	20. $x^2-11x-80$.	21. $x^2-29x-96$
22. $x^2-10x-56$	23. x^2-x-42	24. x^2-x-72
25 $x^2+22x+120$	26. $x^2+16x-80$	27. $x^2-21x-72$.
28. $x^2+5x-84$	29 $x^2-20x+96$	30. $x^2+23x-78$.
31. $x^2-6x-72$	32. $x^2-25x+84$	33. $x^2-26x+88$.
34. $x^2+7x-120$.	35. $x^2-2x-80$.	36. $x^2+8x-84$.
37. a^2-a-56	38. $m^2-9m-90$	39. $a^2+17a-60$
40. $a^2-15a+54$.	41. $p^2-22p-48$	42. m^2+m-72.

43. $m^2+27m-90$.	44 $a^2-29a+120$
45. $x^2+7x-78$	46. $a^2-49a-102$
47. $a^2-19a+60$.	48. $x^2+12x-64$
49. $a^2-26a-120$	50. $x^2+8x-105$
51. $x^2-xy-42y^2$	52. $a^2-12ab+32b^2$.
53. $m^2+mn-30n^2$.	54. $a^2+ab-12b^2$
55. $a^2-2ab-15b^2$.	56. $x^2-7xy-8y^2$
57. $x^2+3xy-40y^2$.	58. $p^2-14pq+48q^2$.
59. $p^2+2pq-80q^2$.	60. $x^2+20xy-96y^2$
61. a^4+4a^2-5.	62. x^4+2x^2-15.
63. x^4+3x^2-28.	64. x^6+2x^3-3
65. a^6-10a^3+16.	66. x^6+26x^3-27
67. a^6+7a^3-8	68. x^8-20x^4+64
69. a^8-11a^4-80	70. $x^{12}-7x^6-8$
71. $(a^2+2a)^2-(a^2+2a)-2$.	72. $(x^2+3x)^2+3(x^2+3x)+2$.
73. $(x^2-2x)^2-2(x^2-2x)-3$	
74. $(a^2-3a)^2-3(a^2-3a)-4$.	

75. $(x^2-4x)^2-4(x^2-4x)-5.$
76. $(x^2-x)^2-8(x^2-x)+12$
77. $(x^2-5x)^2+10(x^2-5x)+24$
78. $(a^2+7a)^2-8(a^2+7a)-180$
79. $(a^2+6a)^2-32(a^2+6a)-320$
80. $(x^2-8x)^2-29(x^2-8x)+180.$
81. $(3x+4y)^2+(3x+4y)(x+2y)-2(x+2y)^2$
82. $(2a-5b)^2+2(2a-5b)(a+2b)-3(a+2b)^2$
83. $(4m-3n)^2+3(4m-3n)(m+5n)-4(m+5n)^2.$
84. $(5a+3b)^2+4(5a+3b)(a+b)-5(a+b)^2.$
85. $(5x+7y)^2-2(5x+7y)(x-y)-8(x-y)^2.$
86. $(7a+3b)^2-3(7a+3b)(a-4b)-10(a-4b)^2.$
87. $(9a-4b)^2-5(9a-4b)(a-3b)-6(a-3b)^2.$
88. $(2m^2+3n^2)^2+5(2m^2+3n^2)(m^2-2n^2)-14(m^2-2n^2)^2.$
89. $(x^2+6xy)^2-3(x^2+6xy)(xy-6y^2)+2(xy-6y^2)^2.$
90. $(a^2-5ab)^2-9(a^2-5ab)(ab-4b^2)+18(ab-4b^2)^2.$

91. $2x^2+x-15$ 92. $6a^2-a-15$
93. $8m^2-6m-9.$ 94. $6x^2+7xy-24y^2.$
95. $10a^2-41ab+21b^2.$ 96. $12m^2-mn-20n^2.$
97. $12x^2+28xy-5y^2.$ 98. $20a^2+ab-30b^2.$
99. $18x^2-51xy+35y^2.$ 100. $12x^2+23xy-24y^2.$

7. Quantities of the form x^2+px+q resolved into factors by expressing them as the difference of two squares.

The method will be best illustrated by the solution of a few typical cases

Example 1. Resolve into factors $x^2-7x+12.$

$$x^2-7x+12 = x^2-7x+(\tfrac{7}{2})^2-(\tfrac{7}{2})^2+12$$

[adding and subtracting $(\frac{7}{2})^2$]

$$= \{x^2-7x+(\tfrac{7}{2})^2\}-(\tfrac{49}{4}-12)$$
$$= (x-\tfrac{7}{2})^2-\tfrac{1}{4}$$
$$= \{(x-\tfrac{7}{2})+\tfrac{1}{2}\}\{(x-\tfrac{7}{2})-\tfrac{1}{2}\}$$
$$= (x-3)(x-4).$$

Note. It must be noticed that we have added to x^2-7x the square of half of 7 (*i.e.*, the square of half the co-efficient of x) to get a perfect square. Generally speaking x^2+2ax or (x^2-2ax), becomes a complete square when a^2 is added to it.

Example 2. Resolve into factors

$$x^2+2xy-8y^2-4z^2+12yz.$$

The given expression

$$\begin{aligned}
&=(x^2+2xy+y^2)-(9y^2+4z^2-12yz)\\
&=(x+y)^2-(3y-2z)^2\\
&=\{(x+y)+(3y-2z)\}\{(x+y)-(3y-2z)\}\\
&=(x+4y-2z)(x-2y+2z)
\end{aligned}$$

Example 3. Resolve into factors $3x^2+11x-4$.

$$\begin{aligned}
3x^2+11x-4 &= 3(x^2+\tfrac{11}{3}x-\tfrac{4}{3})\\
&=3\{x^2+\tfrac{2\cdot 11}{6}x+(\tfrac{11}{6})^2-(\tfrac{11}{6})^2-\tfrac{4}{3}\}\\
&=3\{(x+\tfrac{11}{6})^2-(\tfrac{121}{36}+\tfrac{4}{3})\}\\
&=3\{(x+\tfrac{11}{6})^2-\tfrac{169}{36}\} \qquad [\tfrac{169}{36}=(\tfrac{13}{6})^2]\\
&=3\{(x+\tfrac{11}{6})+\tfrac{13}{6}\}\{(x+\tfrac{11}{6})-\tfrac{13}{6}\}\\
&=3(x+4)(x-\tfrac{1}{3})\\
&=(x+4)(3x-1)
\end{aligned}$$

Example 4. Resolve into factors $8x^2-10x+3$

$$\begin{aligned}
8x^2-10x+3 &= 8\{x^2-\tfrac{10}{8}x+\tfrac{3}{8}\}\\
&=8\{x^2-\tfrac{5}{4}x+(\tfrac{5}{8})^2-(\tfrac{25}{64}-\tfrac{3}{8})\}\\
&=8\{(x-\tfrac{5}{8})^2-\tfrac{1}{64}\}\\
&=8\{(x-\tfrac{5}{8})+\tfrac{1}{8}\}\{(x-\tfrac{5}{8})-\tfrac{1}{8}\}\\
&=8(x-\tfrac{1}{2})(x-\tfrac{3}{4})\\
&=\{2(x-\tfrac{1}{2})\}\{4(x-\tfrac{3}{4})\}\\
&=(2x-1)(4x-3)
\end{aligned}$$

Example 5. Resolve into factors $2a^2+5ab-12b^2$.

$$\begin{aligned}
2a^2+5ab-12b^2 &= 2(a^2+\tfrac{5}{2}ab-6b^2)\\
&=2\left\{a^2+\frac{5}{2}ab+\left(\frac{5b}{4}\right)^2-\left(\frac{25b^2}{16}+6b^2\right)\right\}\\
&=2\{(a+\tfrac{5}{4}b)^2-\tfrac{121}{16}b^2\}\\
&=2\{(a+\tfrac{5}{4}b)+\tfrac{11}{4}b\}\{(a+\tfrac{5}{4}b)-\tfrac{11}{4}b\}\\
&=2(a+4b)(a-\tfrac{3}{2}b)\\
&=(a+4b)(2a-3b)
\end{aligned}$$

Example 6. Resolve into factors $ax^2+(a^2+1)x+a$.

$$ax^2+(a^2+1)x+a = a\left\{x^2+\frac{a^2+1}{a}.x+1\right\}$$
$$= a\left\{x^2+\frac{a^2+1}{a}x+\left(\frac{a^2+1}{2a}\right)^2-\left(\frac{a^4+2a^2+1}{4a^2}-1\right)\right\}$$
$$= a\left\{\left(x+\frac{a^2+1}{2a}\right)^2-\frac{a^4-2a^2+1}{4a^2}\right\}$$
$$= a\left\{\left(x+\frac{a^2+1}{2a}\right)+\frac{a^2-1}{2a}\right\}\left\{\left(x+\frac{a^2+1}{2a}\right)-\frac{a^2-1}{2a}\right\}$$
$$= a\left(x+a\right)\left(x+\frac{1}{a}\right)$$
$$= (x+a)(ax+1).$$

Similarly, it may be shown that

$$ax^2-(a^2+1)x+a = (x-a)(ax-1),$$
$$ax^2+(a^2-1)x-a = (x+a)(ax-1),$$
$$ax^2-(a^2-1)x-a = (x-a)(ax+1).$$

Note It is useful to remember these results as we are thus enabled to write down at once the factors of any expression which agrees in form with any of those considered above For instance, we can at once say that—

$$3x^2-10x+3=(x-3)(3x-1),$$
$$4x^2-15x-4=(x-4)(4x+1),$$
$$5x^2+24x-5=(x+5)(5x-1), \text{ and so on}$$

Example 7. Resolve into factors

$$4(x^2+2x+5)^2+17(x^2+2x+5)(x^2+6x)+4(x^2+6x)^2.$$

Putting a for x^2+2x+5 and b for x^2+6x, the given expression becomes $4a^2+17ab+4b^2$, and it is easy to see that

$$4a^2+17ab+4b^2 = (a+4b)(4a+b)$$

Hence, the given expression

$$= \{(x^2+2x+5)+4(x^2+6x)\}\{4(x^2+2x+5)+(x^2+6x)\}$$
$$= (5x^2+26x+5)(5x^2+14x+20)$$
$$= (x+5)(5x+1)(5x^2+14x+20).$$

Exercise (45).

Resolve the following expressions into factors applying the method of this article —

1. $x^2+4x+3.$ 2. x^2+6x+5 3. $x^2+8x+15$
4. $x^2-10x+21$ 5. $x^2-2x-48$ 6. $x^2-4x-45.$
7. $x^2-12x+32.$ 8. $x^2-6x-55.$ 9. $a^2+2ab-c^2+2bc$
10. $x^2+2x-y^2+2y.$ 11. x^2+6x-y^2+4y+5
12. $a^2+4ab-5b^2-c^2+6bc$ 13. $x^2-6xy+5y^2-z^2+4yz.$
14. $x^2-10xy+16y^2-4z^2+12yz$
15. $a^2-12ab-13b^2-9c^2+42bc$
16. $x^2+12xy-9z^2+36yz$
17. $x^2-14xy-15y^2-25z^2+80yz$ 18. $2x^2-5x-3.$
19. $3x^2-5x-2$ 20. $3x^2+14x+8.$ 21. $4x^2+7x-2$
22. $6x^2+x-2$ 23. $6x^2-5x-4$ 24. $6x^2+7x-3.$
25. $8x^2+2x-15$ 26. $4x^2+4x-35$ 27. $6x^2-x-12$
28. $3x^2-16x-12$ 29. $2x^2-9x-35$ 30. $2x^2+5x-42$
31. $3x^2+13x-30$ 32. $12x^2+x-6.$
33. $2a^2+7ab-15b^2$ 34. $6x^2-13xy+6y^2$
35. $6m^2-11mn-10n^2$ 36. $3p^2+5pq-12q^2$
37. $8a^2-14ab-15b^2$ 38. $10m^2+11mn-6n^2$
39. $12x^2+13xy-4y^2$ 40. $15a^2-11ab-12b^2.$
41. $2a^2-5ab+2b^2$ 42. $3a^2-8ab-3b^2.$
43. $3x^2+8xy-3y^2$ 44. $4a^2+15a-4.$
45. $4a^2-17ab+4b^2$ 46. $5x^2-24x-5$
47. $5x^2-26xy+5y^2.$ 48. $6x^2+37x+6.$
49. $6a^2+35ab-6b^2$ 50. $6a^2-35ab-6b^2.$
51. $7a^2-50ab+7b^2.$ 52. $7a^2+48ab-7b^2.$
53. $7a^2-48ab-7b^2.$ 54. $8x^2+63xy-8y^2.$
55. $9x^2-82xy+9y^2$ 56. $10x^2+99xy-10y^2.$
57. $2(a+b)^2+3(a+b)-2$
58. $2(x^2+y^2)^2-3xy(x^2+y^2)-2x^2y^2.$
59. $2(a^2+b^2)^2+5ab(a^2+b^2)+2a^2b^2.$
60. $4(x^2-4xy+y^2)^2+15xy(x^2-4xy+y^2)-4x^2y^2.$

61. $2x^4-5x^2-12.$
62. $8a^4-14a^2b^2-9b^4.$
63. $9a^4+2a^2b^2-32b^4$
64. $8x^6-65x^3+8.$
65. $4a^8-17a^4b^4+4b^8.$

***8. Factors found by suitable arrangement and grouping of terms.**

There are some expressions of which the factors become obvious after re-arrangement of the terms in a certain way; but there are others again which do not exactly come under this category Hence, no definite method can be specified as applicable to all cases that may be practically included in this article We must, therefore, content ourselves only with directing the student's attention to a few important cases, more or less isolated, which will fairly introduce him to the subject under consideration

Example 1. Resolve into factors

$$(3x^2-4b^2)a+(3a^2-4x^2)b.$$

The given expression $= 3x^2a-4b^2a+3a^2b-4x^2b$

$= (3x^2a+3a^2b)-(4b^2a+4x^2b)$

[taking the 3rd term with the 1st and the 4th with the 2nd]

$= 3a(x^2+ab)-4b(ab+x^2)$

$= (x^2+ab)(3a-4b).$

Example 2. Resolve into factors $x^4+x^2y^2-y^2z^2-z^4.$

Combining the 4th term with the 1st, and the second with the 3rd, we have $x^4+x^2y^2-y^2z^2-z^4$

$= (x^4-z^4)+(x^2y^2-y^2z^2)$

$= (x^2+z^2)(x^2-z^2)+y^2(x^2-z^2)$

$= (x^2-z^2)\{(x^2+z^2)+y^2\}$

$= (x+z)(x-z)(x^2+y^2+z^2).$

Example 3. Resolve into factors $x^3+7x^2-21x-27.$

The given expression $= (x^3-27)+(7x^2-21x)$

$= (x-3)(x^2+3x+9)+7x(x-3)$

$= (x-3)\{(x^2+3x+9)+7x\}$

$= (x-3)(x^2+10x+9)$

$= (x-3)(x+9)(x+1).$

Example 4. Resolve into factors

$$4a^2+12ab+9b^2-8a-12b.$$

The given expression $=(4a^2+12ab+9b^2)-(8a+12b).$

$$=(2a+3b)^2-4(2a+3b)$$
$$=(2a+3b)\{(2a+3b)-4\}$$
$$=(2a+3b)(2a+3b-4).$$

Example 5. Resolve into factors

$$2a^2-2bc+6b^2+ac-7ab.$$

We observe that the 1st, 3rd and 5th terms are of the second degree in a and b, whilst the 2nd and the 4th terms are of the first degree in those letters

Putting the former set of terms in one group and the latter in another, we have the given expression

$$=(2a^2-7ab+6b^2)+c(a-2b)$$
$$=(a-2b)(2a-3b)+c(a-2b)$$
$$=(a-2b)(2a-3b+c)$$

Example 6. Resolve into factors

$$x^2-y^2-z^2+2yz+x+y-z$$

The given expression $=(x^2-y^2-z^2+2yz)+(x+y-z)$

$$=\{x^2-(y-z)^2\}+(x+y-z)$$
$$=(x+y-z)(x-y+z)+(x+y-z)$$
$$=(x+y-z)\{(x-y+z)+1\}$$
$$=(x+y-z)(x-y+z+1).$$

Example 7. Resolve into factors

$$a^2x^3+a^5-2abx^3+b^2x^3+a^3b^2-2a^4b$$

We observe that the 1st, 3rd and 4th terms have got x^3 for a common factor whilst the others have got a^3

Hence, putting the 1st, 3rd and 4th terms in one group and the remaining terms in another, we have the given expression

$$=(a^2a^3-2abx^3+b^2x^3)+(a^5+a^3b^2-2a^4b)$$
$$=x^3(a^2-2ab+b^2)+a^3(a^2+b^2-2ab)$$
$$=(a^2-2ab+b^2)(x^3+a^3)$$
$$=(a-b)^2(x+a)(x^2-xa+a^2)$$

Example 8. Resolve into factors

$$(a+b+c)(ab+bc+ac)-abc.$$

Arranging the expression within brackets according to powers of a, we have the given expression

$$= \{a+(b+c)\}\{a(b+c)+bc\}-abc$$
$$= a^2(b+c)+a(b+c)^2+bc(b+c)$$
$$= (b+c)\{a^2+a(b+c)+bc\}$$
$$= (b+c)(a+b)(a+c).$$

Example 9. Resolve into factors

$$a^3(b-c)+b^3(c-a)+c^3(a-b)$$

$a^3(b-c)+b^3(c-a)+c^3(a-b)$
$$= a^3(b-c)-a(b^3-c^3)+bc(b^2-c^2)$$
[arranged according to powers of a]
$$= (b-c)\{a^3-a(b^2+bc+c^2)+bc(b+c)\}$$
$$= (b-c)\{-b^2(a-c)-bc(a-c)+a(a^2-c^2)\}$$
[arranged according to powers of b]
$$= (b-c)(a-c)\{-b^2-bc+a(a+c)\}$$
$$= (b-c)(a-c)\{c(a-b)+(a^2-b^2)\}$$
[arranged according to powers of c]
$$= (b-c)(a-c)(a-b)(c+a+b)$$

Note—It must be observed that (i) as soon as the given expression is arranged according to powers of a, one of the factors, namely $b-c$, becomes obvious; (ii) when the expression within larger brackets is arranged according to powers of b, the next factor, $a-c$, becomes obvious; (iii) when the expression *now* within larger brackets is arranged according to powers of c, the third factor, $a-b$, becomes obvious. The order in which the letters a, b, c, occur in the given expression should also be noted; evidently we get the second term by changing a, b, c of the first respectively into b, c, a and the third term by changing b c a of the second respectively into c, a, b. The letters in the given expression are hence said to be arranged in ***cyclic order***.

Example 10. Resolve into factors

$$a^3(b^2-c^2)+b^3(c^2-a^2)+c^3(a^2-b^2).$$

In this expression also the letters occur in *cyclic order* and we can at once proceed as in the last example.

$a^3(b^2-c^2)+b^3(c^2-a^2)+c^3(a^2-b^2)$
$$= a^3(b^2-c^2)-a^2(b^3-c^3)+b^2c^2(b-c)$$
[arranged according to powers of a]

Example 3. Resolve into factors $x^3+6x^2+11x+6$

On inspection it is observed that we can split up the expression into parts each of which is divisible by $x+3$ in either of the two following ways —

$$\text{(i)}\quad x^2(x+3)+3x(x+3)+2(x+3);$$
$$\text{(ii)}\quad x(x^2+6x+9)+2(x+3).$$

Hence, choosing the latter way, we have

$$\begin{aligned} x^3+6x^2+11x+6 &= x(x+3)^2+2(x+3) \\ &= (x+3)\{x(x+3)+2\} \\ &= (x+3)(x^2+3x+2) \\ &= (x+3)(x+1)(x+2) \end{aligned}$$

Note Here also the student may observe that the given expression vanishes when -3, -1 or -2 is substituted for x

Example 4. Resolve into factors x^3-2x^2-5x+6.

On inspection we find that the given expression can be split up into parts each of which is divisible by $x-1$ in either of the two following ways —

$$\text{(i)}\quad x(x^2-2x+1)-6(x-1),$$
$$\text{(ii)}\quad x^2(x-1)-x(x-1)-6(x-1).$$

Hence, choosing the former way, we have

$$\begin{aligned} x^3-2x^2-5x+6 &= x(x-1)^2-6(x-1) \\ &= (x-1)\{x(x-1)-6\} \\ &= (x-1)(x^2-x-6) \\ &= (x-1)(x+2)(x-3). \end{aligned}$$

Note Here also it must be noticed that the given expression vanishes when $1, -2,$ or 3 is substituted for x Thus, the student may well remember it as a general rule that if any expression involving x vanishes when $x=a$, $x-a$ is a factor of that expression We shall apply this principle in the next example

Example 5. Resolve into factors $x^3+x^2-21x-38$

By trial we find that the given expression vanishes when $x=-2$ Hence $x+2$ is a factor.

Thus, we have $x^3+x^2-21x-38$

$$\begin{aligned} &= x^2(x+2)-x(x+2)-19(x+2) \\ &= (x+2)(x^2-x-19). \end{aligned}$$

Example 6. Resolve into factors $8x^3+16x-9$.

We find that the given expression can be split up into parts each of which is divisible by $2x-1$ in either of the two following ways :—

(i) $(8x^3-1)+8(2x-1)$;

(ii) $2x(4x^2-1)+9(2x-1)$.

Hence, choosing the former way, we have

$$\begin{aligned} 8x^3+16x-9 &= (8x^3-1)+8(2x-1) \\ &= (2x-1)\{(4x^2+2x+1)+8\} \\ &= (2x-1)(4x^2+2x+9). \end{aligned}$$

Example 7. Resolve into factors $a^3+7ab^2-22b^3$.

We find that the expression can be split up into parts each of which is divisible by $a-2b$ in either of the two following ways —

(i) $(a^3-8b^3)+7b^2(a-2b)$,

(ii) $a(a^2-4b^2)+11b^2(a-2b)$.

Hence, choosing the former way, we have

$$\begin{aligned} a^3+7ab^2-22b^3 &= (a^3-8b^3)+7b^2(a-2b) \,; \\ &= (a-2b)\{(a^2+2ab+4b^2)+7b^2\} \\ &= (a-2b)(a^2+2ab+11b^2). \end{aligned}$$

Example 8. Resolve into factors

$$x^2+2(a^2+b^2)+3ax-b(3x+5a).$$

Arranging the expression according to descending powers of x we have it

$$\begin{aligned} &= x^2+3(a-b)x+(2a^2-5ab+2b^2) \\ &= x^2+3(a-b)x+(2a-b)(a-2b) \\ &= x^2+\{(2a-b)+(a-2b)\}x+(2a-b)(a-2b) \\ &= \{x+(2a-b)\}\{x+(a-2b)\} \\ &= (x+2a-b)(x+a-2b). \end{aligned}$$

Example 9. Resolve into factors $x^2-6xy+8y^2-z^2+2yz$.

The given expression

$$\begin{aligned} &= (x^2-6xy+9y^2)-(y^2+z^2-2yz) \\ &= (x-3y)^2-(y-z)^2 \\ &= \{(x-3y)+(y-z)\}\{(x-3y)-(y-z)\} \\ &= (x-2y-z)(x-4y+z). \end{aligned}$$

Example 10. Resolve into factors

$$(a^2-b^2)(x^2-y^2)+4abxy$$

The given expression

$$\begin{aligned}
&= a^2x^2-a^2y^2-b^2x^2+b^2y^2+4abxy\\
&= (a^2x^2+b^2y^2+2abxy)-(a^2y^2+b^2x^2-2abxy)\\
&= (ax+by)^2-(ay-bx)^2\\
&= \{(ax+by)+(ay-bx)\}\{(ax+by)-(ay-bx)\}\\
&= \{(a-b)x+(a+b)y\}\{(a+b)x-(a-b)y\}
\end{aligned}$$

Example 11 Resolve into factors

$$x^4+6x^3+4x^2-15x+6$$

The given expression

$$\begin{aligned}
&= (x^4+6x^3+9x^2)-(5x^2+15x)+6\\
&= (x^2+3x)^2-5(x^2+3x)+6\\
&= \{(x^2+3x)-2\}\{(x^2+3x)-3\}\\
&= (x^2+3x-2)(x^2+3x-3)
\end{aligned}$$

Example 12. Resolve into factors

$$x^4+2x^3y+3x^2y^2+2xy^3+y^4.$$

The given expression

$$\begin{aligned}
&= (x^4+2x^2y^2+y^4)+x^2y^2+(2x^3y+2xy^3)\\
&= (x^2+y^2)^2+(xy)^2+2(xy)(x^2+y^2)\\
&= \{(x^2+y^2)+xy\}^2\\
&= (x^2+xy+y^2)^2
\end{aligned}$$

Example 13. Resolve into factors

$$(x-1)(x-2)(x+3)(x+4)+4.$$

$$\begin{aligned}
(x-1)(x-2)(x+3)(x+4)\\
&= \{(x-1)(x+3)\}\{(x-2)(x+4)\}\\
&= (x^2+2x-3)(x^2+2x-8)
\end{aligned}$$

Hence, putting z for x^2+2x, the given expression

$$\begin{aligned}
&= (z-3)(z-8)+4\\
&= z^2-11z+28\\
&= (z-4)(z-7)\\
&= (x^2+2x-4)(x^2+2x-7)
\end{aligned}$$

Note The student must carefully notice why in multiplying together the four binomials $x-1$, $x-2$, $x+3$, $x+4$, we combine $x+3$ with $x-1$, and $x+4$ with $x-2$

Example 14. If $x+y=a$ and $xy=b^2$, find the value of (i) x^4+y^4, and (ii) $x^3-x^2y-xy^2+y^3$ in terms of a and b.

$$\begin{aligned}
\text{(i)}\quad x^4+y^4 &= (x^2+y^2)^2-2x^2y^2\\
&= \{(x+y)^2-2xy\}^2-2x^2y^2,\\
\text{and } \therefore \text{ the required value} &= (a^2-2b^2)^2-2b^4\\
&= a^4-4a^2b^2+2b^4\\
\text{(ii)}\quad x^3-x^2y-xy^2+y^3 &= x^2(x-y)-y^2(x-y)\\
&= (x-y)(x^2-y^2)\\
&= (x-y)^2(x+y)\\
&= \{(x+y)^2-4xy\}(x+y)\\
&= (a^2-4b^2)a.
\end{aligned}$$

Example 15. Find the value of $x^4-x^3+x^2+2$, when $x^2+2=2x$.

$$\begin{aligned}
x^4-x^3+x^2+2 &= (x^4+x^3+x^2)-2(x^3-1)\\
&= x^2(x^2+x+1)-2(x-1)(x^2+x+1)\\
&= (x^2+x+1)\{x^2-2(x-1)\}\\
&= (x^2+x+1)(x^2-2x+2),\\
\text{and } \therefore \text{ the required value} &= (x^2+x+1)\times 0\\
&= 0
\end{aligned}$$

Example 16. Find the value of

$$a^4+b^4+c^4-2b^2c^2-2c^2a^2-2a^2b^2, \text{ when } a+b=c.$$

The given expression

$$\begin{aligned}
&= a^4-2(b^2+c^2)a^2+b^4+c^4-2b^2c^2\\
&= \{a^4-2(b^2+c^2)a^2+(b^2+c^2)^2\}\\
&\qquad\qquad -(b^2+c^2)^2+b^4+c^4-2b^2c^2\\
&= \{a^2-(b^2+c^2)\}^2-4b^2c^2\\
&= \{(a^2-b^2-c^2)+2bc\}\{(a^2-b^2-c^2)-2bc\}\\
&= \{a^2-(b-c)^2\}\{a^2-(b+c)^2\}\\
&= (a+b-c)(a-b+c)(a+b+c)(a-b-c),
\end{aligned}$$

and $\therefore$ $= 0$, when $a+b=c$.

Exercise (47).

Find the quotient of —

1. $a^3+b^3-c^3+3abc$ by $a+b-c$
2. $x^3-y^3-1-3xy$ by $x-y-1$
3. $x^3-8y^3+27z^3+18xyz$ by $x-2y+3z$.
4. $8a^3-27b^3-c^3-18abc$ by $4a^2+9b^2+c^2+6ab+2ac-3bc$.
5. x^6-26x^3-27 by $(x^2+3x+9)(x^2-x+1)$.
6. $(a^2-bc)^3+27b^3c^3$ by $a^4-5a^2bc+13b^2c^2$.
7. $125-8a^3+b^3+30ab$ by $(a+b)^2+(a+5)^2+2a^2-5b$.
8. $2x^3-x^2+x+4$ by $x+1$.
9. $x^4-4x^3+7x^2-11x+10$ by $x-2$
10 $x^4-3x^3-13x^2+13x-6$ by $x+3$.
11. $x^3+4x-39$ by $x-3$
12. $a^3-4a^2b+24b^3$ by $a+2b$
13. $a^4-5a^3b+8a^2b^2-7ab^3+3b^4$ by $a-3b$.
14. $x^4+x^3-10x^2+7x-4$ by $y+4$.
15. $x^3+3xy^2-76y^3$ by $x-4y$

Resolve into factors .—

16. $x^3+8x^2+19x+12$
17. $x^3+9x^2+26x+24$.
18. $x^3-6x^2+11x-6$.
19. $x^3+5x^2-2x-24$
20. x^3-4x^2+x+2
21. x^3+5x^2-2x-6.
22. $x^3-6x^2+13x-10$.
23. $x^4-3x^3-9x^2+12x+20$.
24. $x^4-3x^3-x^2+13x-10$
25. $x^4-5x^3+x^2+13x+6$
26. $x^4+5x^3-8x^2-30x+36$
27. $x^4-7x^3+9x^2+26x-56$.
28. $x^3-7x^2+13x-15$.
29. $x^3-5x+12$.
30. x^3-6x^2+32
31. $2x^3-3x^2-4$.
32. $x^3-9xy^2-10y^3$
33 $a^3+4a^2b-9b^3$
34. $5a^3-8a^2b-28b^3$.
35. $8x^3+4x-3$
36. $2x^3+5x^2-4x-3$
37. x^3-3x-2.
38. $2a^3-a^2b-b^3$
39. $3x^3+8x^2-8x-3$
40. $x^3-6xy^2+9y^3$
41. $x^2+bx-(a^2-3ab+2b^2)$.
42. $x^4+4abx^2y^2-(a^2-b^2)^2y^4$.

43. $a^4+2(x^2+y^2)a^2b^2+(x^2-y^2)^2b^4$.
44. $a^2+(x+y)a-2x^2+5xy-2y^2$.
45. $x(x+a)-2a^2+3b(a+x)+2b^2$.
46. $x^2+4xy+3y^2+2yz-z^2$.
47. $4a^2-4ab-3b^2+12bc-9c^2$.
48. $x^4+6x^3+8x^2+6x-9$
49. $a^4-4a^3b-5a^2b^2+6ab^3-b^4$.
50. $4x^4-20x^3+24x^2+6x-9$.
51. $x^4-2x^3+2x^2-2x+1$. 52. $a^4-9a^2+30a-25$.
53. $a^3-2abx-(ac-b^2)x^2+bcx^3$.
54. $x^4y^4+x^2y^2-z^2+2xyz+1$.
55. $x^2(y^2-z^2)+4xyz-y^2+z^2$
56. $(a^2-b^2)(x^2+y^2)+2(a^2+b^2)\,xy$.
57. $x^4-4x^3-x^2+10x+4$ 58. $a^4-6a^3+15a^2-18a+5$.
59. $4x^4+12x^3-5x^2-21x+12$
60. $x^4-5x^3y+6x^2y^2-5xy^3+y^4$.
61. $x^4-5x^3+14x^2-20x+16$
62. $a^4-7a^3b+14a^2b^2-14ab^3+4b^4$.
63. $x^4+4x^3-11x^2+20x+25$.
64. $a^4+4a^3b-10a^2b^2+4ab^3+b^4$.
65. $x^4+8x^3+24x^2+32x-20$
66. $(x+1)(x+3)(x-4)(x-6)+13$.
67. $(x+2)(x+3)(x+4)(x+5)-360$
68 $x(2x+1)(x-2)(2x-3)-63$.

69. Find the value of $xy(x+y)+yz(y+z)+zx(z+x)+3xyz$,
when $x=a(b-c)$, $y=b(c-a)$, $z=c(a-b)$.

70. Find the value of
$$(y-z)(y^2-z^2)-x\{(y-z)^2+x(y+z)\}+x^3,$$
when $x=a^2-b^2$, $y=b^2-c^2$, $z=a^2-c^2$.

71. Find the value of $x^2-2(y-2)x-3y^2+20y-32$,
when $x+y=4$

72. Find the value of $x^2-y^2+4x+14y-45$,
when $x+y=25$ and $x-y=6$

73. Find the value of $x^6+y^6-z^6+3x^2y^2z^2$,
when $x = a^2-b^2$, $y = 2ab$, $z = a^2+b^2$.

74. Find the value of $8xy(x^2+y^2)$, when $\left.\begin{array}{r} x+y = \sqrt{3} \\ \text{and } x-y = \sqrt{2} \end{array}\right\}$

75. Find the value of x^5+y^5 in terms of a and b,
when $x+y = a$ and $xy = b^2$.

76. Find the value of $x^3+y^3+z^3-3xyz$,
when $x = 658$, $y = 668$ and $z = 674$

CHAPTER IX

HIGHEST COMMON FACTORS.

1. Definitions. A *common factor* of two or more algebraical expressions is an expression which divides each of them without a remainder

N B By *expressions* we shall mean *rational* and *integral expressions* only [See note, Art 1, Chap VIII.]

An *elementary* common factor is one which cannot itself be resolved into factors

The product of *all* the *elementary* common factors of two or more expressions is called their *Highest Common Factor*, or, in other words, the Highest Common Factor of two or more expressions is that common factor which is resolvable into the *greatest* number of elementary factors

Thus, since $6a^2b(x^2-1) = 2\times3\times a\times a\times b\times(x+1)\times(x-1)$, and $15ab^2(x^2-3x+2) = 3\times5\times a\times b\times b\times(x-1)\times(x-2)$, the elementary common factors of the two expressions on the left are 3, a, b and $x-1$, hence their H. C. F. $= 3ab(x-1)$

Note 1 Other common factors of the given expressions are $3a$, $b(x-1)$, ab, $3(x-1)$, $3ab$, &c, but none of them is elementary

Note 2 When the expressions considered have no *numerical* common factor, it is easy to comprehend that the Highest Common Factor is an expression of a higher degree than any other common factor Hence, when two or more expressions have no numerical factor common, their Highest Common Factor may be defined to be the expression of the highest degree by which each of them is divisible without a remainder

Note 3 If any expression A divides any other expression B without a remainder, then A is evidently the H C F of A and B

Note 4 If H be the H C F of any number of quantities A, B, C, &c, then the quotient of A, B, C, &c, by H have no common factor

Note 5 If an elementary factor occurs more than once in each of two or more given expressions, then the *highest* power of this factor *common* to the given expressions, and no higher power, *must* occur as a factor in the H C F of these expressions

Note 6 If $A = p \times q$, and $B = p' \times q'$, such that q and q' have no common factor, then the H C F of A and B, if any, will be the same as the H C F of p and p'

Note 7 If $A = m \times n$, and $B = m' \times n'$ where m and m' respectively include *all* the monomial factors of A and B, then the H C F of A and B = (the H. C F of m and m')×(the H. C F of n and n')

Note 8 The H C F of A and B is the same as the H C F of A and mB, if m is not a factor of A

2. Highest Common Factors of simple expressions. Such expressions can be at once resolved into their elementary factors, and so their is no difficulty in finding the H C F of any number of them

Example 1 Find the H. C F of $a^2b^4c^5$, $a^4b^3c^7$ and $a^3b^5c^4$.

The elementary common factors are a, b and c, and the *highest* powers of them *common* to the given expressions are respectivly a^2, b^3 and c^4.

Hence, the H C F required $= a^2b^3c^4$

Example 2. Find the H C F of $24ab^2x^3y^4$, $36a^2x^4z^5$ and $240b^3x^6y^2z$

$$\begin{aligned} \text{We have}\quad 24ab^2x^3y^4 &= 3\times2^3\times ab^2x^3y^4, \\ 36a^2x^4z^5 &= 2^2\times3^2\times a^2x^4z^5, \\ 240b^3x^6y^2z &= 3\times5\times2^4\times b^3x^6y^2z \end{aligned}$$

Evidently then the elementary common factors are 3, 2, and x, and the highest powers of them common to the given expressions are respectively 3, 2^2 and x^3.

Hence, the H. C F. required $= 3\times2^2\times x^3 = 12x^3$.

Note. After exhibiting each expression as a product of powers of different elementary factors, the elementary factors *common* to the given expressions are at once obtained by writing down in succession such of the elementary factors of the first expression as are also found in *every* one of the remaining expressions Thus in the above example the elementary factors of the first expression are 3, 2, a, b, x and y, of which 3, 2 and a only are to be found in *each* of the others

Exercise (48)

Find the H C F of —

1. a^2b^3 and a^3b^2
2. $12a^3b$ and $20a^2c^3$.
3. $9xy^2z^3$ and $24x^3y^4$
4. $20a^3x^4y^5$ and $75a^2y^3$.
5. $18m^2n^4$ and $45m^5n^3$
6. $16a^3x^4y$, $40a^2y^3x$ and $28x^3a$.
7. $24m^2np^5$, $60mn^2p$ and $84m^3p^2$.
8. $45x^3y^2z^4$, $75x^2y^4z^3$ and $90x^4y^3z^2$
9. $36a^2b^3c^4x^5$, $54a^5c^2x^4$ and $90a^4b^3c^5$
10. $72a^3b^4c^5$, $96b^3c^4d^5$ and $120c^3d^4a^5$.
11. $48a^5x^4y^3z^2$, $60x^5y^4z^3b^2$, $72y^5z^4b^3a^2$ and $84z^5b^4a^3x^2$
12. $75m^4n^3p^5q^6$, $90m^3n^5p^6q^4$, $105m^6n^4p^3q^5$ and $135m^5n^6p^4q^3$.
13. $54a^2b^5c^3d^4$, $72a^5b^2c^4d^3$, $108a^3b^4c^5d^2$ and $126a^4b^3c^2d^5$
14. $18a^3x^4y^5$, $42a^4y^3z^4$, $60x^3y^4z^5$ and $78a^2x^4z^3$.
15. $32a^2b^3x^4y^5z^6$, $40a^3x^5y^4z^8$, $56b^3x^2y^7z^4$, $72x^6a^5y^2z^3$ and $96b^4a^8x^3y^3$.

4. Highest Common Factors of compound expressions whose elementary factors can be easily found.

The method illustrated in the last article will also evidently apply in such cases

Example 1. Find the H C F of

$$a^3b^2+2a^2b^3 \text{ and } a^5b-4a^3b^3.$$

$$a^3b^2+2a^2b^3 = a^2b^2(a+2b)$$

and $$a^5b-4a^3b^3 = a^3b(a^2-4b^2)$$
$$= a^3b(a+2b)(a-2b).$$

Hence, the required H C. F $= a^2b(a+2b)$.

Example 2. Find the H C F. of

$$x^4y^2+xy^5 \text{ and } x^4y+2x^3y^2+x^2y^3.$$

$$x^4y^2+xy^5 = xy^2(x^3+y^3)$$
$$= xy^2(x+y)(x^2-xy+y^2).$$

and $\quad x^4y+2x^3y^2+x^2y^3 = x^2y(x^2+2xy+y^2)$
$$= x^2y(x+y)^2.$$

Hence, the required H. C F. $= xy(x+y)$

Example 3. Find the H. C. F. of

$$24(x^4-2ax^3-8a^2x^2) \text{ and } 54(x^5-ax^4-6a^2x^3).$$

The first expression $= 3\times 8 \times x^2(x^2-2ax-8a^2)$
$= 3\times 2^3\times x^2(x+2a)(x-4a).$

The second expression $= 6\times 9 \times x^3(x^2-ax-6a^2)$
$= 2\times 3^3\times x^3(x+2a)(x-3a).$

Hence, the required H C F $= 3\times 2 \times x^2(x+2a)$
$= 6x^2(x+2a).$

Example 4. Find the H. C. F. of

$$a^4-16x^4 \text{ and } a^3+a^2x-13ax^2+8x^3.$$

The first expression $= (a^2+4x^2)(a^2-4x^2)$
$= (a^2+4x^2)(a+2x)(a-2x).$

The second expression $= (a-2x)(a^2+3ax-4x^2)$
$= (a-2x)(a-x)(a+4x)$

Hence, the required H C F. $= a-2x$

Note It may be observed in this example that although the factors of the second expression are not so obvious as those of the first, still there is no great difficulty in discovering them as it may be presumed that the given expression have at least one common factor Hence, after the resolution of the first expression into factors, by a little trial it may be seen that of these $a-2x$ is also a factor of the second expression, thus the factorisation of the 2nd expression is much facilitated

Exercise (49).

Find the H. C F of —

1. a^3-ab^2 and $a^4+2a^3b+a^2b^2$.
2. $x^5y^3-x^3y^5$ and $x^5y^4+x^4y^5$.
3. $6(x^2-9)$ and $15(x^3+27)$.

4. $12(a^6-a^2b^2c^2)$ and $20(a^4b^2c^2+a^2b^3c^3)$
5. $m^6n^2-2m^5n^3+m^4n^4$ and $(m^2n-mn^2)^3$.
6. $4a^4x-9a^2x^3$ and $4a^2x^2+6ax^3$
7. $18a^4b^3-32a^2b^5$ and $18a^4b^2+24a^3b^3$
8. $9x^4y^4-36x^2y^6$ and $24x^4y^2-48x^3y^3$
9. $6a^3b^2-24ab^4$ and $4a^5b+32a^2b^4$
10 $48x^2a^2(x+a)^2(x^2a^2-xa^3)$ and $64(x^5a^2-x^2a^5)(x^3a+x^2a^2)$.
11. $24(x^3-a^3)$ and $40(x^4+x^2a^2+a^4)$
12. $56(x^6a^2-x^2a^6)$ and $72(x^5a^3+3a^5x^3+2a^7)x$.
13. $30(a^2+4ab+3b^2)$ and $42(a^2+ab-6b^2)$
14. $28(x^3-3x^2-10x)$ and $52(x^4-8x^3+15x^2)$
15. $x^4y+3x^3y^2-18x^2y^3$ and $x^3y^2+10x^2y^3+24xy^4$
16. $a^4x^3-4a^3x^4-12a^2x^5$ and $a^5x^2+8a^4x^3+12a^3x^4$
17. $4x^3+12x^2+9x$ and $4x^2-2x-12$
18. $a^2-ab-2b^2$ and $a^3-a^2b-4ab^2+4b^3$.
19. $x^2+3x-10$ and $x^3-x^2-14x+24$
20. $54(x^3+8a^3)$ and $90(x^3+7ax^2+16a^2x+12a^3)$.
21. $(a^3-b^3)(a+b)^2$, a^4-b^4 and $3a^4+2a^3b-5a^2b^2$.
22. $(2x-3)^2(x^2+x-2)$, $4x^2-x-18$ and $2x^3-23x-54$.
23. $8(27a^5b+a^2b^4)$, $12(6a^4b^2-7a^3b^3-3a^2b^4)$ and $40(3a^3b^2+13a^2b^3+4ab^4)$.
24. x^4-13x^2+36, $3x^3+13x^2+8x-12$ and $4x^3+17x^2+9x-18$.

4 The ordinary method of finding the H. C. F. of two multinomial expressions which have no monomial factors

Let A and B stand for two such expressions both arranged according to descending powers of some common letter* and let the index of the highest power of that letter in A be not less than the index of the highest power of that letter in B

* This letter is called the *letter of reference*

Divide A by B, and let Q be the quotient and C the remainder

Then we must have $C = A - BQ$. (1)
or $A = BQ + C$ (2)

From (1) it is clear that *every* common factor of A and B is a factor of C [for if $A = pa$ and $B = pb$, we have $C = p(a - bQ)$] Hence, if H denote the H C F of A and B, H also is a factor of C, and is therefore a *common factor* of B and C.

It is clear therefore that the H C F. of B and C is either H or an expression of higher dimensions than H (*a*)

Now from (2) it is evident that *every* common factor of B and C is a factor of A and is therefore a common factor of B and A Hence the H C F of B and C also is a common factor of B and A and therefore cannot be of higher dimensions than H.

Hence from (*a*), the H. C F of B and C is H.

Thus the H C F. of B and C is the H C F required.

Similarly, if B be divided by C, and D be the new remainder, the H C F of C and D is the same as the H C F of B and C and is therefore the H C F required.

Now divide C by D and let there be no remainder. Then D is the H C F of C and D and is therefore the H C F required

Cor. 1. As the H C F of any divisor and the corresponding dividend is the H C F required, it is clear that, for the sake of convenience, either of them may be multiplied or divided by any monomial expression *which is not a factor of the other.* [See note 8, Art. 1]

Cor. 2. In dividing A by B if we stop before the complete* quotient is obtained so that q is the partial quotient and C′ the corresponding remainder, then the H C F of B and C′ just as the H C F of B and C is the H C F required Hence, by Cor 1, in dividing C′ by B (or if convenient, B by C′ when C′ is *not* of higher degree then B) we can multiply or divide either of them, if necessary, by any monomial expression *which is not a factor of the other.*

---

* The quotient obtained is said to be complete when the remainder is of lower degree in the letter of reference than the di

Hence, we have the following rule —

Arrange the two expressions according to descending powers of some common letter, divide the expression which is of higher degree in that letter by the other, or if they be of the same degree, either of them by the other, if there be any remainder, take it for a new divisor and the preceding divisor for the dividend, and continue the process till there is no remainder The last divisor will be the H C F. required Of any divisor and the corresponding dividend either may be multiplied or divided by any number which is not a factor of the other

Example 1. Find the H C F of $3x^3-7x^2-18x-8$ and $2x^3-3x^2-17x-12$

The H C F required is evidently the H C F of $3x^3-7x^2-18x-8$ and $3(2x^3-3x^2-17x-12)$ [Cor 1]. Let us therefore multiply the 2nd expression by 3 and divide the product by the 1st,

$$
\begin{array}{r|l|l}
 & 2x^3-3x^2-17x-12 & \\
 & 3 & \\
\hline
3x^3-7x^2-18x-8 & 6x^3-9x^2-51x-36 & 2 \\
 & 6x^3-14x^2-36x-16 & \\
\hline
 & \phantom{6x^3-{}}5x^2-15x-20 &
\end{array}
$$

Hence, the H C F required is the H. C. F. of $3x^3-7x^2-18x-8$ and $5x^2-15x-20$ [$= 5(x^2-3x-4)$] and is therefore the H C F of $3x^3-7x^2-18x-8$ and x^2-3x-4. [Cor 1] We must proceed then as follows —

$$
5\,\big)\,5x^2-15x-20
$$

$$
\begin{array}{r|l|l}
x^2-3x-4 & 3x^3-7x^2-18x-8 & 3x+2 \\
 & 3x^3-9x^2-12x & \\
\hline
 & \phantom{3x^3-{}}2x^2-6x-8 & \\
 & \phantom{3x^3-{}}2x^2-6x-8 & \\
\hline
\end{array}
$$

Hence, the H C F required $= x^2-3x-4$

Example 2. Find the H C F of

$22x^6-78x^5-16x^2$ and $2x^5-78x^2-44x$.

The 1st expression $= 2x^2(11x^4-39x^3-8)$

The 2nd expression $= 2x(x^4-39x-22)$

Hence, by note 7, Art 1, the H C. F required
$= $ (the H C F. of $2x^2$ and $2x$) $\times$ (the H. C. F of
$11x^4-39x^3-8$ and $x^4-39x-22$)
$= 2x\times X$, putting X for the H C. F. of the multinomials
Now let us find X, as in the last example —

$$
\begin{array}{r|l|l}
x^4-39x-22 & 11x^4-39x^3-8 & 11 \\
 & 11x^4-429x-242 & \\
\hline
 & -3)-39x^3+429x+234 & \\
\hline
 & 13)13x^3-143x-78 & \\
\hline
 & x^3-11x-6 &
\end{array}
$$

$$
\begin{array}{r|l|l}
x^3-11x-6 & x^4-39x-22 & x \\
 & x^4-11x^2-6x & \\
\hline
 & 11)11x^2-33x-22 & \\
\hline
 & x^2-3x-2 &
\end{array}
$$

$$
\begin{array}{r|l|l}
x^2-3x-2 & x^3-11x-6 & x+3 \\
 & x^3-3x^2-2x & \\
\hline
 & 3x^2-9x-6 & \\
 & 3x^2-9x-6 & \\
\hline
\end{array}
$$

Thus $X = x^2-3x-2$

Hence, the H C F required $= 2x(x^2-3x-2)$.

Example 3. Find the H C F of
$12x^4a^2+54x^3a^3+6x^2a^4-72xa^5$ and
$8x^6a+60x^5a^2+160x^4a^3+180x^3a^4+72x^2a^5$.

The first expression $= 6xa^2(2x^3+9x^2a+xa^2-12a^3)$

The 2nd expression $= 4x^2a(2x^4+15x^3a+40x^2a^2+45xa^3+18a^4)$.

Hence, if X denote the H. C F of the multinomial factors of the given expressions, we must have the required H. C. F. $=$ (the H C. F. of $6xa^2$ and $4x^2a$) $\times X = 2xa\times X$.

Now to find X.

$$
\begin{array}{r|l}
2x^3+9x^2a+xa^2-12a^3 & 2x^4+15x^3a+40x^2a^2+45xa^3+18a^4\,(x \\
 & 2x^4+\ \ 9x^3a+\ \ \ x^2a^2-12xa^3 \\
\hline
 & 3a)\,6x^3a+39x^2a^2+57xa^3+18a^4 \\
\hline
 & \quad 2x^3+13x^2a+19xa^2+\ 6a^3
\end{array}
$$

Hence, X is the H. C F of $2x^3+9x^2a+xa^2-12a^3$ and $2x^3+13x^2a+19xa^2+6a^3$, and as they are both of the same degree we can divide either of them by the other

$$
\begin{array}{r|l}
2x^3+9x^2a+xa^2-12a^3 & 2x^3+13x^2a+19xa^2+\ 6a^3\,(1 \\
 & 2x^3+\ \ 9x^2a+\ \ \ xa^2-12a^3 \\
\hline
 & 2a)\,4x^2a+18xa^2+18a^3 \\
\hline
 & \quad 2x^2\ +\ 9xa\ +\ 9a^2
\end{array}
$$

$$
\begin{array}{r|l}
2x^2+9xa+9a^2 & 2x^3+9x^2a+\ \ xa^2-12a^3\,(x \\
 & 2x^3+9x^2a+\ 9xa^2 \\
\hline
 & -4a^2)-8xa^2-12a^3 \\
\hline
 & \qquad 2x\ \ +\ \ 3a
\end{array}
$$

$$
\begin{array}{r|l}
2x+3a & 2x^2+9xa+9a^2\,(x+3a \\
 & 2x^2+3xa \\
\hline
 & \quad 6xa+9a^2 \\
 & \quad 6xa+9a^2 \\
\hline
\end{array}
$$

Thus $X=2x+3a$.

Hence, the H. C F required $=2xa(2x+3a)$.

Example 4. Find the H. C. F. of $4x^4+11x^3+27x^2+17x+5$ and $6x^4+14x^3+36x^2+14x+10$

The second expression $=2(3x^4+7x^3+18x^2+7x+5)$, but 2 is not a factor of the 1st expression. Hence, the H C. F. required is the H C. F. of the 1st expression and $3x^4+7x^3+18x^2+7x+5$.

$$
\begin{array}{r}
4x^4+11x^3+27x^2+17x+5 \\
3 \\
\hline
3x^4+7x^3+18x^2+7x+5\,\big)\,12x^4+33x^3+81x^2+51x+15\,\big(4 \\
12x^4+28x^3+72x^2+28x+20 \\
\hline
5x^3+\ 9x^2+23x-5
\end{array}
$$

$$
\begin{array}{r}
3x^4+\ 7x^3+18x^2+\ 7x+\ 5 \\
5 \\
\hline
5x^3+9x^2+23x-5\,\big)\,15x^4+35x^3+90x^2+35x+25\,\big)\,3x \\
15x^4+27x^3+69x^2-15x \qquad \\
\hline
8x^3+21x^2+50x+25 \\
5 \\
\hline
40x^3+105x^2+250x+125\,\big(8 \\
40x^3+\ 72x^2+184x-\ 40 \\
\hline
33\,\big)\,33x^2+\ 66x+165 \\
\hline
x^2+\ 2x+5
\end{array}
$$

$$
\begin{array}{r}
x^2+2x+5\,\big)\,5x^3+\ 9x^2+23x-5\,\big(5x-1 \\
5x^3+10x^2+25x \qquad \\
\hline
-x^2-\ 2x-5 \\
-x^2-\ 2x-5 \\
\hline
\end{array}
$$

Thus the required H. C F. $= x^2+2x+5$

Example 5 Find the H C F. of

$$4x^4-16x^3+108 \text{ and } 6x^5-14x^3-40x^2+36.$$

The first expression $= 4(x^4-4x^3+27)$.

The second expression $= 2(3x^5-7x^3-20x^2+18)$.

Hence, if X denote the H C. F of the multinomial factors of the given expressions, the H C F required $= 2X$.

Let us then find X.

$$
\begin{array}{r}
x^4-4x^3+27\,\big)\,3x^5-\ 7x^3-20x^2+18\,\big(3x+12 \\
3x^5-12x^4+81x \qquad\qquad \\
\hline
12x^4-\ 7x^3-20x^2-81x+\ 18 \\
12x^4-48x^3 \qquad\qquad\qquad +324 \\
\hline
41x^3-20x^2-81x-306
\end{array}
$$

$$
\begin{array}{r}
x^4-4x^3+27 \\
41 \\
\hline
41x^3-20x^2-81x-306\,)\,41x^4-164x^3+1107\,(\,x \\
41x^4-\ 20x^3-81x^2-306x \\
\hline
-9\,)\,-144x^3+81x^2+306x+1107 \\
\hline
16x^3-9x^2-34x-\ 123 \\
41 \\
\hline
656x^3-369x^2-1394x-5043\,(\,16 \\
656x^3-320x^2-1296x-4896 \\
\hline
-49\,)\,-\ 49x^2-\ \ 98x-147 \\
\hline
x^2+\ \ 2x+\ \ 3
\end{array}
$$

$$
\begin{array}{r}
x^2+2x+3\,)\,41x^3-\ 20x^2-\ 81x-306\,(\,41x-102 \\
41x^3+\ 82x^2+123x \\
\hline
-102x^2-204x-306 \\
-102x^2-204x-306 \\
\hline
\end{array}
$$

Hence, the H C F. required $= 2(x^2+2x+3)$.

Exercise (50).

Find the H. C. F of —

1. $2x^2+5x-3$ and $2x^3+3x^2-32x+15$
2. $3x^2+16x-12$ and $3x^3+4x^2-28x+16$
3. $2x^2-3ax-20a^2$ and $2x^3+3ax^2-45a^2x-100a^3$.
4. $3x^4+7x^3-14x^2-24x$ and $6x^4-10x^3-24x^2$.
5. $6a^3-11a^2-3a+2$ and $3a^3+20a^2+23a-10$
6. $6a^3-25a^2b+32ab^2-12b^3$ and $4a^3+12a^2b-7ab^2-30b^3$.
7. $3x^3+5x^2+5x+2$ and $2x^3+5x^2+5x+3$.
8. $4x^3-7x^2y+7xy^2-3y^3$ and $3x^3-7x^2y+7xy^2-4y^3$
9. $6x^4+7x^3+5x^2+2x$ and $4x^5-18x^4-8x^3-10x^2$.
10. $3x^4+10x^3+7x^2+4x+1$ and $2x^3+3x^2-7x-3$.

11. $4x^3+13x^2-8x-3$ and $3x^4+13x^3+9x^2+9x+2$.

12. $12a^3+11a^2x+6ax^2+x^3$ and $21a^3+17a^2x+9ax^2+x^3$.

13. $35a^3+31a^2x+13ax^2+2x^3$ and $65a^3+54a^2x+22ax^2+3x^3$.

14. $70x^3-9ax^2+11a^2x+6a^3$ and $91x^3-25ax^2+20a^2x+4a^3$.

15. $75x^3-35x^2+24x+4$ and $85x^3-36x^2+25x+6$.

16. $35x^3-34x^2+3x+2$ and $49x^3-49x^2+5x+3$.

17. $4x^6+2ax^5+14a^2x^4+10a^3x^3+24a^4x^2$ and $6x^6+21ax^5+30a^2x^4+24a^3x^3$.

18. $4a^4+32a^3+72a^2+44a+8$ and $6a^4+54a^3+138a^2+78a+12$.

19. $2x^4-19x^2+21x-6$ and $6x^4+21x^3+3x-6$

20. $12x^4-30x^2+126x+90$ and $15x^4-25x^3+145x-75$.

21. $18x^4+117x^3+162x^2+72x+9$ and $12x^4+68x^3+72x^2+108x+20$.

22. $x^5-5x^2+6x+12$ and $x^4-8x^2-24x-32$.

23. $x^4+5x^3+3x^2-14x-40$ and $x^5-4x^3+45x+75$.

24. $4x^5-8x^3a^2+28x^2a^3-24xa^4+24a^5$ and $6x^4+24x^3a-12x^2a^2-24xa^3+96a^4$.

25. $9x^4-18x^3y-13x^2y^2-38xy^3-12y^4$ and $6x^5+4x^4y+5x^3y^2+4x^2y^3+8y^5$.

26. $2x^6-11x^3-9$ and $4x^6+11x^4+81$.

27. $32a^4+104a^3-20a^2-122a+30$ and $60a^5+10a^4-45a^3+45a^2-50a$

28. $x^5+2x^4-5x^2-7x+3$ and $3x^6-3x^4-18x^3+x^2+2x+3$.

*5. In some cases the H. C. F. may be found more easily by the application of the following principle :—

If A and B denote two expressions having no monomial factors, and if m, n, p, q be any four *numerical* quantities such that $mq-np$ is *not* equal to zero, then the H. C. F. of A and B is the same as the H. C. F. of $mA+nB$ and $pA+qB$, *numerical common factors, if any, being left out.* This may be proved as follows :—

Let H denote the H C F. of A and B, and H′ the H C F of $mA+nB$ and $pA+qB$, after removal from them of any numerical common factors that may occur

Now, since *every* common factor of A and B is a factor of $mA+nB$ and also of $pA+qB$, therefore H is a common factor of $mA+nB$ and $pA+qB$ Hence H′ is either equal to H or is an expression of higher dimensions than H (a)

Again, since $q(mA+nB)-n(pA+qB) = (mq-np)A$,

and $m(pA+qB)-p(mA+nB) = (mq-np)B$,

it is clear that *every* common factor of $mA+nB$ and $pA+qB$ is a factor of $(mq-np)A$, and also of $(mq-np)B$ Hence, as $mq-np$ is only a numerical quantity, *every* common factor of those expressions *other than numerical* must be a factor of A as well as of B Hence H′ is a common factor of A and B and therefore cannot be of higher dimensions than H

Hence, by (a), H′ = H, which proves the proposition

Cor. 1 The H C F of A and B is the same as the H C F of A+B and A−B Here $m = 1$, $n = 1$, $p = 1$ and $q = -1$

Cor. 2 The H C F. of A and B is the same as the H C. F of A±B and B, here $m = 1$, $n = \pm 1$, $p = 0$ and $q = 1$. Similarly it is the same as the H C F. of A±B and A.

Example 1. Find the H C. F of

$$x^4+x^3-5x^2-3x+2 \text{ and } x^4-3x^3+x^2+3x-2$$

Let $A = x^4+x^3-5x^2-3x+2$,

and $B = x^4-3x^3+x^2+3x-2$,

then $A+B = 2x^4-2x^3-4x^2 = 2x^2(x^2-x-2)$,

and $A-B = 4x^3-6x^2-6x+4 = 2(2x^3-3x^2-3x+2)$.

Hence, by Cor 1, the required H C F is the H C F. of $x^2(x^2-x-2)$ and $2x^3-3x^2-3x+2$, and therefore of x^2-x-2 and $2x^3-3x^2-3x+2$.

Let $A' = x^2-x-2$

and $B' = 2x^3-3x^2-3x+2$.

Then $A'+B' = 2x^3-2x^2-4x = 2x(x^2-x-2)$.

Hence, the required H. C F.

$$= \text{the H C F. of } A' \text{ and } A'+B' \quad [\text{Cor } 2]$$
$$= x^2-x-2$$

Example 2 Find the H C F. of

$4x^4+11x^3+27x^2+17x+5$ and $3x^4+7x^3+18x^2+7x+5$.

Let $\quad A = 4x^4+11x^3+27x^2+17x+5$

and $\quad B = 3x^4+7x^3+18x^2+7x+5$

Then $\quad A-B = x^4+4x^3+9x^2+10x$

$\quad = x(x^3+4x^2+9x+10)$,

and $\quad 3A-4B = 5x^3+9x^2+23x-5$

Hence, the H. C F of $x^3+4x^2+9x+10$ and $5x^3+9x^2+23x-5$ is the H C F. required

Let $\quad A' = x^3+4x^2+9x+10$

and $\quad B' = 5x^3+9x^2+23x-5$.

Then $\quad A'+2B' = 11x^3+22x^2+55x = 11x(x^2+2x+5)$,

and $\quad 5A'-B' = 11x^2+22x+55 \quad = 11(x^2+2x+5)$.

Hence, the H C F. required is the H. C. F. of $x(x^2+2x+5)$ and x^2+2x+5, and is therefore $= x^2+2x+5$.

Example 3 Find the H. C F of $2x^5-11x^2-9$ and $4x^5+11x^4+81$.

Let $\quad A = 4x^5+11x^4+81$

and $\quad B = 2x^5-11x^2-9$

Then $\quad A-2B = 11x^4+22x^2+99 \quad = 11(x^4+2x^2+9)$,

and $\quad A+9B = 22x^5+11x^4-99x^2 = 11x^2(2x^3+x^2-9)$.

Hence, the required H. C. F. is the same as the H. C. F. of x^4+2x^2+9 and $x^2(2x^3+x^2-9)$, and therefore of x^4+2x^2+9 and $2x^3+x^2-9$

Let $\quad A' = x^4+2x^2+9$

and $\quad B' = 2x^3+x^2-9$

Then $\quad A'+B' = x^4+2x^3+3x^2 = x^2(x^2+2x+3)$.

Hence, the H. C F of

$$\left.\begin{array}{r} 2x^3+x^2-9(=B') \\ \text{and } x^2+2x+3(=C' \text{ say}) \end{array}\right\} \text{ is the H. C. F. required.}$$

Now, since $B'+3C' = 2x^3+4x^2+6x$
$= 2x(x^2+2x+3)$,

$\therefore$ the H C F required $=$ the H C F of C' and $B'+3C$
$= x^2+2x+3$

Exercise (51).

Find the H C F. of —

1. $x^3-3x^2-4x+12$ and $x^3-7x^2+16x-12$.
2. $2x^3-17x+12$ and $4x^4-2x^3-34x^2+41x-12$
3. $4x^3+13x^2+19x+4$ and $2x^3+5x^2+5x-4$
4. $3x^3-5x^2+7$ and $6x^4-7x^3-5x^2+14x+7$.
5. $6x^4-11x^3+16x^2-22x+8$ and $6x^4-11x^3-8x^2+22x-8$.
6. $2x^4+19x^3+20x^2-31x+8$ and $2x^4+7x^3-64x^2+62x-16$
7. $3x^4-7x^3-27x^2-6x+2$ and $3x^4-13x^3-40x^2-9x+3$
8. $5x^4-18x^3-7x^2+12x+3$ and $5x^4-23x^3-9x^2+16x+4$
9. $2x^4-5x^3-17x^2-2x+2$ and $6x^5+23x^4+34x^3+21x^2-2x-2$.
10. $6x^5+9x^4-13x^3-4x^2+9x-3$ and $9x^5+12x^4-18x^3-5x^2+12x-4$
11. x^5-x^3+8 and x^5-x^2+4
12. $3x^5+189x^2-44$ and $39x^5+189x^4-16$

*6. The H C F of three or more expressions whose factors cannot be easily found.

Let A, B, C stand for any three expressions of which the H C F. is to be found

Let G denote the H C F of A and B, and H that of G and C

Then G being the product of *all* the elementary common factors of A and B, every factor of G is a common factor of A and B, and therefore every common factor of G and C is a common factor of A, B and C

Hence, H also is a common factor of A, B and C. Therefore the H. C F required is either H or an expression of higher dimensions than H. .. .. (β)

But, since every common factor of A and B is a factor of G, *every* common factor of A, B and C is a common factor of G and C Hence, the H. C F required is a common factor of G and C and therefore cannot be of higher dimensions than H.

Hence, by (β), the H C. F required = H.

By a similar reasoning it follows that if D be a fourth expression, then the H. C F. of H and D is the H. C. F. of A, B, C and D.

Thus we have the following rule.—*To find the H. C. F. of any number of expressions A, B, C, D, &c, first find the H C. F of A and B, then the H. C. F of this result and C, and so on; the result obtained last of all is the H. C F required*

Example Find the H. C F of $2x^4-7x^3-17x^2+58x-24$, $3x^4+14x^3-11x^2-70x+24$ and $5x^4+9x^3-47x^2-81x+18$

Let us first find the H C F of the first two expressions.

Put $\qquad A = 2x^4-7x^3-17x^2+58x-24,$

and $\qquad B = 3x^4+14x^3-11x^2-70x+24$

Then, $\qquad A+B = 5x^4+7x^3-28x^2-12x$

$\qquad = x(5x^3+7x^2-28x-12),$

and $\qquad -3A+2B = 49x^3+29x^2-314x+120.$

Hence, the H C F of A and B is the H C F. of $5x^3+7x^2-28x-12$ and $49x^3+29x^2-314x+120$.

Let $\qquad A' = 5x^3+7x^2-28x-12$

and $\qquad B' = 49x^3+29x^2-314x+120.$

Then, $\qquad 10A'+B' = 99x^3+99x^2-594x$

$\qquad = 99x(x^2+x-6)$

Hence, the H C F of A and B is the same as the H. C F. of $\left.\begin{array}{l} 5x^3+7x^2-28x-12(=A') \\ \text{and } x^2+x-6(=C' \text{ say}). \end{array}\right\}$

Now, $\qquad A'-2C' = 5x^3+5x^2-30x = 5x(x^2+x-6).$

$\therefore$ the H. C. F of A and B = the H. C F. of C′ and
$$A'-2C' = x^2+x-6$$

Hence, the H C F required is the H C F. of x^2+x-6 and $5x^4+9x^3-47x^2-81x-18$, which can be found as follows —

$$x^2+x-6\,)\,5x^4+9x^3-47x^2-81x+18\,(\,5x^2+4x$$
$$5x^4+5x^3-30x^2$$
$$\overline{\quad 4x^3-17x^2-81x+18}$$
$$4x^3+\ \ 4x^2-24x$$
$$-3\,)\,\overline{-21x^2-57x+18}$$
$$\overline{7x^2+19x-\ 6}\,(\,7$$
$$7x^2+\ \ 7x-42$$
$$12\,)\,\overline{12x+36}$$
$$\overline{x+3}$$

$$x+3\,)\,x^2+\ x-6\,(\,x-2$$
$$x^2+3x$$
$$\overline{-2x-6}$$
$$-2x-6$$
$$\overline{\qquad\qquad}$$

Thus the required H. C F $= x+3$

Exercise (52).

Find the H. C. F of —

1. $2x^3+7x^2-5x-4$, $x^3+8x^2+11x-20$ and $2x^3+19x^2+49x+20$.
2. $2x^4+3x^3+8x^2+15x-10$, $2x^4-5x^3+12x^2-25x+10$ and $2x^4-5x^3+10x^2-20x+8$
3. $2x^4+7x^3-19x^2-14x+30$, $2x^4+5x^3-16x^2-10x+24$ and $2x^4+5x^3-10x^2+5x-12$
4. $2x^4-4x^3-69x^2-2x-35$, $2x^4-6x^3-55x^2-3x-28$ and $2x^4+18x^3+41x^2+9x+20$.

5. $3a^3+28a^2b+52ab^2-48b^3$, $3a^3+4a^2b-28ab^2+16b^3$ and $3a^3+10a^2b-44ab^2+24b^3$

6. $6a^3+5a^2b-34ab^2+15b^3$, $6a^3-37a^2b+57ab^2-20b^3$ and $8a^3-8a^2b-31ab^2+60b^3$.

7. $3x^4+11x^3-32x^2-44x+80$, $3x^4-x^3-52x^2+124x-80$, $3x^4+2x^3-20x^2-8x+32$ and $3x^4+2x^3-83x^2-50x+200$.

8. $6x^5+14x^4-53x^3-37x^2+66x+24$, $6x^5-28x^4+17x^3+54x^2-39x-18$, $6x^5+8x^4-79x^3-36x^2+105x+36$ and $2x^5-2x^4-31x^3+51x^2+42x-72$

CHAPTER X.

LOWEST COMMON MULTIPLE.

1. Definitions. One expression is said to be a *multiple* of another when the former is exactly divisible by the latter

One expression is said to be a *common multiple* of two or more other when it is exactly divisible by *each* of these latter

Of the different common multiples of two or more expressions that which consists of the *least* number of elementary factors is called the *Lowest Common Multiple* of those expressions In other words, a common multiple of two or more expressions is said to be their *Lowest Common Multiple* when it is the product of *just* as many elementary factors as it *must necessarily* have and no more.

Thus the common multiples of a and b are ab, $2ab$, a^2b, ab^2, a^2b^2, &c ; but of these ab consists of the least number of elementary factors, and hence it is called the lowest common multiple of the quantities a and b.

Cor. Hence every common multiple of two or more expressions is divisible by their Lowest Common Multiple.

Note The Letters L C M are usually written for "Lowest Common Multiple "

2. L. C. M. of simple expressions or such compound expressions as can be easily resolved into their elementary factors.

In such cases the L C M can be written down by inspection The following examples will illustrate the process —

Example 1 Find the L C M of $4a^2bc$ and $6ab^2d$

The 1st expression $= 2^2 \times a^2 \times b \times c$

The 2nd expression $= 2 \times 3 \times a \times b^2 \times d$

Hence, $2^2 \times 3 \times a^2 \times b^2 \times c \times d$ *must necessarily* be a factor of *every* common multiple of them

Hence, the L C. M required

$$= 2^2 \times 3 \times a^2 \times b^2 \times c \times d$$
$$= 12a^2b^2cd$$

Example 2 Find the L C M of $24x^2yz$, $18xy^3z^2$ and $27x^4y^2z^2$.

The 1st expression $= 2^3 \times 3 \times x^2 \times y \times z$

The 2nd expression $= 2 \times 3^2 \times x \times y^3 \times z^2$.

The 3rd expression $= 3^3 \times x^4 \times y^2 \times z^2$

Hence, $2^3 \times 3^3 \times x^4 \times y^3 \times z^2$ *must necessarily* be a factor of *every* common multiple of them.

Hence, the L C M required

$$= 2^3 \times 3^3 \times x^4 \times y^3 \times z^2$$
$$= 216x^4y^3z^2.$$

Example 3 Find the L C M of

$$4x^2(x+a)^2,\ 6a^2x(x^2-a^2) \text{ and } 9x^3(x^3-a^3).$$

The 1st expression $= 2^2 \times x^2 \times (x+a)^2$

The 2nd expression $= 2 \times 3 \times a^2 \times x \times (x+a)(x-a)$

The 3rd expression $= 3^2 \times x^3 \times (x-a)(x^2+ax+a^2)$

Hence, $2^2 \times 3^2 \times a^2 \times x^3 \times (x+a)^2(x-a)(x^2+ax+a^2)$ *must necessarily* be a factor of *every* common multiple of them.

Hence, the required L C M

$$= 2^2 \times 3^2 \times a^2 \times x^3 \times (x+a)^2(x-a)(x^2+ax+a^2)$$
$$= 36a^2x^3(x+a)^2(x^3-a^3)$$

Example 4. Find the L C M of
$$x^2-3x+2,\ x^3+2x^2-3x \text{ and } x^4+x^3-6x^2.$$

The 1st expression $=(x-1)(x-2)$.
The 2nd expression $=x(x^2+2x-3)=x(x-1)(x+3)$.
The 3rd expression $=x^2(x^2+x-6)=x^2(x-2)(x+3)$.

Hence, $x^2(x-1)(x-2)(x+3)$ *must necessarily* be a factor of *every* common multiple of the given expressions

Hence, the L C M required $=x^2(x-1)(x-2)(x+3)$

Example 5. Find the L. C. M of x^3-3x^2+3x-1, x^3-x^2-x+1, x^4-2x^3+2x-1 and $x^4-2x^3+2x^2-2x+1$

$$\begin{aligned}
x^3-3x^2+3x-1 &= (x-1)^3 \\
x^3-x^2-x+1 &= x^2(x-1)-(x-1) \\
&= (x-1)(x^2-1) = (x-1)^2(x+1). \\
x^4-2x^3+2x-1 &= (x^4-1)-2x(x^2-1) \\
&= (x^2-1)\{(x^2+1)-2x\} \\
&= (x^2-1)(x-1)^2 \\
&= (x-1)^3(x+1) \\
x^4-2x^3+2x^2-2x+1 &= x^2(x^2-2x+1)+(x^2-2x+1) \\
&= (x^2-2x+1)(x^2+1) \\
&= (x-1)^2(x^2+1)
\end{aligned}$$

Hence, $(x-1)^3(x+1)(x^2+1)$ *must necessarily* be a factor of *every* common multiple of the given expressions

Hence, the L C M required $=(x-1)^3(x+1)(x^2+1)$

Exercise (53).

Find the L. C. M. of —

1. a^2b and ab^2
2. a^3b^2 and a^2bc.
3. $6x^2y^4$ and $10xy^2$
4. $4m^2n^3$ and $14m^4n^2p$
5. $8x^2y^3z$ and $12x^3y^2z^2$
6. $4a^2bc$, $10ab^2c$ and $14abc^2$.
7. $8a^3b^2c$, $12ab^3c^2$ and $20a^2bc^3$.
8. $6x^4y$, $9x^2y^2z$, $12a^2xy^3$ and $15axz^2$
9. a^3b-ab^3 and $a^3b^2+a^2b^3$.

10. $4(x-y)^2$, $6(x^2-y^2)$ and $8(x+y)^2$.
11. x^2-4x+3 and x^2-5x+6
12. $a^3+2a^2x-3ax^2$ and $a^4+a^3x-6a^2x^2$.
13. $a^2(a^2-4)$ and $a+2a^3-8a^2$.
14. $4a^2x^2$, $2x(x^2-a^2)$ and $6a^3x(x^3+a^3)$.
15. $12(x^2+3x-10)$ and $16(x^2+4x-12)$.
16. $x^2+2x-15$, $x^2+9x+20$ and $x^2+4x-21$
17. $12a^4-27a^2b^2$, $2a^2+ab-3b^2$ and $2a^2-ab-3b^2$
18 $8a^3+27b^3$, $8a^3-27b^3$ and $16a^4+36a^2b^2+81b^4$.
19. $8x^4-50x^2y^2$, $12x^3+24x^2y-15xy^2$ and $16x^2-48xy+20y^2$
20. $4x^2-12ax+9a^2$, $6x^2-7ax-3a^2$ and $6x^2-11ax+3a^2$
21. $2x^2+6x+9$, $4x^3-12x^2+18x$ and $4x^4+81$.
22. $9a^2-6ax+x^2$, $6a^2+10ax-4x^2$ and $9a^2-21ax+6x^2$.
23. $8x^3-12x^2+6x-1$, $8x^3-4x^2-2x+1$ and $2x^2+5x-3$
24. $x^2-6xy+8y^2$, $x^2-7xy+12y^2$, $x^2+2xy-15y^2$ and $x^2+xy-20y^2$.
25. $6x^2-x-1$, $3x^2+7x+2$ and $2x^2+3x-2$
26. $1+4x+4x^2-16x^4$ and $1+2x-8x^3-16x^4$

(Calcutta University Entrance Paper, 1874)

27. $9x^4-28x^2+3$, $27x^4-12x^2+1$, $27x^4+6x^2-1$ and x^4-6x^2+9 (Calcutta University Entrance Paper, 1886)

[The factors of the last expression suggest a factor of the first]

3 L C M. of two expressions whose factors are not obvious by inspection.

Let A and B stand for two such expressions, and suppose their H C F is found to be H

Divide A and B by H and let the respective quotients be a and b Then we have

$$\left.\begin{aligned} A &= aH \\ B &= bH \end{aligned}\right\}$$

Hence, since a and b have no common factors, *every* common multiple of A and B must *necessarily* contain $a\times H\times b$ as a factor

Hence, the L. C M. required $= a\text{H}b$.

But $$\left.\begin{aligned} a\text{H}b &= a(\text{H}b) = \frac{\text{A}}{\text{H}} \times \text{B} \\ \text{or } &= (a\text{H})b = \text{A} \times \frac{\text{B}}{\text{H}} \end{aligned}\right\}$$

Hence, the required L C M $= \dfrac{\text{A}}{\text{H}} \times \text{B}$, or $= \text{A} \times \dfrac{\text{B}}{\text{H}}$.

Thus, *to find the L. C. M. of any two expressions we have to divide one of them by their H C. F. and multiply the quotient by the other*

Cor. If L denote the L C M of A and B, we have $\text{L} \times \text{H} = \text{A} \times \text{B}$, that is, *the product of the L. C. M. and H C F. of any two expressions is equal to the product of these expressions.*

Note If any two expressions have no common factor, their L C M. is evidently equal to their product.

Example Find the L. C. M. of

$$6x^3 + 25x^2 + 16x + 7 \text{ and } 6x^3 - 11x^2 - 8x - 5$$

$$\begin{array}{r|l|l} 6x^3 - 11x^2 - 8x - 5 & 6x^3 + 25x^2 + 16x + 7 & 1 \\ & \underline{6x^3 - 11x^2 - 8x - 5} & \\ 12 & \underline{36x^2 + 24x + 12} & \\ & 3x^2 + 2x + 1 & \end{array}$$

$$\begin{array}{r|l|l} 3x^2 + 2x + 1 & 6x^3 - 11x^2 - 8x - 5 & 2x - 5 \\ & \underline{6x^3 + 4x^2 + 2x} & \\ & -15x^2 - 10x - 5 & \\ & \underline{-15x^2 - 10x - 5} & \end{array}$$

Thus, the H. C F of the given expressions $= 3x^2 + 2x + 1$.
Hence, the L C M. required

$$\begin{aligned} &= \frac{6x^3 - 11x^2 - 8x - 5}{3x^2 + 2x + 1}(6x^3 + 25x^2 + 16x + 7) \\ &= (2x - 5)(6x^3 + 25x^2 + 16x + 7) \\ &= 12x^4 + 20x^3 - 98x^2 - 66x - 35. \end{aligned}$$

Exercise (54).

Find the L C M of —

1. $3x^3+2x^2-11x+4$ and $3x^3+14x^2+13x-8$.
2. $6x^3+17x^2+9x-4$ and $6x^3-7x^2-27x+8$
3. $12x^3-4x^2-25x+12$ and $12x^3-28x^2+7x+12$.
4. $9x^3-12x^2-15x+20$ and $15x^3+12x^2-25x-20$
5. $4x^3-10x^2-18x+45$ and $6x^3+8x^2-27x-36$
6. $4x^4+4x^3+7x^2+11x+4$ and $6x^4+7x^3+4x^2+5x+2$
7. $8x^4-6x^3-8x^2+9x-6$ and $16x^4-12x^3+20x^2-9x+6$
8. $4x^4+8x^3+21x^2+18x+27$ and $3x^4+6x^3+17x^2+16x+24$
9. If h be the Highest Common Divisor and l the Lowest Common Multiple of two quantities x and y, and if $h+l=x+y$, prove that $h^3+l^3=x^3+y^3$

(Punjab University Entrance Paper, 1891)

***4 L. C. M. of three or more expressions whose factors are not obvious by inspection**

Let A, B, C stand for three such expressions, to find their L C M

Let L denote the L C M of A and B, and M that of L and C.

Then evidently *every* common multiple of L and C is a common multiple of A, B, C, (1)

also *every* common multiple of A, B, C is a common multiple of L and C . (2)

From (1), M is a common multiple of A, B C Hence, either M or an expression of a lower degree than M is the L C M of A, B, C

But an expression of a lower degree than M cannot be the L C M of A, B, C, because, from (2), the L C M of A, B, C is a common multiple of L and C

Hence, the required L C M = M

Thus, *to find the L C M of any number of expressions A, B, C, D, &c, we have first to find the L C M of A and B, then the L C M of this result and C, and so on, the last result thus obtained is the L C. M required*

Example Find the L. C. M. of

$6x^2-11x+3$, $4x^2-4x-3$ and $6x^2+25x-9$

$$
\begin{array}{r|r}
6x^2-11x+3\,\big)\,6x^2+25x-9\,\big(\,1 & 3x-1\,\big)\,6x^2-11x+3\,\big(\,2x-3 \\
6x^2-11x+3 & 6x^2-\ \ 2x \\
\hline
12\,\big)\,36x-12 & -\ 9x+3 \\
\hline
3x-1 & -\ 9x+3 \\
\end{array}
$$

Thus the H. C. F. of $6x^2-11x+3$ and

$$6x^2+25x-9 = 3x-1.$$

Hence, the L. C. M. of these expressions

$$= \frac{6x^2-11x+3}{3x-1}(6x^2+25x-9)$$

$$= (2x-3)(6x^2+25x-9)$$

$$= 12x^3+32x^2-93x+27.$$

Now to find the L. C. M. of this expression and $4x^2-4x-3$.

$$
\begin{array}{r}
4x^2-4x-3\,\big)\,12x^3+32x^2-93x+27\,\big(\,3x+11 \\
12x^3-12x^2-9x \\
\hline
44x^2-84x+27 \\
44x^2-44x-33 \\
\hline
-20\,\big)\ -40x+60 \\
\hline
2x-3
\end{array}
$$

$$
\begin{array}{r}
2x-3\,\big)\,4x^2-4x-3\,\big(\,2x+1 \\
4x^2-6x \\
\hline
2x-3 \\
2x-3 \\
\hline
\end{array}
$$

Thus the H. C. F. of the expressions considered $= 2x -$

Hence their L. C. M.

$$= \frac{4x^2-4x-3}{2x-3}(12x^3+32x^2-93x+27)$$

$$= (2x+1)(12x^3+32x^2-93x+27)$$

$$= 24x^4+76x^3-154x^2-39x+27.$$

Exercise (55).

Find the L C M. of —

1. $3x^2-10x-8$, $4x^2-20x+9$ and $6x^2+x-2$
2. $3x^2-23x-8$, $6x^2-7x-3$ and $2x^2-11x+12$
3. $6x^2-19x+10$, $12x^2-11x+2$ and $8x^2+10x-3$.
4. $2x^4+4x^3+x^2+6x-3$, $4x^4+8x^3-7x^2-6x+3$ and $8x^4+4x^3-2x^2-3x-3$.

Miscellaneous Exercises (2).

I.

1. When is one number said to be multiplied by another ? From the definition deduce the result when -8 is multiplied by -4

2 Arrange the following expression (i) according to descending powers of y and (ii) according to ascending powers of z —

$$x^3z+xy^3-x^3y-xy^2z-xz^3+xyz^2-2yz^3-2y^3z$$

3. If $x-\frac{1}{x}=p$, prove that $x^3-\frac{1}{x^3}=p^3+3p$.

4. Write down the quotient of x^5-y^5 by $x-y$.

5. Simplify

$(a+b+c)^2-(a-b+c)^2+(a+b-c)^2-(b+c-a)^2$, and find its numerical value when $a=b=c=-4$.

6. Reduce, to the form of $(A+B)(A-B)$,

$$(a-b+c+d)(a+b+c-d).$$

7. Resolve into factors $4x^2+12xy+9y^2-8x-12y$

8. Find the H C F of

$2x^4-12x^3+19x^2-6x+9$ and $4x^3-18x^2+19x-3$.

II

1. Prove that $a\times b=b\times a$, when a and b are any two positive integers

2. Multiply $x^{\frac{2}{3}}+2x^{\frac{1}{2}}+3x^{\frac{1}{3}}+2x^{\frac{1}{6}}+1$ by $x^{\frac{1}{3}}-2x^{\frac{1}{6}}+1$.

3. Prove that

$$\{(ac+bd)x+(ad-bc)y\}^2+\{(ac+bd)y-(ad-bc)x\}^2$$
$$=(a^2+b^2)(c^2+d^2)(x^2+y^2).$$

4. Write down the cube of x^2-2x+1.

5. Divide $x^5-px^4+qx^3-qx^2+px-1$ by $x-1$.

6. Resolve into factors —

(i) $ab-ac-b^2+bc$, (ii) $b^2-12ac-4a^2-9c^2$.

7. Find the H. C. F. of

$$2x^3-15x+14 \text{ and } x^4-15x^2+28x-12.$$

8. Find the L. C. M. of

$$x^3+bx^2+ax+ab \text{ and } x^2-(a-b)x-ab.$$

III.

1. If $a+b=8$ and $ab=5$, find the value of a^3+b^3.

2. Find the value of $49c^2+9(a+b)^2-42(a+b)c$,

when $a=89$, $b=-69$, $c=8$.

3. Divide $x^3(y-z)+y^3(z-x)+z^3(x-y)$ by

$$y^2-xz-z^2+xy.$$

4. Resolve into factors $4a-3+16a^2+64a^3$, after reducing it to the form of $(A-B)+(A^2-B^2)+(A^3-B^3)$.

5. Show that $(1+x+x^2)^2-(1-x+x^2)^2=4x(1+x^2)$.

6. If $a_1+a_2+a_3+\dots+a_n=\frac{n}{2}s$, show that

$$(s-a_1)^2+(s-a_2)^2+(s-a_3)^2 \dots +(s-a_n)^2$$
$$=a_1{}^2+a_2{}^2+a_3{}^2+\dots+a_n{}^2.$$

7. Find the H. C. F. of $x^4-px^3+(q-1)x^2+px-q$

and $x^4-qx^3+(p-1)x^2+qx-p$.

8. Find the L. C. M. of

$$x^3-7x^2+16x-12 \text{ and } 3x^3-14x^2+16x.$$

IV.

1. Multiply $a^{\frac{5}{2}}-2a^2b^{\frac{1}{3}}+4a^{\frac{3}{2}}b^{\frac{2}{3}}-8ab+16a^{\frac{1}{2}}b^{\frac{4}{3}}-32b^{\frac{5}{3}}$ by $a^{\frac{1}{2}}+2b^{\frac{1}{3}}$

2. Arrange the following expressions according to descending powers of a —

(i) $a^3+b^3+c^3-3abc$,

(ii) $a^2(b-c)+b^2(c-a)+c^2(a-b)$;

(iii) $a^4(b-c)+b^4(c-a)+c^4(a-b)$

3 Write down the product of $x+a$, $x+b$ and $x+c$ Hence, deduce the co-efficients of x^2 and x in

$$(x-7)(x+8)(x-12)$$

4. Prove that

$$(ab+cd+ac+bd)(ab+cd-ac-bd) = a^2b^2+c^2d^2-a^2c^2-b^2d^2$$

5. If $a = q+r+s$, $b = r+s-p$, $c = p+q-r$, prove that $a^2+b^2+c^2-2ab-2ac+2bc = r^2$

6 Divide $a^3+8b^3+27c^3-18abc$ by

$$a^2+4b^2+9c^2-6bc-3ca-2ab$$

7. Define the *Highest Common Factor* of two or more algebraical expressions

Find the H. C F of

$$36x^2a^4c^5,\ 24xy^2a^3b^4 \text{ and } 240y^3a^6b^2c$$

8. Find the L C M. of x^2-y^2, $(x-y)^2$ and x^3-y^3

V.

1. Find the value of $49a^2+126ab+81b^2$, when $a = 46$, $b = -37$.

2. If $2s = a+b+c$, show that

$$(s-a)^2+(s-b)^2+(s-c)^2+s^2 = a^2+b^2+c^2.$$

3. Simplify

$$(5a-7c)^3+(8c-3a)^3+3(2a+c)(5a-7c)(8c-3a)$$

4. Divide $a-b$ by $a^{\frac{1}{4}}-b^{\frac{1}{4}}$

4. Resolve into factors —

(i) $6a^4x^2+a^3x-6a^3x^3-a^2x^2$,

(ii) $xy(1+z^2)+z(x^2+y^2)$.

6. Find the H C F of $ab+2a^2-3b^2-4bc-ac-c^2$ and $9ac+2a^2-5ab+4c^2+8bc-12b^2$.

7. If H and L respectively denote the H. C. F and the L. C. M. of any two algebraical expressions A and B, prove that $L \times H = A \times B$.

8. Find the L. C. M. of

$$6x^3-11x^2+5x-3 \text{ and } 9x^3-9x^2+5x-2.$$

VI.

1. Find the value of

$$8765943 \times 8765943 - 8765938 \times 8765938.$$

2. Find the value of $27a^3+108a^2b+144ab^2+64b^3$, when $a = 29$, $b = -23$.

3. Divide $a^3+b^3+c^3-3abc$ by $a+b+c$, and hence show that

$$a^3+b^3+c^3-3abc=\tfrac{1}{2}(a+b+c)\{(a-b)^2+(b-c)^2+(c-a)^2\}.$$

4. Show that

$$(a-b)(a-c)(b-c) = a^2(b-c)+b^2(c-a)+c^2(a-b);$$

hence prove that

$$(x+2y+3z)^2(x-2y+z)+(y+2z+3x)^2(y-2z+x)+$$
$$(z+2x+3y)^2(z-2x+y)+(x-2y+z)(y-2z+x)(z-2x+y)=0.$$

5. Resolve $(a^2-b^2-c^2+d^2)^2-4(ad-bc)^2$ into four factors.

6. Resolve into factors :—

(i) $a^2-2ab+b^2+2a-2b$,

(ii) $6a^2-ab-b^2+6a-3b$;

(iii) $15x^2-4xy-4y^2+10x+4y$.

7. Divide $(2x-y)^2a^4-(x+y)^2a^2x^2+2(x+y)ax^4-x^6$ by $(2x-y)a^2-(x+y)ax+x^3$.

8. Find the H. C F. of $x^4-(a^2+b^2)x^2+a^2b^2$ and $x^4-(a+b)^2x^2+2ab(a+b)x-a^2b^2$.

VII.

1. If $x+y+z = 8$ and $x^2+y^2+z^2 = 50$, find the value of $xy+yz+zx$.

2. Prove that

$$(2a-3b)^2+(3b-5c)^2+(5c-2a)^2$$
$$= 2(2a-3b)(2a-5c)+2(3b-5c)(3b-2a)+2(5c-2a)(5c-3b).$$

3. Divide $a^2(x^2-a^2)-ab(x+a)^2+b(x^3+a^3)$
by $a^2(x-a)+bx(x-2a)$

4. Resolve into factors —

(i) $6x^2+x-15$,

(ii) $35x(x-y)^2-41(x-y)+12$,

(iii) $11x^2-54xy^2+63y^4$.

5. Show that
$$(x+y)(y+z)(z+x) = (x+y+z)(xy+yz+zx)-xyz$$

6. If $x+y+z = 0$, show that
$(x+y)(y+z)(z+x) = -xyz$, and $x^3+y^3+z^3 = 3xyz$

7. Resolve into factors
$$(a-b)(b+c)(c+a)+(b-c)(c+a)(a+b)+(c-a)(a+b)(b+c)$$

8. Find the L. C. M of
$35x^2-11x-6$ and $40x^2-29x+3$

VIII.

1. If $x+y+z = 15$ and $xy+yz+zx = 85$,
find the value of $x^2+y^2+z^2$.

2. Show that $(a+b-2c)^3+(b+c-2a)^3+(c+a-2b)^3$
$= 3(a+b-2c)(b+c-2a)(c+a-2b)$

3. If $a^2+b^2 = 1 = c^2+d^2$, show that
$(ad-bc)(ad+bc) = (a-c)(a+c)$.

4 Find the L C M of
$3x^2-11x+6$, $2x^2-7x+3$ and $6x^2-7x+2$.

5. Find the H C. F of
$a^2x^3+a^5-2abx^3+b^2x^3+a^3b^2-2a^4b$ and
$2a^2x^4-5a^4x^2+3a^6-2b^2x^4+5a^2b^2x^2-3a^4b^2$

6. Divide $(ax+by)^3+(ax-ay)^3+(bx-ay)^3+(bx+ay)^3$
by $(a+b)^2x^2-3ab(x^2-y^2)$

7. Evaluate $x^2+\frac{1}{x^2}$, $x^3+\frac{1}{x^3}$, and $x^4+\frac{1}{x^4}$,
when $x+\frac{1}{x}=a$

8. Show that $(x^2+y^2)(x^2+z^2)+2x(x^2+yz)(y+z)+$
$4x^2yz = (x^2+xy+xz+yz)^2$.

IX.

1. Show that $(a^2+2ab+b^2-c^2)(a^2-2ab+b^2+c^2)$
$$= (a^2-b^2)^2+(4ab-c^2)c^2.$$

2. Write down the expansion of $\left(x+\dfrac{2}{x}\right)^6$.

3. Find the H C F of $x^5+11x-12$ and x^5+11x^3+54.

4. Divide $a^3(b^2-c^2)+b^3(c^2-a^2)+c^3(a^2-b^2)$
by $ab+bc+ca$.

5 Find the L C M of
$x^2-3xy-10y^2$, $x^2+2xy-35y^2$ and $x^2-8xy+15y^2$,

6. Show that
$$(x-a)^2(b-c)+(x-b)^2(c-a)+(x-c)^2(a-b)$$
$$= (a-b)(a-c)(b-c)$$

7. Show that
$$(a+b+c)^3-a^3-b^3-c^3 = 3(a+b)(b+c)(c+a).$$
Hence prove that
$$(x+y+z)^3-(y+z-x)^3-(z+x-y)^3-(x+y-z)^3 = 24xyz.$$

8. Find the value of $a^3-b^3+c^3+3abc$,
when $a = 4278$, $b = 1{\cdot}2345$ and $c = {\cdot}8067$.

X.

1. Show that
$$(1+a)^2(1+c^2)-(1+c)^2(1+a^2) = 2(a-c)(1-ac).$$

2. If $a+b = 2$ and $ab = 7$, find the value of a^5+b^5.

3. Divide $a+b+c+3(b^{\frac{1}{3}}+c^{\frac{1}{3}})(c^{\frac{1}{3}}+a^{\frac{1}{3}})(a^{\frac{1}{3}}+b^{\frac{1}{3}})$
by $a^{\frac{1}{3}}+b^{\frac{1}{3}}+c^{\frac{1}{3}}$.

4. Resolve $2(a^6+b^6)-ab(a^2+b^2)(2ab-3a^2+3b^2)$ into five simple factors

5. Resolve into factors .—
(i) $6a^4+43a^3b-56a^2b^2+43ab^3+6b^4$;
(ii) $12x^4-37x^3+45x^2-37x+12$,
(iii) $abx^4+(ac+b^2)x^3+(2ab+bc)x^2+(ac+b^2)x+ab$.

6. Show that $(b-c)(b+c-2a)^2+(c-a)(c+a-2b)^2$
$$+(a-b)(a+b-2c)^2 = 9(a-b)(a-c)(b-c).$$

7. Find the value of $a^3+b^3+c^3-3abc$,
when $a = 2658$, $b = 2664$, and $c = 2678$.

8. If $x = b+c-a$, $y = c+a-b$ and $z = a+b-c$,
prove that $x^3+y^3+z^3-3xyz = 4(a^3+b^3+c^3-3abc)$.

CHAPTER XI

FRACTIONS.

1. Definition. The Algebraical fraction $\frac{a}{b}$, where a and b may have *any* numerical values, is defined to be a quantity which, when multiplied by b, becomes equal to a. In other words, $\frac{a}{b}$ is defined to be equivalent to $a \div b$. In $\frac{a}{b}$, a is called the *numerator* and b the *denominator*.

Note Thus an *algebraical fraction* is no other than the quotient of one expression by another, *expressed by placing the dividend over the divisor with a horizontal line between them*, and the dividend and the divisor so placed are respectively called the *numerator* and the *denominator* of the fraction

2. The value of a fraction is not altered if both its numerator and denominator are multiplied or divided by any the same quantity.

If a, b and m stand for any quantities whatever, to prove that

$$\frac{a}{b} = \frac{am}{bm}$$

Let $$x = \frac{a}{b},$$

then $$x \times b = \frac{a}{b} \times b = a \quad \text{[by definition]};$$

$$\therefore \quad x \times b \times m = a \times m,$$
$$\text{or,} \quad x \times bm = am.$$
$$\text{Hence,} \quad x = am \div bm,$$
$$\text{i.e.,} \quad \frac{a}{b} = \frac{am}{bm}.$$

Conversely, we have $\dfrac{am}{bm} = \dfrac{a}{b}$;

$$\text{i.e.,} \quad \frac{am}{bm} = \frac{am \div m}{bm \div m}.$$

Thus the proposition is established

Cor. $\dfrac{a}{b} = \dfrac{a \times (-1)}{b \times (-1)} = \dfrac{-a}{-b}$. Thus the value of a fraction is not altered if the signs of both the numerator and the denominator be changed

3. Reduction of a fraction to its lowest terms.

A fraction is said to be in its lowest terms, when its numerator and denominator have no common factor.

Hence, to reduce a fraction to its lowest terms, or more briefly to *simplify* it, is no other than to find an equivalent fraction whose numerator and denominator have no common factor, and this is evidently done by dividing the numerator and the denominator of the fraction by their highest common factor

Note In all cases where the numerator and the denominator can be factorised by inspection the reduction is at once effected by simply removing the common factors

Example 1. Reduce $\dfrac{4a^2b^3c^2}{10ab^4c^2}$ to its lowest terms.

$$\frac{4a^2b^3c^2}{10ab^4c^2} = \frac{2 \times 2 \times a^2 \times b^3 \times c^2}{2 \times 5 \times a \times b^4 \times c^2} = \frac{2a}{5b}.$$

Example 2. Simplify $\dfrac{a^2b^3(a^2-b^2)}{3ab^4(a^3+b^3)}$.

$$\frac{a^2b^3(a^2-b^2)}{3ab^4(a^3+b^3)} = \frac{a^2b^3(a+b)(a-b)}{3ab^4(a+b)(a^2-ab+b^2)}$$
$$= \frac{a(a-b)}{3b(a^2-ab+b^2)}.$$

Example 3. Reduce $\frac{x^2+3x-40}{x^2+4x-32}$ to its lowest terms.

The numerator $= (x+8)(x-5)$.

The denominator $= (x+8)(x-4)$

Hence, the given fraction $= \frac{(x+8)(x-5)}{(x+8)(x-4)} = \frac{x-5}{x-4}$.

Example 4. Simplify $\frac{2a^2+3ax-2ab-3bx}{3a^2-2ax-3ab+2bx}$

The numerator $= 2a(a-b)+3x(a-b)$

$= (a-b)(2a+3x)$

The denominator $= 3a(a-b)-2x(a-b)$

$= (a-b)(3a-2x)$.

Hence, the given expression $= \frac{(a-b)(2a+3x)}{(a-b)(3a-2x)} = \frac{2a+3x}{3a-2x}$.

Example 5. Reduce to its lowest terms

$$\frac{3x^3-27ax^2+78a^2x-72a^3}{2x^3+10ax^2-4a^2x-48a^3}$$

(Calcutta University Entrance Paper, 1889)

The numerator $= 3(x^3-9ax^2+26a^2x-24a^3)$

The denominator $= 2(x^3+5ax^2-2a^2x-24a^3)$

Now to find their H C F

$$\begin{array}{r} x^3+5ax^2-2a^2x-24a^3 \\ x^3-9ax^2+26a^2x-24a^3 \\ \hline 14ax\,\big)\,14ax^2-28a^2x \\ \hline x-2a \end{array}$$

$$\begin{array}{l} x-2a\,\big)\,x^3-9ax^2+26a^2x-24a^3\,\big(\,x^2-7ax+12a \\ \qquad\quad x^3-2ax^2 \\ \hline \qquad\quad -7ax^2+26a^2x-24a^3 \\ \qquad\quad -7ax^2+14a^2x \\ \hline \qquad\qquad\qquad 12a^2x-24a^3 \\ \qquad\qquad\qquad 12a^2x-24a^3 \\ \hline \end{array}$$

Thus the H. C. F. required $= x-2a$.

Hence, the required result

$$= \frac{3(x^3-9ax^2+26a^2x-24a^3)\div(x-2a)}{2(x^3+5ax^2-2a^2x-24a^3)\div(x-2a)}$$

$$= \frac{3(x^2-7ax+12a^2)}{2(x^2+7ax+12a^2)}.$$

Example 6. Reduce $\dfrac{2x^4-x^3-9x^2+13x-5}{7x^3-19x^2+17x-5}$ to its lowest terms (Calcutta University Entrance Paper, 1870)

The H. C F. of the numerator and denominator of the given fraction can be found as follows —

$$\begin{array}{r}
2x^4-x^3-9x^2+13x-5\\
7x^3-19x^2+17x-5
\end{array}$$

[See Cor 2 Art 5, Chap IX]

$$\begin{array}{rl}
 & \overline{}\\
2x\big) & 2x^4-8x^3+10x^2-4x\\
 & \overline{x^3-4x^2+5x-2}
\end{array}$$

$$\begin{array}{r|l|l}
x^3-4x^2+5x-2 & 7x^3-19x^2+17x-5 & 7\\
 & 7x^3-28x^2+35x-14 & \\
\hline
9\big) & 9x^2-18x+9 & \\
\hline
 & x^2-2x+1 &
\end{array}$$

$$\begin{array}{r|l|l}
x^2-2x+1 & x^3-4x^2+5x-2 & x-2\\
 & x^3-2x^2+x & \\
\hline
 & -2x^2+4x-2 & \\
 & -2x^2+4x-2 & \\
\hline
\end{array}$$

Thus the H C F. required $= x^2-2x+1$.

Hence, the required result

$$= \frac{(2x^4-x^3-9x^2+13x-5)\div(x^2-2x+1)}{(7x^3-19x^2+17x-5)\div(x^2-2x+1)}$$

Exercise (56).

Reduce to lowest terms —

1. $\dfrac{2a^2b^3}{4a^2b^4}$. 2. $\dfrac{6x^2y^3}{8xy^4}$. 3. $\dfrac{4a^2xy^2}{10ax^2y^2}$.

4. $\dfrac{15x^3y^2z^4}{25x^2y^4z^3}$. 5. $\dfrac{18a^2bc^4d^5}{27a^3b^4c^4d^4}$. 6. $\dfrac{16x^2a^4y^3z^5}{40a^3z^4x^3y^4}$.

7. $\dfrac{70a^2b^3c^4d^7}{105c^4d^2a^3b^3}$. 8. $\dfrac{39m^2n^5p^3q^6}{65p^4m^3n^4q^6}$. 9. $\dfrac{x^2-a^2}{x^2+ax}$.

10. $\dfrac{x^2-3x}{9x-x^3}$. 11. $\dfrac{4x^2-9a^2}{4x^2+6ax}$. 12. $\dfrac{3a^2-12ab}{48b^2-3a^2}$.

13. $\dfrac{3ax-12a^2}{x^2-16a^2}$. 14. $\dfrac{2x^4-4a^2x^2}{x^4-4a^2x^2+4a^4}$. 15. $\dfrac{4x^2+8x}{x^2+5x+6}$.

16. $\dfrac{x^2+2x-8}{x^2+x-12}$. 17. $\dfrac{x^2+2x-15}{x^2+9x+20}$.

18. $\dfrac{a^2-3ab-4b^2}{a^2-4ab-5b^2}$. 19. $\dfrac{a^4-a^3b+a^2b^2}{a^3+b^3}$.

20. $\dfrac{1-7x+12x^2}{1-8x+15x^2}$. 21. $\dfrac{x^2-6xy+5y^2}{x^2+2xy-35y^2}$.

22. $\dfrac{1-9a^2+14a^4}{1-4a^2-21a^4}$. 23. $\dfrac{x^4-8x^2-65}{x^4+x^2-20}$.

24. $\dfrac{3a^3x+9a^2x^2+27ax^3}{a^3-27x^3}$. 25. $\dfrac{2x^2-x-6}{3x^2-2x-8}$.

26. $\dfrac{3x^2-5ax+2a^2}{3x^2+ax-2a^2}$. 27. $\dfrac{3x^2+16ax+5a^2}{3x^2+22ax+7a^2}$.

28. $\dfrac{6x^2-7x-20}{9x^2+6x-8}$. 29. $\dfrac{2x^2+3ax-20a^2}{3x^2+5ax-28a^2}$.

30. $\dfrac{10-17ax+3a^2x^2}{5-26ax+5a^2x^2}$. 31. $\dfrac{x^2-(a-b)x-ab}{x^3+bx^2+ax+ab}$.

32. $\dfrac{6ac+10bc+9ax+15bx}{6c^2+9cx-2c-3x}$. 33. $\dfrac{8bx+12ab+6xy+9ay}{12bx+8ab+9xy+6ay}$.

34. $\dfrac{2a^2+ab-b^2}{a^3+a^2b-a-b}$. 35. $\dfrac{x^3+4x^2+x-6}{x^2+x-2}$.

36. $\dfrac{a^2-b^2-2bc-c^2}{a^2+2ab+b^2-c^2}$. 37. $\dfrac{x^3-7x+6}{x^3+2x^2-13x+10}$.

38. $\dfrac{a^3+2a^2b-2ab^2+3b^3}{a^3-5a^2b+5ab^2-4b^3}$. 39. $\dfrac{x^4+(2b^2-a^2)x^2+b^4}{x^4+2ax^3+a^2x^2-b^4}$.

40. $\dfrac{3x^3+4x^2y-7xy^2+2y^3}{2x^3+9x^2y+8xy^2-5y^3}$. 41. $\dfrac{1+3x-x^3-3x^4}{1-x+2x^2+x^3+3x^4}$.

42. $\dfrac{x^4-x^3-x+1}{x^4-2x^3-x^2-2x+1}$. 43. $\dfrac{x^4+x^2+25}{x^4-9x^2+30x-25}$.

44. $\dfrac{2x^3+3ax^2+5a^2x-21a^3}{4x^3-12ax^2+19a^2x-15a^3}$.

45. $\dfrac{2x^4+x^3-3x^2+2x+3}{3x^4+x^3-4x^2+3x+4}$.

46. $\dfrac{9x^3-7a^2x-2a^3}{9x^3+6ax^2-5a^2x-2a^3}$.

47. $\dfrac{2a^3-16a^2b+44ab^2-42b^3}{3a^3+6a^2b-24ab^2-63b^3}$.

48. $\dfrac{9x^4+30x^3+12x^2-6x-45}{8x^4+28x^3+16x^2-4x-48}$.

49. $\dfrac{6a^6-9a^5b+a^4b^2+3a^3b^3-a^2b^4}{4a^5-6a^4b+3a^2b^3-ab^4}$.

50. $\dfrac{24x^5+16x^4y-28x^3y^2-24x^2y^3-12xy^4}{45x^4y+30x^3y^2-15x^2y^3-20xy^4-10y^5}$.

4. Reduction of two or more fractions to a common denominator.

Let $\dfrac{a}{b}$, $\dfrac{c}{d}$, $\dfrac{e}{f}$, &c., stand for any number of fractions.

Let L denote the L. C. M. of the denominators, *i.e.*, of b, d, f, &c. Then, since the value of a fraction is not altered when its numerator and denominator are *both* multiplied by the same quantity, we must have

$$\frac{a}{b}=\frac{a\times(\mathrm{L}\div b)}{b\times(\mathrm{L}\div b)}=\frac{a\times(\mathrm{L}\div b)}{\mathrm{L}},$$

$$\frac{c}{d}=\frac{c\times(\mathrm{L}\div d)}{d\times(\mathrm{L}\div d)}=\frac{c\times(\mathrm{L}\div d)}{\mathrm{L}},$$

$$\frac{e}{f}=\frac{e\times(\mathrm{L}\div f)}{f\times(\mathrm{L}\div f)}=\frac{e\times(\mathrm{L}\div f)}{\mathrm{L}}.$$

Thus the fractions in the third column are respectively equivalent to the given fractions, and they have all got the same denominator, namely, L

Hence, we have the following rule for reducing fractions to a common denominator —*Find the L C M of the denominators and multiply the numerator and denominator of each fraction by the quotient of the L C M. thus found by the denominator of that fraction.*

Example 1. Reduce $\frac{x}{a+b}$, $\frac{x^2}{a(a-b)}$ and $\frac{x^2}{a(a^2-b^2)}$ to a common denominator.

The L C M of the denominators $= ab(a^2-b^2)$, and the quotients obtained by dividing it by the denominators are respectively $ab(a-b)$, $b(a+b)$ and a

Hence, we have $\frac{x}{a+b}=\frac{x\times ab(a-b)}{(a+b)\times ab(a-b)}=\frac{xab(a-b)}{ab(a^2-b^2)}$

$$\frac{x^2}{a(a-b)}=\frac{x^2\times b(a+b)}{a(a-b)\times b(a+b)}=\frac{x^2b(a+b)}{ab(a^2-b^2)},$$

$$\frac{x^2}{b(a^2-b^2)}=\frac{x^2\times a}{b(a^2-b^2)\times a}=\frac{x^2a}{ab(a^2-b^2)}.$$

Example 2. Reduce $\frac{x-1}{x^2-5x+6}$, $\frac{x-2}{x^2-4x+3}$ and $\frac{x-3}{x^2-3x+2}$ to a common denominator.

The denominators are respectively

$(x-2)(x-3)$, $(x-1)(x-3)$ and $(x-1)(x-2)$

Hence, their L C M $=(x-1)(x-2)(x-3)$, and the quotients obtained by dividing it by the denominators are respectively, $x-1$, $x-2$, and $x-3$ Hence, we have

$$\frac{x-1}{x^2-5x+6}=\frac{(x-1)(x-1)}{(x^2-5x+6)(x-1)}=\frac{x^2-2x+1}{x^3-6x^2+11x-6},$$

$$\frac{x-2}{x^2-4x+3}=\frac{(x-2)(x-2)}{(x^2-4x+3)(x-2)}=\frac{x^2-4x+4}{x^3-6x^2+11x-6};$$

$$\frac{x-3}{x^2-3x+2}=\frac{(x-3)(x-3)}{(x^2-3x+2)(x-3)}=\frac{x^2-6x+9}{x^3-6x^2+11x-6}$$

Exercise (57).

Reduce to a common denominator —

1. $\frac{a}{2b}$, $\frac{3c}{4d}$ and $\frac{e}{f}$.

2. $\frac{x^2}{2bc}$, $\frac{y^2}{3ca}$, $\frac{z^2}{4ab}$.

3. $\frac{ab}{4xy^2}$, $\frac{bc}{6x^2y}$, $\frac{ca}{10x^3}$.

4. $\frac{a}{a-b}$, $\frac{b}{a+b}$, $\frac{c}{a(a+b)}$,

5. $\frac{x^2}{a^2+2ab}$, $\frac{y^2}{a-2b}$.

6. $\frac{2a}{a-b}$, $\frac{a-c}{ab-a^2}$

7. $\frac{2a}{a-b}$, $\frac{3b}{b-a}$, $\frac{4c}{a+b}$.

8. $\frac{2x}{a^2(a+x)}$, $\frac{3y}{b^2(a-x)}$, $\frac{4z}{c^2(a^2-x^2)}$.

9. $\frac{a^2}{2xy-3y^2}$, $\frac{b^2}{2x^2+3xy}$, $\frac{c^2}{4x^3y-9xy^3}$.

10. $\frac{a^2}{x^2+x+1}$, $\frac{b^2}{x^2-x+1}$.

11. $\frac{3}{x^2-x-2}$, $\frac{4}{x^2+x-6}$.

12. $\frac{a-2b}{a(a^2-2ab+4b^2)}$, $\frac{bc}{a^3+8b^3}$.

13. $\frac{a}{a-3b}$, $\frac{b}{a^2+3ab+9b^2}$, $\frac{c}{a^3-27b^3}$.

14. $\frac{a}{b(a-b-c)}$, $\frac{b}{a(a-b+c)}$, $\frac{c}{a^2+b^2-c^2-2ab}$.

15. $\frac{c-a}{(a-b)(b-c)}$, $\frac{b-a}{(a-c)(b-c)}$, $\frac{b-c}{(c-a)(a-b)}$.

5. Addition of fractions.

From Cor. 3, Art 8, Chap. V, we know that

$a(b+c+d+e)=ab+ac+ad+ae$, where a, b, c, d, e are any quantities whatever Hence, conversely,

$$\frac{ab+ac+ad+ae}{a}=b+c+d+e=\frac{ab}{a}+\frac{ac}{a}+\frac{ad}{a}+\frac{ae}{a}$$

Hence, putting p, q, r, s respectively for ab, ac, ad, ae, we have

$$\frac{p+q+r+s}{a}=\frac{p}{a}+\frac{q}{a}+\frac{r}{a}+\frac{s}{a},$$ where p, q, r, s and a are any quantities whatever.

Thus the sum of any number of fractions which have a common denominator is a fraction whose denominator is the same and whose numerator is the sum of the numerators of the given fractions

Hence, to obtain the sum of any number of fractions which have not the same denominator we must first reduce them to equivalent fractions having a common denominator, and then proceed as above

Example 1. Find the value of $\frac{a}{a-b}+\frac{b}{b-a}$.

Since $\frac{b}{b-a}=\frac{b\times(-1)}{(b-a)\times(-1)}=\frac{-b}{a-b}$,

we have $\frac{a}{a-b}+\frac{b}{b-a}=\frac{a}{a-b}+\frac{-b}{a-b}$

$$=\frac{a+(-b)}{a-b}=\frac{a-b}{a-b}=1$$

Example 2. Find the value of $\frac{x}{x+a}+\frac{a}{x-a}$.

Since the L C M of the denominators $=x^2-a^2$,

we have $\frac{x}{x+a}=\frac{x(x-a)}{x^2-a^2}$ and $\frac{a}{x-a}=\frac{a(x+a)}{x^2-a^2}$.

Hence, the required value $=\frac{x(x-a)}{x^2-a^2}+\frac{a(x+a)}{x^2-a^2}$

$$=\frac{x(x-a)+a(x+a)}{x^2-a^2}$$

$$=\frac{x^2+a^2}{x^2-a^2}.$$

Example 3. Find the value of $\frac{1}{a+b}+\frac{b}{a^2-b^2}-\frac{a}{a^2+b^2}$.

In the present example it is not convenient to reduce all the fractions to a common denominator at once. We can proceed best as follows —

We have $\frac{1}{a+b}+\frac{b}{a^2-b^2}=\frac{(a-b)+b}{a^2-b^2}=\frac{a}{a^2-b^2}$.

Hence, the required value $= \dfrac{a}{a^2-b^2} + \dfrac{a}{a^2+b^2}$

$$= \frac{a(a^2+b^2)-a(a^2-b^2)}{a^4-b^4}$$

$$= \frac{2ab^2}{a^4-b^4}.$$

Example 4. Simplify $\dfrac{1}{x-2} - \dfrac{1}{x+2} - \dfrac{4}{x^2+4} + \dfrac{32}{x^4+16}$.

We have $\dfrac{1}{x-2} - \dfrac{1}{x+2} = \dfrac{(x+2)-(x-2)}{x^2-4} = \dfrac{4}{x^2-4}$;

$$\frac{4}{x^2-4} - \frac{4}{x^2+4} = \frac{4(x^2+4)-4(x^2-4)}{x^4-16} = \frac{32}{x^4-16};$$

lastly, $\dfrac{32}{x^4-16} + \dfrac{32}{x^4+16} = \dfrac{32(x^4+16)+32(x^4-16)}{x^8-256}$

$= \dfrac{64x^4}{x^8-256}$, which is the required result.

Example 5. Simplify $\dfrac{1}{a+b} - \dfrac{1}{a+2b} - \dfrac{1}{a+3b} + \dfrac{1}{a+4b}$.

The given expression

$$= \left\{\frac{1}{a+b} - \frac{1}{a+2b}\right\} - \left\{\frac{1}{a+3b} - \frac{1}{a+4b}\right\}.$$

Now, we have $\dfrac{1}{a+b} - \dfrac{1}{a+2b} = \dfrac{(a+2b)-(a+b)}{(a+b)(a+2b)}$

$$= \frac{b}{(a+b)(a+2b)},$$

and $\dfrac{1}{a+3b} - \dfrac{1}{a+4b} = \dfrac{(a+4b)-(a+3b)}{(a+3b)(a+4b)}$

$$= \frac{b}{(a+3b)(a+4b)}.$$

Lastly, $\dfrac{b}{(a+b)(a+2b)} - \dfrac{b}{(a+3b)(a+4b)}$

$$= \frac{b(a+3b)(a+4b)-b(a+b)(a+2b)}{(a+b)(a+2b)(a+3b)(a+4b)},$$

of which the numerator $= b(a^2+7ab+12b^2)$
$$-b(a^2+3ab+2b^2)$$
$$= b(4ab+10b^2) = 2b^2(2a+5b).$$

Hence, the required result $= \dfrac{2b^2(2a+5b)}{(a+b)(a+2b)(a+3b)(a+4b)}$.

Exercise (58)

Find the value of —

1. $\dfrac{a+b}{a}+\dfrac{a-b}{b}$.

2. $\dfrac{x-y}{xy}+\dfrac{y-z}{yz}+\dfrac{z-x}{zx}$

3. $\dfrac{a}{a-x}+\dfrac{x}{x-a}$

4. $\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}$

5. $\dfrac{a^2+b^2}{a^2-b^2}-\dfrac{a-b}{2(a+b)}$

6. $\dfrac{4x^2+9y^2}{4x^2-9y^2}-\dfrac{2x-3y}{2x+3y}$.

7. $\dfrac{a}{(a+b)^2}-\dfrac{b}{a^2-b^2}$

8. $\dfrac{a^2+ab+b^2}{a+b}+\dfrac{a^2-ab+b^2}{a-b}$.

9. $\dfrac{1}{(a-b)(a-c)}+\dfrac{1}{(a-c)(b-c)}$.

10. $\dfrac{1}{x^2-3x+2}+\dfrac{1}{x^2-5x+6}$.

11. $\dfrac{1}{x^2+7x+10}+\dfrac{1}{x^2+13x+40}$

12. $\dfrac{1}{2x+3y}-\dfrac{(2x-3y)^2}{8x^3+27y^3}$.

13. $\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}+\dfrac{2ab}{b^2-a^2}$

14. $\dfrac{1}{a+2b}+\dfrac{1}{a-2b}+\dfrac{2a}{4b^2-a^2}$.

15. $\dfrac{x+y}{x-y}+\dfrac{x-y}{x+y}-\dfrac{2(x^2-y^2)}{x^2+y^2}$.

16. $\dfrac{a-2x}{a+2x}-\dfrac{a+2x}{a-2x}+\dfrac{8ax}{a^2+4x^2}$.

17. $\dfrac{3x+1}{x-3}-\dfrac{x-3}{3x+9}+\dfrac{5x^2+24x}{18-2x^2}$.

18. $\dfrac{4a-b}{1-4ab}-\dfrac{4a+b}{1+4ab}-\dfrac{4b(1-8a^2)}{16a^2b^2-1}$.

19. $\dfrac{x}{x-2a}+\dfrac{x}{x+2a}+\dfrac{2x^2}{x^2+4a^2}$.

20. $\dfrac{b}{a-b}+\dfrac{b}{a+b}+\dfrac{2ab}{a^2+b^2}+\dfrac{4a^3b}{a^4+b^4}$.

21. $\dfrac{x}{3x-y}+\dfrac{x}{3x+y}+\dfrac{6x^2}{9x^2+y^2}$.

22. $\dfrac{1}{x-3a}-\dfrac{1}{2x+6a}-\dfrac{x-9a}{2x^2+18a^2}$.

23. $\dfrac{(a^2+b^2)^2}{ab(a-b)^2}-\dfrac{a}{b}-\dfrac{b}{a}-2$ 24. $\dfrac{1}{x-1}-\dfrac{1}{x+1}+\dfrac{1}{x-2}-\dfrac{1}{x+2}$

25. $\dfrac{1}{x-a}-\dfrac{2}{2x+a}+\dfrac{1}{x+a}-\dfrac{2}{2x-a}$.

26. $\dfrac{3}{a-x}-\dfrac{1}{x+3a}+\dfrac{3}{a+x}+\dfrac{1}{x-3a}$.

27. $\dfrac{2}{x-1}-\dfrac{x}{x^2+1}-\dfrac{1}{x+1}+\dfrac{3}{1-x^2}$.

28. $\dfrac{a-c}{(a-b)(x-a)}+\dfrac{b-c}{(b-a)(x-b)}$

29. $\dfrac{1}{x^2-3x+2}+\dfrac{1}{x^2-5x+6}+\dfrac{2}{x^2-8x+15}$

30. $\dfrac{1}{x^2+5ax+4a^2}+\dfrac{1}{x^2+11ax+28a^2}+\dfrac{2}{x^2+20ax+91a^2}$.

31. $\dfrac{1}{x^2+3x+2}+\dfrac{2x}{x^2+4x+3}+\dfrac{1}{x^2+5x+6}$.

32. $\dfrac{1}{1-x+x^2}-\dfrac{1}{1+x+x^2}-\dfrac{2x}{1+x^2+x^4}$.

33. $\dfrac{1}{1+x+x^2}-\dfrac{1}{1-x+x^2}+\dfrac{2x}{1-x^2+x^4}$

34. $\dfrac{1}{x-2}-\dfrac{x-2}{x^2+2x+4}+\dfrac{6x}{x^3+8}$

35. $\dfrac{1}{2x^2-6ax+9a^2}-\dfrac{1}{2x^2+6ax+9a^2}+\dfrac{12ax}{4x^4-81a^4}$

6. Multiplication of fractions.

Let $\dfrac{a}{b}$ and $\dfrac{c}{d}$ be any two fractions; to find the value of $\dfrac{a}{b}\times\dfrac{c}{d}$.

Let $x=\dfrac{a}{b}\times\dfrac{c}{d}$.

Then we have $x \times b \times d = \frac{a}{b} \times \frac{c}{d} \times b \times d$

$$= \frac{a}{b} \times b \times \frac{c}{d} \times d$$

$$= \left(\frac{a}{b} \times b\right) \times \left(\frac{c}{d} \times d\right)$$

$$= a \times c,$$

or, $\quad x \times bd = ac,$

$\therefore \quad x = \frac{ac}{bd},$

i e., $\quad \frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}.$

Hence, $\quad \frac{a}{b} \times \frac{c}{d} \times \frac{e}{f} = \frac{ac}{bd} \times \frac{e}{f} = \frac{ace}{bdf},$

$\frac{a}{b} \times \frac{c}{d} \times \frac{e}{f} \times \frac{g}{h} = \frac{ace}{bdf} \times \frac{g}{h} = \frac{aceg}{bdfh}$, and so on.

Thus, *the product of any number of fractions is a fraction whose numerator is the product of the numerators of the given fractions and whose denominator is the product of their denominators.*

Cor Since $c = \frac{c}{1}$, we have $\frac{a}{b} \times c = \frac{a}{b} \times \frac{c}{1} = \frac{ac}{b}$.

Example 1. Multiply together $\frac{x^2}{yz}$, $\frac{y^2}{zx}$ and $\frac{z^2}{xy}$.

The required product $= \frac{x^2 \times y^2 \times z^2}{yz \times zx \times xy} = \frac{x^2 \times y^2 \times z^2}{y^2 \times z^2 \times x^2} = 1.$

Example 2. Multiply $\frac{x(a-x)}{a^2+2ax+x^2}$ by $\frac{a(a+x)}{a^2-2ax+x^2}$.

The required product $= \frac{x(a-x) \times a(a+x)}{(a^2+2ax+x^2)(a^2-2ax+x^2)}$

$$= \frac{ax(a-x)(a+x)}{(a+x)^2(a-x)^2}$$

$$= \frac{ax}{(a+x)(a-x)} = \frac{ax}{a^2-x^2}.$$

Example 3. Multiply together

$$\frac{1-x^2}{1+y},\ \frac{1-y^2}{x+x^2} \text{ and } 1+\frac{x}{1-x}.$$

Since $$1+\frac{x}{1-x}=\frac{1-x+x}{1-x}=\frac{1}{1+x},$$

the required product $$=\frac{(1+x)(1-x)}{1+y}\times\frac{(1+y)(1-y)}{x(1+x)}\times\frac{1}{1-x}$$

$$=\frac{(1+x)(1-x)(1+y)(1-y)}{(1+y)x(1+x)(1-x)}=\frac{1-y}{x}.$$

Exercise (59).

Multiply together :—

1. $\dfrac{2a^2}{3ab},\ \dfrac{9b^2}{16ac}$ and $\dfrac{8c^2}{9bc}$.
2. $\dfrac{4a^2b^2}{3c^2},\ \dfrac{9c^2}{16d^2}$ and $\dfrac{4d^2}{27b^2}$.
3. $\dfrac{x^3}{yz},\ \dfrac{y^3}{zx}$ and $\dfrac{z^3}{xy}$.
4. $\dfrac{7a^2b^2c^2}{12xyz}$ and $\dfrac{4x^3y^3z^3}{21a^4b^4c^4}$.
5. $\dfrac{12m^2n^3}{7xy^2z}$ and $\dfrac{35x^3yz}{96m^3n}$.

Simplify the following :—

6. $\dfrac{x+1}{x-1}\times\dfrac{x^2+x-2}{x^2+x}$.
7. $\dfrac{a^2-9b^2}{a^2+3ab}\times\dfrac{3a^2}{a^2-3ab}$.
8. $\dfrac{a^3-b^3}{a^2+ab}\times\dfrac{(a+b)^2}{a^2+ab+b^2}$.
9. $\dfrac{a^3+8x^3}{a^3-2a^2x}\times\dfrac{a^2-4ax+4x^2}{a^2-2ax+4x^2}$.
10. $\dfrac{x^2+4x+3}{x^2-4}\times\dfrac{x^2-3x+2}{x^2-9}$.
11. $\dfrac{x^2-7x+10}{x^2-2x-15}\times\dfrac{x^2-3x-18}{x^2-8x+12}$.
12. $\dfrac{x^2-4x+3}{x^2-6x+5}\times\dfrac{x^2-7x+10}{x^2-5x+6}$.
13. $\dfrac{a^4-b^4}{a^2-2ab+b^2}\times\dfrac{a-b}{a^2+ab}$.
14. $\dfrac{2x^2-5x+2}{3x^2-5x-2}\times\dfrac{3x^2+x}{4x-2}$.
15. $\dfrac{x^2-6x-16}{x^2-4x-21}\times\dfrac{x^2-11x+28}{x^2-12x+32}$.

16. $\dfrac{a^2-x^2}{a+b} \times \dfrac{a^2-b^2}{ax+x^2} \times \left(a+\dfrac{ax}{a-x}\right)$.

17. $\left(\dfrac{x^2}{a^2}-\dfrac{x}{a}+1\right)\left(\dfrac{x^2}{a^2}+\dfrac{x}{a}+1\right)$ 18. $\left(\dfrac{4a}{3x}+\dfrac{3x}{2b}\right)\left(\dfrac{2b}{3x}+\dfrac{3x}{4a}\right)$.

19. $\left(\dfrac{a}{b}+\dfrac{b}{a}\right)\left(\dfrac{c}{d}+\dfrac{d}{c}\right)-\left(\dfrac{a}{b}-\dfrac{b}{a}\right)\left(\dfrac{c}{d}-\dfrac{d}{c}\right)$

20. $\dfrac{2x^2-7x+3}{2x^2+7x-4} \times \dfrac{3x^2+11x-4}{3x^2+8x-3} \times \dfrac{2x^2+x-15}{2x^2-11x+15}$.

21. $\dfrac{b^2-c^2-a^2+2ac}{c^2+a^2-b^2+2ac} \times \dfrac{b^2+c^2-a^2-2bc}{a^2-b^2+c^2-2ac}$.

22. $\dfrac{c^2-a^2-b^2+2ab}{b^2-c^2-a^2+2ac} \times \dfrac{a^2-b^2+c^2-2ac}{a^2+b^2-c^2-2ab}$.

7. Division of fractions.

Let $\dfrac{a}{b}$ and $\dfrac{c}{d}$ be any two fractions, to find the value of $\dfrac{a}{b} \div \dfrac{c}{d}$

$$\text{Let } x = \frac{a}{b} \div \frac{c}{d}.$$

Then we have $x \times \dfrac{c}{d} = \dfrac{a}{b} \div \dfrac{c}{d} \times \dfrac{c}{d}$

$= \dfrac{a}{b}$ [$\because$ $m \div n \times n = m$, whatever m and n may be]

$$\therefore \quad x \times \frac{c}{d} \times \frac{d}{c} = \frac{a}{b} \times \frac{d}{c},$$

or, $x = \dfrac{a}{b} \times \dfrac{d}{c}$ [$\because$ $\dfrac{c}{d} \times \dfrac{d}{c} = 1$]

Thus, *to divide one fraction by another we have to multiply the former by the reciprocal of the latter.*

Cor. $\dfrac{a}{b} \div c = \dfrac{a}{b} \div \dfrac{c}{1} = \dfrac{a}{b} \times \dfrac{1}{c} = \dfrac{a}{bc}$.

Example 1. Simplify $\dfrac{a^3+b^3}{a^2-b^2} \div \dfrac{a^2-ab+b}{a-b}$

The required result $= \dfrac{a^3+b^3}{a^2-b^2} \times \dfrac{a-b}{a^2-ab+}$

$$= \frac{(a^3+b^3)(a-b)}{(a^2-b^2)(a^2-ab+b^2)}$$

$$= \frac{(a+b)(a^2-ab+b^2)(a-b)}{(a+b)(a-b)(a^2-ab+b^2)}$$

$$= 1.$$

Example 2 Simplify

$$\frac{x^2+x-2}{x^2+7x+12} \div \frac{x^2-3x-10}{x^2+\ x-12} \times \frac{x^2-4x-5}{x^2-4x+3}.$$

The required result

$$= \frac{x^2+x-2}{x^2+7x+12} \times \frac{x^2+\ x-12}{x^2-3x-10} \times \frac{x^2-4x-5}{x^2-4x+3}$$

$$= \frac{(x-1)(x+2)}{(x+3)(x+4)} \times \frac{(x+4)(x-3)}{(x-5)(x+2)} \times \frac{(x-5)(x+1)}{(x-3)(x-1)}$$

$$= \frac{(x-1)(x+2)(x+4)(x-3)(x-5)(x+1)}{(x+3)(x+4)(x-5)(x+2)(x-3)(x-1)}$$

$$= \frac{x+1}{x+3}.$$

Example 3. Simplify

$$\frac{\dfrac{a}{a-b}-\dfrac{a}{a+b}}{\dfrac{b}{a-b}-\dfrac{b}{a+b}} \div \frac{\dfrac{a+b}{a-b}+\dfrac{a-b}{a+b}}{\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}} \times \frac{a^2}{a^2+b^2}.$$

(Calcutta University Entrance Paper, 1876)

We have $\dfrac{\dfrac{a}{a-b}-\dfrac{a}{a+b}}{\dfrac{b}{a-b}-\dfrac{b}{a+b}} = \dfrac{\dfrac{a(a+b)-a(a-b)}{a^2-b^2}}{\dfrac{b(a+b)-b(a-b)}{a^2-b^2}}$

$$= \frac{2ab}{a^2-b^2} \div \frac{2b^2}{a^2-b^2}$$

$$= \frac{2ab}{a^2-b^2} \times \frac{a^2-b^2}{2b^2}$$

$$= \frac{a}{b} \; ; \quad \dots \quad \dots \quad \dots \quad \dots \quad (1)$$

and $$\frac{\dfrac{a+b}{a-b}+\dfrac{a-b}{a+b}}{\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}}=\frac{\dfrac{(a+b)^2+(a-b)^2}{a^2-b^2}}{\dfrac{(a+b)^2-(a-b)^2}{a^2-b^2}}$$

$$=\frac{2(a^2+b^2)}{a^2-b^2}\div\frac{4ab}{a^2-b^2}$$

$$=\frac{2(a^2+b^2)}{a^2-b^2}\times\frac{a^2-b^2}{4ab}$$

$$=\frac{a^2+b^2}{2ab} \qquad . \quad . \, . \qquad (2)$$

Hence, from (1) and (2) the given expression

$$=\frac{a}{b}\div\frac{a^2+b^2}{2ab}\times\frac{a^2}{a^2+b^2}$$

$$=\frac{a}{b}\times\frac{2ab}{a^2+b^2}\times\frac{a^2}{a^2+b^2}$$

$$=\frac{2a^4}{(a^2+b^2)^2}$$

Exercise (60).

Simplify

1. $\dfrac{4a^2bc}{15xy^2z}\div\dfrac{8ab^2c}{25x^2yz}$

2. $\dfrac{a^2+ab}{a-b}\div\dfrac{ab}{a^2-b^2}$.

3. $\dfrac{x^2-49}{x^2-25}\div\dfrac{x+7}{x+5}$

4. $\dfrac{a^4-b^4}{a^2+2ab+b^2}\div\dfrac{a^2+b^2}{a+b}$

5. $\dfrac{m^2-9n^2}{m^2+5mn+6n^2}\div\dfrac{m^2-2mn-3n^2}{m^2-n^2}$.

6. $\dfrac{m^3-n^3}{m+n}\div\dfrac{m^2+mn+n^2}{m^2-n^2}$

7. $\left(\dfrac{2x+y}{x+y}-1\right)\div\left(1-\dfrac{x}{x+y}\right)$

8. $\left(\dfrac{a}{a+b}+\dfrac{b}{a-b}\right)\div\left(\dfrac{a}{a-b}-\dfrac{b}{a+b}\right)$

9. $\left(\dfrac{x+y}{x-y}+\dfrac{x-y}{x+y}\right)\div\left(\dfrac{x+y}{x-y}-\dfrac{x-y}{x+y}\right)$.

10. $\dfrac{x^2-4}{x^2+3x-18} \div \dfrac{x^2-5x-14}{x^2-36}$.

11. $\left\{1-\dfrac{2pq}{p^2+q^2}\right\} \div \left\{\dfrac{p^3-q^3}{p-q}-3pq\right\}$.

12. $\dfrac{a^3+b^3+3ab(a+b)}{(a+b)^2-4ab} \div \dfrac{(a-b)^2+4ab}{a^3-b^3-3ab(a-b)}$.

13. $\dfrac{x^3+y^3}{(x-y)^2+3xy} \div \dfrac{(x+y)^2-3xy}{x^3-y^3} \times \dfrac{xy}{x^2-y^2}$.

14. $\dfrac{a(a-b)^2+4a^2b}{ab+b^2} \div \dfrac{a^2-b^2}{ab} \times \dfrac{b(a+b)^2-4ab^2}{a^2-ab}$.

15. $\dfrac{x^2-x-30}{x^2-36} \div \dfrac{x^2+3x-10}{x^2+2x-8} \div \dfrac{x+4}{2x^2+12x}$.

16. $\dfrac{x^2+3x-108}{x^2-64} \div \dfrac{x^2+6x-72}{x^2+x-56} \div \dfrac{x^2-16x+63}{x^2-14x+48}$.

17. $\left(\dfrac{x^2+y^2}{x^2-y^2}-\dfrac{x^2-y^2}{x^2+y^2}\right) \div \left(\dfrac{x+y}{x-y}-\dfrac{x-y}{x+y}\right)$.

18. $\left\{\dfrac{a+b}{a-b}+\dfrac{a^2+b^2}{a^2-b^2}\right\} \div \left\{\dfrac{a-b}{a+b}-\dfrac{a^3-b^3}{a^3+b^3}\right\}$.

19. $\dfrac{a^4-b^4}{(a+b)^3-3ab(a+b)} \div \dfrac{(a+b)^2-4ab}{(a+b)^2-3ab} \times \dfrac{a}{(a+b)^2-2ab}$.

20. $\dfrac{(a-b)\{(a+b)^2-ab\}}{(a-b)^2+2ab} \div \dfrac{(a-b)^2+3ab}{(a+b)\{(a-b)^2+ab\}}$

$$\times \dfrac{(a+b)^2-2ab}{(a+b)^2-3ab}.$$

21. $$\dfrac{\dfrac{a^3}{b^3}-\dfrac{b^3}{a^3}}{\left(\dfrac{a}{b}-\dfrac{b}{a}\right)\left(\dfrac{a}{b}+\dfrac{b}{a}-1\right)} \times \dfrac{\dfrac{1}{b}-\dfrac{1}{a}}{\dfrac{1}{a^2}+\dfrac{1}{b^2}+\dfrac{1}{ab}}.$$

(Calcutta University Entrance Paper, 1874.)

*8. Miscellaneous Examples.

Example 1 Simplify $-1+\dfrac{a}{2(a+b)-\dfrac{a+b}{1-\dfrac{b}{a+b}}}$.

Since $1-\frac{b}{a+b}=\frac{a+b-b}{a+b}=\frac{a}{a+b}$, we have the given expression $=-1+\cfrac{a}{2(a+b)-\cfrac{a+b}{\left(\frac{a}{a+b}\right)}}$

$$=-1+\cfrac{a}{2(a+b)-\cfrac{(a+b)^2}{a}}$$

$$=-1+\cfrac{a}{\cfrac{(2a^2+2ab)-(a^2+2ab+b^2)}{a}}$$

$$=-1+\frac{a^2}{a^2-b^2}=\frac{-a^2+b^2+a^2}{a^2-b^2}=\frac{b^2}{a^2-b^2}.$$

Example 2 Show that

$$\frac{x}{x^2+a^2}=\frac{1}{x}-\frac{a^2}{x^3}+\frac{a^4}{x^5}-\frac{a^6}{x^7}+\frac{a^8}{x^7(x^2+a^2)}.$$

Let us divide x by x^2+a^2 —

$$x^2+a^2\Big)\,x\,\Big(\frac{1}{x}-\frac{a^2}{x^3}+\frac{a^4}{x^5}-\frac{a^6}{x^7}$$

$$\begin{array}{l}
x+\frac{a^2}{x} \\
\hline
-\frac{a^2}{x} \\
-\frac{a^2}{x}-\frac{a^4}{x^3} \\
\hline
\qquad\frac{a^4}{x^3} \\
\qquad\frac{a^4}{x^3}+\frac{a^6}{x^5} \\
\hline
\qquad\quad-\frac{a^6}{x^5} \\
\qquad\quad-\frac{a^6}{x^5}-\frac{a^8}{x^7} \\
\hline
\qquad\qquad\quad\frac{a^8}{x^7}
\end{array}$$

Hence, proceeding no further with the division, we have

$$\frac{x}{x^2+a^2} = \frac{1}{x}-\frac{a^2}{x^3}+\frac{a^4}{x^5}-\frac{a^6}{x^7}+\frac{\left(\frac{a^8}{x^7}\right)}{x^2+a^2}$$
$$= \frac{1}{x}-\frac{a^2}{x^3}+\frac{a^4}{x^5}-\frac{a^6}{x^7}+\frac{a^8}{x^7(x^2+a^2)}.$$

Example 3. Find the value of

$$\frac{x+2a}{x-2a}+\frac{x+2b}{x-2b}, \quad \text{when } x = \frac{4ab}{a+b}.$$

The given expression

$$= \left(\frac{x+2a}{x-2a}-1\right)+\left(\frac{x+2b}{x-2b}-1\right)+2$$
$$= \frac{4a}{x-2a}+\frac{4b}{x-2b}+2$$
$$= \frac{4}{(x-2a)(x-2b)}\left\{a\left(x-2b\right)+b\left(x-2a\right)\right\}+2$$
$$= \frac{4}{(x-2a)(x-2b)}\left\{\left(a+b\right)x-4ab\right\}+2$$
$$= 0+2 \qquad [\because (a+b)x = 4ab]$$
$$= 2$$

Example 4. Find the value of

$$\frac{x^{3n}}{x^n-1}-\frac{x^{2n}}{x^n+1}-\frac{1}{x^n-1}+\frac{1}{x^n+1}.$$

The given expression $= \left(\frac{x^{3n}}{x^n-1}-\frac{1}{x^n-1}\right)-\left(\frac{x^{2n}}{x^n+1}-\frac{1}{x^n+1}\right)$

$$= \frac{x^{3n}-1}{x^n-1}-\frac{x^{2n}-1}{x^n+1}$$
$$= \frac{(x^n-1)(x^{2n}+x^n+1)}{x^n-1}-\frac{(x^n+1)(x^n-1)}{x^n+1}$$
$$= (x^{2n}+x^n+1)-(x^n-1) = x^{2n}+2.$$

Example 5. Reduce to its simplest form

$$\frac{x^2-(y-z)^2}{(x+z)^2-y^2}+\frac{y^2-(x-z)^2}{(x+y)^2-z^2}+\frac{z^2-(x-y)^2}{(y+z)^2-x^2}.$$

We have 1st fraction $= \dfrac{\{x+(y-z)\}\{x-(y-z)\}}{\{(x+z)+y\}\{(x+z)-y\}}$

$$= \frac{(x+y-z)(x-y+z)}{(x+z+y)(x+z-y)} = \frac{x+y-z}{x+y+z}.$$

Similarly, 2nd fraction $= \dfrac{(y+x-z)(y-x+z)}{(x+y+z)(x+y-z)} = \dfrac{y-x+z}{x+y+z}.$

and 3rd fraction $= \dfrac{(z+x-y)(z-x+y)}{(y+z+x)(y+z-x)} = \dfrac{z+x-y}{x+y+z}.$

Hence, the given expression

$$= \frac{(x+y-z)+(y-x+z)+(z+x-y)}{x+y+z}$$

$$= \frac{x+y+z}{x+y+z} = 1.$$

Example 6. Show that $\left(\dfrac{b}{c}+\dfrac{c}{b}\right)^2+\left(\dfrac{c}{a}+\dfrac{a}{c}\right)^2$

$$+\left(\frac{a}{b}+\frac{b}{a}\right)^2 = 4+\left(\frac{b}{c}+\frac{c}{b}\right)\left(\frac{c}{a}+\frac{a}{c}\right)\left(\frac{a}{b}+\frac{b}{a}\right).$$

We have $\left(\dfrac{c}{a}+\dfrac{a}{c}\right)^2+\left(\dfrac{a}{b}+\dfrac{b}{a}\right)^2$

$$= \left\{\frac{c^2}{a^2}+2+\frac{a^2}{c^2}\right\}+\left\{\frac{a^2}{b^2}+2+\frac{b^2}{a^2}\right\}$$

$$= 4+a^2\left(\frac{1}{b^2}+\frac{1}{c^2}\right)+\frac{1}{a^2}\left(b^2+c^2\right)$$

$$= 4+\frac{a^2}{bc}\left(\frac{bc}{b^2}+\frac{bc}{c^2}\right)+\frac{bc}{a^2}\left(\frac{b^2}{bc}+\frac{c^2}{bc}\right)$$

$$= 4+\frac{a^2}{bc}\left(\frac{c}{b}+\frac{b}{c}\right)+\frac{bc}{a^2}\left(\frac{b}{c}+\frac{c}{b}\right)$$

$$= 4+\left(\frac{b}{c}+\frac{c}{b}\right)\left(\frac{a^2}{bc}+\frac{bc}{a^2}\right).$$

$\therefore$ the given expression

$$= 4+\left(\frac{b}{c}+\frac{c}{b}\right)\left\{\left(\frac{b}{c}+\frac{c}{b}\right)+\left(\frac{a^2}{bc}+\frac{bc}{a^2}\right)\right\}$$

$$= 4+\left(\frac{b}{c}+\frac{c}{b}\right)\left\{\left(\frac{b}{c}+\frac{bc}{a^2}\right)+\left(\frac{c}{b}+\frac{a^2}{bc}\right)\right\}$$

$$= 4+\left(\frac{b}{c}+\frac{c}{b}\right)\left\{\frac{b}{a}\left(\frac{a}{c}+\frac{c}{a}\right)+\frac{a}{b}\left(\frac{c}{a}+\frac{a}{c}\right)\right\}$$

$$= 4+\left(\frac{b}{c}+\frac{c}{b}\right)\left(\frac{c}{a}+\frac{a}{c}\right)\left(\frac{a}{b}+\frac{b}{a}\right).$$

Example 7. If $2s = a+b+c$, show that

$$\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c}-\frac{1}{s} = \frac{abc}{s(s-a)(s-b)(s-c)}.$$

We have $$\frac{1}{s-a}+\frac{1}{s-b} = \frac{2s-a-b}{(s-a)(s-b)} = \frac{c}{(s-a)(s-b)}.$$

and $$\frac{1}{s-c}-\frac{1}{s} = \frac{s-(s-c)}{s(s-c)} = \frac{c}{s(s-c)}.$$

Hence, the given expression

$$= \frac{c}{(s-a)(s-b)}+\frac{c}{s(s-c)}$$

$$= c\,\frac{s(s-c)+(s-a)(s-b)}{s(s-a)(s-b)(s-c)}$$

$$= c\,\frac{2s^2-s(a+b+c)+ab}{s(s-a)(s-b)(s-c)}$$

$$= \frac{abc}{s(s-a)(s-b)(s-c)} \qquad [\; 2s^2-s(a+b+c) = 2s^2-2s^2 = 0\;]$$

Example 8. Simplify

$$\frac{a+b}{ab}\left(a^2+b^2-c^2\right)+\frac{b+c}{bc}\left(b^2+c^2-a^2\right)+\frac{c+a}{ca}\left(c^2+a^2-b^2\right).$$

Putting $2s^2$ for $a^2+b^2+c^2$, we have

$$a^2+b^2-c^2 = (a^2+b^2+c^2)-2c^2 = 2(s^2-c^2),$$
$$b^2+c^2-a^2 = (a^2+b^2+c^2)-2a^2 = 2(s^2-a^2),$$
$$c^2+a^2-b^2 = (a^2+b^2+c^2)-2b^2 = 2(s^2-b^2).$$

Hence, the given expression

$$= 2\left(\frac{1}{b}+\frac{1}{a}\right)\left(s^2-c^2\right)+2\left(\frac{1}{c}+\frac{1}{b}\right)\left(s^2-a^2\right)$$
$$+2\left(\frac{1}{a}+\frac{1}{c}\right)\left(s^2-b^2\right)$$

$$= 2\left\{\frac{1}{a}\left(2s^2-b^2-c^2\right)+\frac{1}{b}\left(2s^2-c^2-a^2\right)+\frac{1}{c}\left(2s^2-a^2-b^2\right)\right\}$$

$$= 2\left\{\frac{1}{a}a^2+\frac{1}{b}b^2+\frac{1}{c}c^2\right\}$$

$$= 2(a+b+c).$$

Example 9 Show that

$$\frac{a}{a^2-1}+\frac{a^2}{a^4-1}+\frac{a^4}{a^8-1} = \frac{1}{2}\left(\frac{a+1}{a-1}-\frac{a^8+1}{a^8-1}\right).$$

$$\frac{a}{a^2-1} = \frac{1}{2}\frac{2a}{a^2-1} = \frac{1}{2}\cdot\frac{(a+1)^2-(a^2+1)}{a^2-1}$$

$$= \frac{1}{2}\cdot\left(\frac{a+1}{a-1}-\frac{a^2+1}{a^2-1}\right);$$

$$\frac{a^2}{a^4-1} = \frac{1}{2}\frac{2a^2}{a^4-1} = \frac{1}{2}\cdot\frac{(a^2+1)^2-(a^4+1)}{a^4-1}$$

$$= \frac{1}{2}\left(\frac{a^2+1}{a^2-1}-\frac{a^4+1}{a^4-1}\right);$$

$$\frac{a^4}{a^8-1} = \frac{1}{2}\frac{2a^4}{a^8-1} = \frac{1}{2}\frac{(a^4+1)^2-(a^8+1)}{a^8-1}$$

$$= \frac{1}{2}\left(\frac{a^4+1}{a^4-1}-\frac{a^8+1}{a^8-1}\right).$$

Hence, the given expression

$$= \frac{1}{2}\cdot\left\{\left(\frac{a+1}{a-1}-\frac{a^2+1}{a^2-1}\right)+\left(\frac{a^2+1}{a^2-1}-\frac{a^4+1}{a^4-1}\right)+\left(\frac{a^4+1}{a^4-1}-\frac{a^8+1}{a^8-1}\right)\right\}$$

$$= \frac{1}{2}\cdot\left\{\frac{a+1}{a-1}-\frac{a^8+1}{a^8-1}\right\}.$$

Example 10. Reduce to its simplest form

$$bc\frac{a+d}{(a-b)(a-c)}+ac\frac{b+d}{(b-a)(b-c)}+ab.\frac{c+d}{(c-a)(c-b)}.$$

Since $b-a=-(a-b)$

and $(c-a)(c-b)=[-(a-c)]\times[-(b-c)]=(a-c)(b-c).$

the given expression

$$= bc.\frac{a+d}{(a-b)(a-c)}+ac\frac{-(b+d)}{(a-b)(b-c)}+ab\frac{c+d}{(a-c)(b-c)}$$

$$= \frac{bc(a+d)(b-c)-ac(b+d)(a-c)+ab(c+d)(a-b)}{(a-b)(a-c)(b-c)}.$$

Now, the numerator $= abc\{(b-c)-(a-c)+(a-b)\}$

$$+d\{bc(b-c)-ac(a-c)+ab(a-b)\}$$

$$= d\{bc(b-c)-ac(a-c)+ab(a-b)\}$$

$$= d\{a^2(b-c)+b^2(c-a)+c^2(a-b)\}$$

$$= d(a-b)(a-c)(b-c).$$

[Formula 19, page 96]

Hence, the given expression $= d$

Example 11. Simplify

$$\frac{a^2}{(a-b)(a-c)(x+a)}+\frac{b^2}{(b-a)(b-c)(x+b)}+\frac{c^2}{(c-a)(c-b)(x+c)}.$$

The given expression

$$= \frac{a^2}{(a-b)(a-c)(x+a)}+\frac{b^2}{(a-b)(b-c)(x+b)}+\frac{c^2}{(a-c)(b-c)(x+c)}$$

$$= \frac{a^2(b-c)(x+b)(x+c)-b^2(a-c)(x+c)(x+a)+c^2(a-b)(x+a)(x+b)}{(a-b)(a-c)(b-c)(x+a)(x+b)(x+c)}.$$

Now, the numerator

$$= a^2(b-c)\{x^2+x(b+c)+ac\}$$

$$+b^2(c-a)\{x^2+x(c+a)+ca\}$$

$$+c^2(a-b)\{x^2+x(a+b)+ab\}$$

$$= x^2\{a^2(b-c)+b^2(c-a)+c^2(a-b)\}$$

$$+x\{a^2(b^2-c^2)+b^2(c^2-a^2)+c^2(a^2-b^2)\}$$

$$+abc\{a(b-c)+b(c-a)+c(a-b)\}$$

$$= x^2\{a^2(b-c)+b^2(c-a)+c^2(a-b)\}$$

$$= x^2(a-b)(a-c)(b-c).$$

[Formula 19, page 96]

Hence, the given expression $= \dfrac{x^2}{(x+a)(x+b)(x+c)}$.

Example 12. Simplify

$$\frac{a^3}{(a-b)(a-c)}+\frac{b^3}{(b-c)(b-a)}+\frac{c^3}{(c-a)(c-b)}.$$

(Calcutta University Entrance Paper, 1887)

The given expression

$$=\frac{a^3}{(a-b)(a-c)}+\frac{-b^3}{(b-c)(a-b)}+\frac{c^3}{(a-c)(b-c)}$$

$$=\frac{a^3(b-c)-b^3(a-c)+c^3(a-b)}{(a-b)(a-c)(b-c)}$$

Now, the numerator $= a^3(b-c)+b^3(c-a)+c^3(a-b)$

$= (a-b)(a-c)(b-c)(a+b+c)$ [See Example 9, page 125]

Hence, the given expression $= a+b+c$.

† Alternative Method —

Since $\dfrac{1}{(a-b)(a-c)}=\dfrac{1}{(a-b)(b-c)}-\dfrac{1}{(a-c)(b-c)}$,

∴ the given expression

$$=\left\{\frac{a^3}{(a-b)(b-c)}-\frac{a^3}{(a-c)(b-c)}\right\}+\frac{-b^3}{(a-b)(b-c)}+\frac{c^3}{(a-c)(b-c)}$$

$$=\frac{a^3-b^3}{(a-b)(b-c)}-\frac{a^3-c^3}{(a-c)(b-c)}$$

$$=\frac{a^2+ab+b^2}{b-c}-\frac{a^2+ac+c^2}{b-c}$$

$$=\frac{a(b-c)+(b^2-c^2)}{b-c}=a+b+c.$$

Exercise (61).

Simplify —

1. $\dfrac{a}{b+\dfrac{c}{d+\dfrac{e}{f}}}$.

2. $\dfrac{x}{x-\dfrac{x-1}{1-\dfrac{1}{x+1}}}$.

3. $a^2+\dfrac{b^4}{a^2-\dfrac{a^3+b^3}{a+\dfrac{b^2}{a-b}}}$.

4. $\dfrac{m}{m^2-\dfrac{m^3-1}{m+\dfrac{1}{m+1}}}$.

† This method is due to my friend and pupil Babu Bimala Charan Shome, Head assistant, Forest Surveys, Dehra Dun

5. $$\frac{x^2-2xy+y^2}{\dfrac{x+y}{x+y-\dfrac{(x-y)^2}{x+y}}-1}.$$ 6. $$\frac{x^2(x+2)}{4x+8+\dfrac{2x^4-32}{x+2+\dfrac{(x-2)(x-2)}{x+2}}}.$$

N B In the following 14 examples the expression on the left of "=" in each case is to be transformed into that on the right

Prove that :—

7. $$\frac{a^3+b^3}{a-b} = a^2+ab+b^2+\frac{2b^3}{a-b}.$$

8. $$\frac{a^3-b^3}{a+b} = a^2-ab+b^2-\frac{2b^3}{a+b}.$$

9. $$\frac{a^2+2ab+3b^2}{a+b} = a+b+\frac{2b^2}{a+b}$$

10. $$\frac{a^2+2ab-3b^2}{a+b} = a+b-\frac{4b^2}{a+b}$$

11. $$\frac{x^6}{x^2+y^2} = x^4-x^2y^2+y^4-\frac{y^6}{x^2+y^2}$$

12. $$\frac{x^6}{x^2-y^2} = x^4+x^2y^2+y^4+\frac{y^6}{x^2-y^2}.$$

13. $$\frac{x^2yz+xy^2z+xyz^2}{x^2y^2z^2} = \frac{1}{yz}+\frac{1}{zx}+\frac{1}{xy}$$

14. $$\frac{xy^2z^2+yz^2x^2+zx^2y^2}{x^2y^2z^2} = \frac{1}{x}+\frac{1}{y}+\frac{1}{z}.$$

15. $$\frac{3a-6}{(a-1)(a-2)(a-3)}$$
$$=\frac{1}{(a-2)(a-3)}+\frac{1}{(a-3)(a-1)}+\frac{1}{(a-1)(a-2)}.$$

16. $$\frac{3x^2-14}{(x+1)(x+2)(x+3)}$$
$$=\frac{x-1}{(x+2)(x+3)}+\frac{x-2}{(x+3)(x+1)}+\frac{x-3}{(x+1)(x+2)}.$$

17. $$\frac{1-x}{1+x} = 1-2x+2x^2-2x^3+\frac{2x^4}{1+x}.$$

18. $$\frac{a}{x^2-a^2} = \frac{a}{x^2}+\frac{a^3}{x^4}+\frac{a^5}{x^6}+\frac{a^7}{x^6(x^2-a^2)}.$$

19. $$\frac{a^3}{x^3+a^3} = \frac{a^3}{x^3} - \frac{a^6}{x^6} + \frac{a^9}{x^9} - \frac{a^{12}}{x^9(x^3+a^3)}$$
$$= 1 - \frac{x^3}{a^3} + \frac{x^6}{a^6} - \frac{x^9}{a^9} + \frac{x^{12}}{a^9(x^3+a^3)}$$

20. $$\frac{x^4-1}{x+a} = x^3 - ax^2 + a^2x - a^3 + \frac{a^4-1}{x+a}$$

21. Find the value of
$$\frac{x+2a}{2b-x} + \frac{x-2a}{2b+x} + \frac{4ab}{x^2-4b^2}, \text{ when } x = \frac{ab}{a+b}.$$

22. Show that $\frac{x+a}{x-a} + \frac{x+b}{x-b} + \frac{x+c}{x-c} - 3$
$$= \frac{6abc}{(x-a)(x-b)(x-c)}, \text{ if } x = \frac{2(ab+bc+ca)}{a+b+c}.$$

23. If $x = \frac{ab+bc+ca}{a+b+c}$, show that
$$\frac{a+2x}{a-2x} + \frac{b+2x}{b-2x} + \frac{c+2x}{c-2x} + 3 = \frac{6abc}{(a-2x)(b-2x)(c-2x)}.$$

24. Find the value of
$$\frac{x^2-(b+c)x}{(x-b)(x-c)} + \frac{x^2-(c+a)x}{(x-c)(x-a)} + \frac{x^2-(a+b)x}{(x-a)(x-b)},$$
$$\text{when } x = \frac{3abc}{ab+bc+ca}.$$

25. Find the value of
$$\frac{x^2-y^2+x}{y^2-x^2+y}, \quad \text{when } x = \frac{a-b}{a+b} \text{ and } y = \frac{a+b}{a-b}$$

(Calcutta University Entrance Paper, 1883.)

[The given expression $= \frac{x(x+1)-y^2}{y(y+1)-x^2} =$ &c.]

26. Find the value of $\frac{x^4+3abx^2-10a^2b^2}{x^4+7abx^2+10a^2b^2} \times \frac{a^2+2ab+b^2}{a^2-2ab+b^2}$,

when $x^2 = a^2+b^2$.

27. Find the value of $\frac{x^2y^2+3(2x^2-y^2)ab-18a^2b^2}{y^4+9aby^2+18a^2b^2}$

$\times \frac{a^3-b^3}{a^3+b^3}$, when $x = a+b$ and $y = a-b$.

28. Find the value of

$$\frac{x^4+abx^2-2a^2b^2}{x^2y^2+(x^2+2y^2)ab+2a^2b^2}-\frac{a^2+ab+b^2}{a^2-ab+b^2},$$

when $x=a+b$ and $y=a-b$.

29. Simplify $\dfrac{x^6}{x^3+1}+\dfrac{x^6}{x^3-1}+\dfrac{1}{x^3+1}-\dfrac{1}{x^3-1}$

30. Simplify $\dfrac{x^2-(a-b)^2}{(x+b)^2-a^2}+\dfrac{a^2-(x-b)^2}{(x+a)^2-b^2}+\dfrac{b^2-(x-a)^2}{(a+b)^2-x^2}$.

31. Simplify $\dfrac{(a+2b)^2-b^2}{(a+b)^2-4b^2}+\dfrac{(a-b)^2-4b^2}{(a-2b)^2-b^2}+\dfrac{(2a+3b)^2-b^2}{(2a+b)^2-9b^2}$.

32. Simplify $\dfrac{x^4-(x-1)^2}{(x^2+1)^2-x^2}+\dfrac{x^2-(x^2-1)^2}{x^2(x+1)^2-1}+\dfrac{x^2(x-1)^2-1}{x^4-(x+1)^2}$.

33. If $2s=a+b+c$, shew that

$$1-\frac{a^2+b^2-c^2}{2ab}=\frac{2(s-a)(s-b)}{ab}.$$

34. Simplify $\dfrac{b-c}{a^2-(b-c)^2}+\dfrac{c-a}{b^2-(c-a)^2}+\dfrac{a-b}{c^2-(a-b)^2}$.

35. Simplify $\dfrac{a+b}{2ab}(a+b-c)+\dfrac{b+c}{2bc}(b+c-a)+\dfrac{c+a}{2ca}(c+a-b)$.

36. Simplify

$$\frac{x+y}{2xy}(x^2+y^2-z^2)+\frac{y+z}{2yz}(y^2+z^2-x^2)+\frac{z+x}{2zx}(z^2+x^2-y^2).$$

37. Simplify

$$\frac{a+b}{2ab}\left(a^3+b^3-c^3\right)+\frac{b+c}{2bc}\left(b^3+c^3-a^3\right)+\frac{c+a}{2ca}\left(c^3+a^3-b^3\right).$$

38. If $x=\dfrac{b^2+c^2-a^2}{2bc}$, $y=\dfrac{a^2+c^2-b^2}{2ac}$ and

$$z=\frac{a^2+b^2-c^2}{2ab},$$

find in its simplest form the value of $(b+c)x+(c+a)y+(a+b)z$.

39. If $p=\dfrac{a-b}{x-c}$, $q=\dfrac{b-c}{x-a}$, $r=\dfrac{c-a}{x-b}$, find the value of $p+q+r+pqr$.

40. Show that

$$\left(\frac{1}{b-c}+\frac{1}{c-a}+\frac{1}{a-b}\right)^2=\frac{1}{(b-c)^2}+\frac{1}{(c-a)^2}+\frac{1}{(a-b)^2}.$$

41. Show that $\dfrac{1}{1+x}+\dfrac{2x}{1+x^2}+\dfrac{4x^3}{1+x^4}+\dfrac{8x^7}{1+x^8}$

$$=\frac{1}{1-x}-\frac{16x^{16}}{1-x^{16}}.$$

Simplify —

42. $$\frac{1}{(a-b)(a-c)}+\frac{1}{(b-a)(b-c)}+\frac{1}{(c-a)(c-b)}.$$

43. $$\frac{a^2}{(a-b)(a-c)}+\frac{b^2}{(b-a)(b-c)}+\frac{c^2}{(c-a)(c-b)}.$$

44. $$\frac{x^2+yz}{(x-y)(x-z)}+\frac{y^2+zx}{(y-z)(y-x)}+\frac{z^2+xy}{(z-x)(z-y)}.$$

45. $$\frac{bc}{(a-b)(a-c)}+\frac{ca}{(b-c)(b-a)}+\frac{ab}{(c-a)(c-b)}.$$

46. $$\frac{x^2-yz}{(x-y)(x-z)}+\frac{y^2+zx}{(y+z)(y-x)}+\frac{z^2+xy}{(z-x)(z+y)}$$

47. $$\frac{1}{x(x-y)(x-z)}+\frac{1}{y(y-x)(y-z)}+\frac{1}{z(z-x)(z-y)}.$$

(Calcutta University Entrance Paper, 1872.)

48 $$\frac{1}{(a-b)(a-c)(x-a)}+\frac{1}{(b-a)(b-c)(x-b)}$$

$$+\frac{1}{(c-a)(c-b)(x-c)}.$$

49. $$\frac{a}{(a-b)(a-c)(x-a)}+\frac{b}{(b-a)(b-c)(x-b)}$$

$$+\frac{c}{(c-a)(c-b)(x-c)}.$$

50. $$\frac{a^2}{(a-b)(a-c)(x-a)}+\frac{b^2}{(b-a)(b-c)(x-b)}$$

$$+\frac{c^2}{(c-a)(c-b)(x-c)}.$$

51. $\dfrac{a^2+ha+k}{(a-b)(a-c)(x-a)}+\dfrac{b^2+hb+k}{(b-a)(b-c)(x-b)}$
$$+\frac{c^2+hc+k}{(c-a)(c-b)(x-c)}.$$

52. Show that
$$\frac{a^2\left(\frac{1}{b}-\frac{1}{c}\right)+b^2\left(\frac{1}{c}-\frac{1}{a}\right)+c^2\left(\frac{1}{a}-\frac{1}{b}\right)}{a\left(\frac{1}{b}-\frac{1}{c}\right)+b\left(\frac{1}{c}-\frac{1}{a}\right)+c\left(\frac{1}{a}-\frac{1}{b}\right)}=a+b+c.$$

53. Show that
$$\frac{a^3(b^2-c^2)+b^3(c^2-a^2)+c^3(a^2-b^2)}{a^2(b-c)+b^2(c-a)+c^2(a-b)}=ab+bc+ca.$$

54. Show that
$$\frac{a(a+b)(a+c)}{(a-b)(a-c)}+\frac{b(b+a)(b+c)}{(b-a)(b-c)}+\frac{c(c+a)(c+b)}{(c-a)(c-b)}=a+b+c.$$

55. Prove that $\dfrac{bc}{a(a^2-b^2)(a^2-c^2)}+\dfrac{ac}{b(b^2-a^2)(b^2-c^2)}$
$$+\frac{ab}{c(c^2-b^2)(c^2-a^2)}=\frac{1}{abc}.$$

56. Simplify
$$\frac{bc(x-a)^2}{(a-b)(a-c)}+\frac{ca(x-b)^2}{(b-c)(b-a)}+\frac{ab(x-c)^2}{(c-a)(c-b)}.$$

CHAPTER XII

SIMPLE EQUATIONS.

1. Definitions. Any two expressions connected by the sign of equality constitute an *equation*, and each of the expressions thus connected is called a *side* or *member* of the equation

The term equation, however, is hardly used in this extended sense. When one expression is put equal to another

the equality may hold *either* for all values of the letters involved as in $(a+b)(a-b) = a^2-b^2$ *or* for some particular values of the letters only, as in $4x = 8$ (which is true only when $x = 2$) The latter class of equations alone are called *equations* (more correctly *Equation of Condition*), whilst any equation of the former class is called an *Identity* (or an Identical Equation)

Thus $(x+1)+(2x+3) = 3x+4$ is an *Identity*,

whereas $(x+1)+(x+3) = 3x+2$ is an *Equation* ,

the former being true for *all values* of x, and the latter, *only when* $x = 2$

The letter, to which a particular value or values must be given in order that an equation may be true, is called the *unknown quantity* It is usually represented by one of the last letters of the alphabet x, y, z, &c

Any particular value of the unknown quantity, for which an equation is true, is said to satisfy the equation and is called a *root* or a *solution* of the equation.

To *solve* an equation is to find its root or roots

An equation containing only one unknown quantity is said to be an equation of the first degree or a *simple equation*, when the unknown quantity occurs only in the first power.

2. Axioms The process of solving an equation is primarily based upon the following axioms —

(1) If to equals the same quantity or equal quantities be added, the sums are equal.

(2) If from equals the same quantity or equal quantities be taken, the remainders are equal

(3) If equals be multiplied by the same quantity, or by equal quantities, the products are equal.

Cor 1. From axioms (1) and (2) we deduce an important principle which is of great use in solving equations, and which may be enunciated as follows —

Any term may be transposed from one side of an equation to the other by simply changing its sign.

For, let $x-a = b+c$;

then adding a to both sides we must have

$$x-a+a = b+c+a \qquad \text{[Axiom (1)]}$$

or, $$x = b+c+a;$$

again, subtracting c from both sides we have

$$x-a-c = b+c-c \qquad \text{[Axiom (2)]}$$
$$= b$$

Thus $-a$ removed from the left side appears as $+a$ on the right, and $+c$ removed from the right side appears as $-c$ on the left

Similarly, if $x-a = b-c+d$, we have

$$x-a-b+c-d = 0.$$

Such removal of terms is called **Transposition.**

***Cor* 2.** The sign of every term of an equation may be changed without destroying the equality

For, let $x-a = b+c$,

then $(x-a)\times(-1) = (b+c)\times(-1)$ [Axiom (3)]

or, $-x+a = -b-c$.

3. Simple Examples. We shall now work out some examples illustrating the general method of solving a simple equation by the application of the foregoing principles The unknown quantity will always be donoted by x

Example 1. Solve $3x+5 = x+19$

N. B. The question may be otherwise put as follows — "If $3x+5 = x+19$, what is the value of x?"

Since $$3x+5 = x+19,$$

by transposition, we must have

$$3x-x = 19-5$$

or, $$2x = 14,$$

and therefore, (dividing both sides by 2),

$$x = 7. \qquad \text{[Axiom (4)]}$$

Thus the required value of x is 7.

Example 2. Solve $(x+2)(3x+4)-6x = 10+(3x+2)(x+1)$.

The left side $= 3x^2+10x+8-6x$

$= 3x^2+4x+8$,

and the right side $= 10+3x^2+5x+2$

$= 3x^2+5x+12$

Hence the equation stands thus —

$$3x^2+4x+8 = 3x^2+5x+12.$$

Removing $3x^2$ from both sides, we have

$$4x+8 = 5x+12 \qquad \text{[Axiom (2)]}$$

Hence, by transposition,

$$4x-5x = 12-8$$

or, $-x = 4$

and $\therefore \; x = -4$ [Cor 2, last article]

Thus the required value of x is -4

Note The student can easily see for himself that when x has this value, each side of the given equation becomes equal to 40

Example 3. Given $\frac{x}{6}+5 = \frac{x}{3}+\frac{x}{4}$, find x

Since $\frac{x}{6}+5 = \frac{x}{3}+\frac{x}{4}$,

Multiplying both sides by 12, (which is the L C. M. of the denominators) we have

$$12\left(\frac{x}{6}+5\right) = 12\left(\frac{x}{3}+\frac{x}{4}\right) \qquad \text{[Axiom (3)]}$$

or, $2x+60 = 4x+3x = 7x$

Hence, by transposition,

$$2x-7x = -60$$

or, $-5x = -60$

and therefore, (dividing both sides by -5),

$$x = 12$$

Thus the required root is 12.

Example 4. Given $\frac{x-6}{8}-\frac{2x-15}{9}+1 = \frac{x}{15}-\frac{x-12}{6}$;

find x

Multiplying both sides by $8\times 9\times 5$ or 360, which is the C. M of the denominators, we have

$$\frac{360(x-6)}{8}-\frac{360(2x-15)}{9}+360=\frac{360x}{15}-\frac{360(x-12)}{6}$$

or, $45(x-6)-40(2x-15)+360=24x-60(x-12)$

or, $45x-270-80x+600+360=24x-60x+720$

or, $-35x+690=-36x+720.$

Hence, by transposition,

$$-35x+36x=720-690$$

or, $x=30.$

Example 5. Given $\dfrac{x}{a+b}+1=\dfrac{x}{a-b}+\dfrac{a-b}{a+b}$, find x.

Multiplying both sides by a^2-b^2, which is the L C. M. of the denominators, we have

$$(a-b)x+(a^2-b^2)=(a+b)x+(a-b)^2.$$

Hence, by transposition,

$$(a-b)x-(a+b)x=(a-b)^2-(a^2-b^2)$$

or, $\{(a-b)-(a+b)\}x=-2ab+2b^2$

or, $-2bx=-2b(a-b).$

Therefore, dividing both sides by $-2b$, we have

$$x=a-b.$$

Exercise (62).

Solve the following equations :—

1. $4x+3=2x+5$ **2.** $3x+2=x+6.$

3. $5x-6=2x+3.$ **4.** $15x-9=11x-25.$

5 $4(x-3)=2(x-6).$ **6.** $2(x-15)=5(x-11)+4.$

7. $19-3x=5x+35$

8. $3(x-2)+7(2x-3)=5(1-2x)-59.$

9. $13x-4(5x-8)+17=0$

10. $14(x-4)+3(x+5)=6(7-2x)+4$

11. $8(2x-7)-9(3x-14)=15.$

12. $3x-13(2x-13)=4x-20.$

13. $49+13(5x+27) = 8(5+x)-3x$

14. $16-5(7x-2) = 13(x-2)+4(13-x)$.

15. $8x+5(x+7)+9(2x+23)-3(x+6) = 0$.

16. $(x-7)(4x-29) = (2x-5)(2x-17)+1$

17. $(3x+2)(2x-6) = (4-3x)(1-2x)-10$

18. $(3x+5)(6x-7) = (3x+2)(9x-13)-(3x+1)(3x-1)$.

19. $(x+2)(2x+5) = 2(x+1)^2+13$

20. $(x+1)(4x-7)-(x-1)(x+5) = 3(x+2)^2+5$.

21. $3(x-4)^2+5(x-3)^2 = (2x-5)(4x-1)+24$

22. $(6x+9)^2+(8x-7)^2 = (10x+3)^2-71$.

(Calcutta University Entrance Paper. 1882.)

23. $5(x+1)^2+7(x+3)^2 = 12(x+2)^2$.

24. $(3x-14)^2+(4x-19)^2-(5x-23)^2 = 22$.

25. $(5x-8)^2+(12x-7)^2 = (13x-10)^2+37$.

26. $(x-1)^3+(x+1)^3 = 2x(x^2-1)+4$.

27. $(x-2)^3+2x^3+(x+2)^3 = 4x^2(x+2)$

28. $(x+2)(x+3)(x+4)+96 = x^2(x+9)+5(3x+13)$

29. $3(x^2-14) = (x+1)^2+(x-2)^2+(x-5)^2$

30. $a(x-a) = b(x-b)$. 31. $3(x-a)+5(2x-3a) = 8a$.

32. $(x+a)(x+b)-(a+b)^2 = (x-a)(x-b)$.

33. $a^2(x-a)+b^2(x-b) = abx$.

34. $m^2(x-m)+n^2(x+n)+mnx = 0$.

35. $b(x-2a)+a(x-2b) = (a-b)^2$.

36. $a(4x-a)+b(4x-b)-2ab = 0$.

37. $x(x-a)+x(x-b)-2(x-a)(x-b) = 0$

38. $(x+3a)(x-3b)+3(x-3a)(x+3b) = 4(x-3a)(x-3b)$.

39. $(2b+2c-x)^2+(2b-2c+x)^2 = (2b+2d-x)^2 + (2b-2d+x)^2$

40. $(x-a)^3+(x-b)^3+(x-c)^3 = 3(x-a)(x-b)(x-c)$.

41. $(x+a)^3+(x+b)^3+(x+c)^3 = 3(x+a)(x+b)(x+c)$.

42. $\frac{x}{2}+5 = \frac{x}{3}+7$. 43. $\frac{x}{a}+a = \frac{x}{b}+b$.

44. $\frac{x}{6}-\frac{x}{5} = \frac{x}{15}-\frac{x}{3}+7$. 45. $\frac{x}{2}-\frac{x}{3}+\frac{x}{4} = 2-\frac{x}{6}+\frac{5x}{12}$.

46. $\dfrac{a}{bx}-\dfrac{b}{ax}=a^2-b^2$

47. $\frac{1}{2}(x+1)+\frac{1}{3}(x+2)+\frac{1}{4}(x+3)=16.$

48. $\dfrac{x-6}{5}+\dfrac{x-4}{3}=8-\dfrac{x-2}{7}$

49. $\dfrac{x}{10}+\dfrac{2x-13}{9}=8-\dfrac{4x-35}{15}$

50. $\dfrac{x+7}{2}+\dfrac{x+13}{5}+\dfrac{x+17}{7}=\dfrac{x+27}{4},$

51. $6\frac{1}{3}-\dfrac{x-7}{3}=\dfrac{4x-2}{5}.$

52. $\dfrac{x-1}{3}-\dfrac{x-9}{2}+\dfrac{3x-2(x-2)}{7}=4\frac{1}{2}$

53. $\dfrac{2x-9}{27}+\dfrac{x}{18}-\dfrac{x-3}{4}=8\frac{1}{3}-x$

54. $\dfrac{9x+7}{2}-\left\{x-\dfrac{x-2}{7}\right\}=36$

55. $\dfrac{7x+9}{4}-\left\{x-\dfrac{2x-1}{9}\right\}=7$

56. $\dfrac{x+7}{3}-5\frac{3}{4}=\dfrac{2x+5}{7}+\dfrac{10-5x}{8}$

57. $x-\left(3x-\dfrac{2x-5}{10}\right)=\dfrac{1}{6}\left(2x-57\right)-\dfrac{5}{3}.$ (C U Entr Paper, 1889)

58. $\dfrac{4x-21}{7}+7\frac{5}{6}+\dfrac{7x-28}{3}=x+3\frac{3}{4}-\dfrac{9-7x}{8}+\dfrac{1}{12}.$

59. $\frac{1}{2}\left(x-\frac{a}{3}\right)-\frac{1}{3}\left(x-\frac{a}{4}\right)+\frac{1}{4}\left(x-\frac{a}{5}\right)=0.$

60. $\dfrac{x-3}{7}-\dfrac{\frac{1}{2}x-3}{3}=\dfrac{\frac{1}{4}x+2}{2}-\dfrac{x-6}{3}+\dfrac{x}{8}.$

61. $\frac{1}{8}(x-2)-\frac{1}{7}(x-4)=\frac{1}{12}(2x-3)-2\frac{3}{4}$

62. $\dfrac{a-x}{a}+\dfrac{2a-x}{2a}=\dfrac{3a-x}{3a}$ (C U Entr Paper, 1870)

63. $\dfrac{2x-13}{9}-\dfrac{x-1}{11}=\dfrac{x}{8}+\dfrac{x}{7}-9$ (C U Entr Paper, 1876).

64. $\dfrac{2x-3}{6}+\dfrac{3x-8}{11}=\dfrac{4x+15}{33}+\dfrac{1}{2}$ (C U Entr Paper, 1877).

65. $\dfrac{4x+3}{9}+\dfrac{13x}{108}=\dfrac{8x+19}{18}$ (C U Entr Paper, 1878).

66. $\dfrac{x^2-2\frac{1}{2}}{4}-\dfrac{x-3\frac{1}{2}}{5}=\dfrac{2x^2-3}{8}-\dfrac{x-5\frac{1}{2}}{3}$. (C U Entr Paper, 1883).

67. $\dfrac{a-x^2}{bx}-\dfrac{b-x}{c}=\dfrac{c-x}{b}-\dfrac{b-x^2}{cx}$. (C U Entr Paper, 1886).

68. $\dfrac{x+2\frac{1}{2}}{15}+\dfrac{x+3\frac{1}{2}}{25}=\dfrac{x+4\frac{1}{5}}{55}$. (C U Entr Paper, 1888).

69. $\dfrac{11x-13}{25}+\dfrac{19x+3}{7}-\dfrac{5x-25\frac{1}{3}}{4}=28\frac{1}{7}-\dfrac{17x+4}{21}$.

70. $\dfrac{x-1\frac{25}{36}}{2}-\dfrac{2-6x}{13}=x-\dfrac{5x-\frac{1}{4}(10-3x)}{39}$.

71. $\dfrac{3x-\frac{2}{3}(1+x)}{4}+\dfrac{1-\frac{1}{5}x}{5\frac{1}{2}}=\dfrac{2\frac{2}{5}+\frac{1}{25}(x-1)}{2\frac{1}{5}}$

72. $\frac{1}{3}(x-a)-\frac{1}{5}(2x-3b)-\frac{1}{2}(a-x)=10a+11b$.

73. $\dfrac{2x+a}{b}-\dfrac{x-b}{a}=\dfrac{3ax+(a-b)^2}{ab}$.

74. $\dfrac{2x+1}{29}-\dfrac{402-3x}{12}=9-\dfrac{471-6x}{2}$.

75. $\dfrac{x+5}{4x-9}=\dfrac{x+10\frac{1}{3}}{4x-7}$

76. $\dfrac{7x+2}{17x+14}=\dfrac{7x+6}{17x+26}$.

77. $\dfrac{2x-7}{3x-8}=\dfrac{3(2x-1)}{9x-2}$.

78. $\dfrac{13-6x}{3+4x}=\dfrac{7-9x}{7+6x}$.

79. $\dfrac{10(7+4x)}{7+15x}=\dfrac{27+8x}{4+3x}$.

80. $\dfrac{15-\frac{2}{3}x}{5}-\dfrac{2x+5}{2\frac{1}{2}}=\dfrac{17-\frac{14}{3}x}{3}$.

(Calcutta University Entrance Paper, 1874)

4. Equations involving Decimals.

The decimals, if necessary, may be converted into vulgar fractions.

Example 1. Solve $\frac{x-2}{{\cdot}05} - \frac{x-4}{{\cdot}0625} = 56.$

Since, ${\cdot}05 = \frac{5}{90} = \frac{1}{18}$, and ${\cdot}0625 = \frac{625}{10000} = \frac{1}{16}$,

we have $$\frac{x-2}{\frac{1}{18}} - \frac{x-4}{\frac{1}{16}} = 56,$$

or, $$18(x-2)-16(x-4) = 56,$$

or, $$2x+28 = 56,$$

$$\therefore \quad 2x = 28, \quad \text{or,} \quad x = 14.$$

Example 2. Solve

$${\cdot}65x + \frac{{\cdot}585x - {\cdot}975}{{\cdot}6} = \frac{1{\cdot}56}{{\cdot}2} - \frac{{\cdot}39x - {\cdot}78}{{\cdot}9}.$$

Since $$\frac{{\cdot}585x - {\cdot}975}{{\cdot}6} = \frac{5{\cdot}85x - 9{\cdot}75}{6} = \frac{1{\cdot}95x - 3{\cdot}25}{2},$$

$$\frac{1{\cdot}56}{{\cdot}2} = \frac{15{\cdot}6}{2} = 7{\cdot}8.$$

and $$\frac{{\cdot}39x - {\cdot}78}{{\cdot}9} = \frac{3{\cdot}9x - 7{\cdot}8}{9} = \frac{1{\cdot}3x - 2{\cdot}6}{3},$$

the equation stands thus —

$${\cdot}65x + \frac{1{\cdot}95x - 3{\cdot}25}{2} = 7{\cdot}8 - \frac{1{\cdot}3x - 2{\cdot}6}{3}.$$

Hence, multiplying both sides by 6, we have

$$3{\cdot}9x + (5{\cdot}85x - 9{\cdot}75) = 46{\cdot}8 - (2{\cdot}6x - 5{\cdot}2).$$

By transposition,

$$(3{\cdot}9 + 5{\cdot}85 + 2{\cdot}6)x = 46{\cdot}8 + 5{\cdot}2 + 9{\cdot}75$$

or, $$12{\cdot}35x = 61{\cdot}75$$

$$x = \frac{61{\cdot}75}{12{\cdot}35} = 5$$

Exercise (63).

Solve the following equations :—

1. ${\cdot}5x - {\cdot}2x = {\cdot}3x - 1{\cdot}5.$ 2. $3{\cdot}75x + {\cdot}5 = 2{\cdot}25x + 8.$

3. $1\ 2x-\frac{18x-\cdot05}{5} = 4x+8\ 9$

4. $\frac{x+\ 75}{\cdot125}-\frac{x-\ 25}{25} = 15.$

5. $\frac{x}{5}-\frac{1}{\cdot05}+\frac{x}{005}-\frac{1}{0005} = 0.$ (C U Entr Paper, 1883)

6. $5x+\frac{\cdot45x-\ 75}{6} = \frac{1\ 2}{2}-\frac{\cdot3x-\ 6}{\cdot9}.$

7. $\frac{x+2}{x-2}+6\ 8 = 5\ 3\dot{8}$ 8. $\cdot7x+\ 4 = \ 67x+\ 5.$

9. $\frac{2x-3}{\cdot3x-\ 4} = \frac{4x-\ 6}{06x-\ 07}$

10. $\cdot15x+\frac{135x-\ 225}{\cdot6} = \frac{36}{2}-\frac{\cdot09x-\cdot18}{9}.$

11. $\cdot5x+\frac{\cdot02x+\cdot07}{03}-\frac{x+2}{9} = 9\ 5.$

12. $\frac{4\ 05}{9x}-\frac{\cdot3}{8-2x}=\frac{1\ 8}{x}-\frac{3\ 6}{2\ 4-6x}$ (C U Entr Paper, 1881)

13. $\cdot011x+\frac{001x-\ 125}{6} = \frac{5-x}{\cdot03}-\cdot145$ (C U Entr Paper, 1886)

5 Solution of fractional equations facilitated by suitable transposition and combination of terms.

Example 1. Solve

$$\frac{23x-29}{12}+\frac{19x+13}{7} = \frac{97x+72\frac{1}{2}}{35}+\frac{7x-8\frac{1}{3}}{4}.$$

By transposition, we have

$$\frac{23x-29}{12}-\frac{7x-8\frac{1}{3}}{4} = \frac{97x+72\frac{1}{2}}{35}-\frac{19x+13}{7}$$

$$\frac{(23x-29)-(21x-25)}{12} = \frac{(97x+72\frac{1}{2})-(95x+65)}{35}$$

$$\text{or,}\quad \frac{x-2}{6} = \frac{2x+7\frac{1}{2}}{35}.$$

Hence, multiplying both sides by 6×35,

$$35x-70 = 12x+45.$$

Hence, $\qquad 23x = 115, \quad \text{or,} \quad x = 5.$

Example 2. Solve $\qquad \dfrac{7x-11}{6} = \dfrac{31x-41}{24} - \dfrac{7x^2-4}{56x-47}.$

By transposition, we have

$$\frac{7x^2-4}{56x-47} = \frac{31x-41}{24} - \frac{7x-11}{6}$$

$$= \frac{(31x-41)-(28x-44)}{24}$$

$$= \frac{3(x+1)}{24} = \frac{x+1}{8}.$$

Multiplying both sides by $8(56x-47)$, we have

$$8(7x^2-4) = (x+1)(56x-47)$$

or, $\qquad 56x^2-32 = 56x^2+9x-47;$

$\therefore \qquad -32 = 9x-47$

Hence, $\qquad 9x = -32+47 = 15,$

$\therefore \qquad x = \frac{15}{9} = 1\frac{2}{3}$

Example 3. Solve $\quad \dfrac{25-\frac{1}{3}x}{x+1} + \dfrac{16x+4\frac{1}{5}}{3x+2} = \dfrac{23}{x+1}+5.$

By transposition, we have

$$\frac{16x+4\frac{1}{5}}{3x+2} - 5 = \frac{23}{x+1} - \frac{25-\frac{1}{3}x}{x+1},$$

or, $\qquad \dfrac{x-5\frac{4}{5}}{3x+2} = \dfrac{\frac{1}{3}x-2}{x+1}.$

Hence, $\quad (x-5\frac{4}{5})(x+1) = (\frac{1}{3}x-2)(3x+2)$

or, $\quad x^2-(4\frac{4}{5})x-5\frac{4}{5} = x^2-(5\frac{1}{3})x-4$

Hence, $\quad (5\frac{1}{3}-4\frac{4}{5})x = 5\frac{4}{5}-4$

or, $\quad \frac{8}{15}x = 1\frac{4}{5} = \frac{9}{5},$

$$x = \frac{9}{5} \times \frac{15}{8} = \frac{27}{8} = 3\frac{3}{8}$$

Example 4. Solve $\frac{3}{x-2}+\frac{5}{x-6}=\frac{8}{x+3}$.

Since $$\frac{8}{x+3}=\frac{3}{x+3}+\frac{5}{x+3},$$

we have $$\frac{3}{x-2}+\frac{5}{x-6}=\frac{3}{x+3}+\frac{5}{x+3}.$$

Hence, by transposition,
$$\frac{3}{x-2}-\frac{3}{x+3}=\frac{5}{x+3}-\frac{5}{x-6}$$

or, $$\frac{15}{(x-2)(x-3)}=\frac{-45}{(x+3)(x-6)}.$$

Multiplying both sides by $x+3$, and dividing by 15,

we have $$\frac{1}{x-2}=\frac{-3}{x-6}.$$

Hence, $$x-6=-3(x-2),$$

$\therefore$ $$4x=12, \quad \text{or,} \quad x=3.$$

Example 5 Solve $\frac{8}{2x-1}+\frac{9}{3x-1}=\frac{7}{x+1}$.

We have $$\frac{8}{2x-1}+\frac{9}{3x-1}=\frac{4}{x+1}+\frac{3}{x+1}.$$

Hence, $$\left\{\frac{8}{2x-1}-\frac{4}{x+1}\right\}+\left\{\frac{9}{3x-1}-\frac{3}{x+1}\right\}=0.$$ [By transposition.]

or, $$\frac{12}{(2x-1)(x+1)}+\frac{12}{(3x-1)(x+1)}=0.$$

Hence, $$\frac{1}{2x-1}+\frac{1}{3x-1}=0.$$

Multiplying both sides by $(2x-1)(3x-1)$,

we have $$(3x-1)+(2x-1)=0$$

Therefore, $5x=2$, or, $x=\frac{2}{5}$

Example 6. Solve $\frac{a-c}{2b+x}+\frac{b-c}{2a+x}=\frac{a+b-2c}{a+b+x}$.

We have $$\frac{a-c}{2b+x}+\frac{b-c}{2a+x}=\frac{(a-c)+(b-c)}{a+b+x}$$
$$=\frac{a-c}{a+b+x}+\frac{b-c}{a+b+x}.$$

Hence, by transposition,

$$(a-c)\left\{\frac{1}{2b+x}-\frac{1}{a+b+x}\right\} = (b-c)\left\{\frac{1}{a+b+x}-\frac{1}{2a+x}\right\}$$

$$\text{or,}\quad (a-c)\,\frac{a-b}{(2b+x)(a+b+x)} = (b-c)\,\frac{a-b}{(a+b+x)(2a+x)}$$

$$\text{Hence,}\qquad \frac{a-c}{2b+x} = \frac{b-c}{2a+x}.$$

$$\begin{aligned}
\therefore\quad (a-c)(2a+x) &= (b-c)(2b+x),\\
\therefore\quad x\{(a-c)-(b-c)\} &= 2b(b-c)-2a(a-c)\\
\text{or,}\quad x(a-b) &= 2(b^2-a^2)-2c(b-a)\\
&= 2(b-a)(b+a-c)\\
&= 2(a-b)(c-a-b),\\
\therefore\quad x &= 2(c-a-b).
\end{aligned}$$

Exercise (64).

Solve the following equations :—

1. $\dfrac{5x+6}{4}+\dfrac{64x-35}{15} = \dfrac{20x+23}{16}+\dfrac{13x-7}{3}.$

2. $\dfrac{17x-13}{9}+\dfrac{108x+75}{32} = \dfrac{27x+19}{8}+\dfrac{(50\frac{7}{17})x-39}{27}.$

3. $\dfrac{29x-18}{8}+\dfrac{189x-93}{49} = \dfrac{(86\frac{13}{21})x-54}{24}+\dfrac{27x-13}{7}.$

4. $\dfrac{16x-17}{9}-\dfrac{23x-15}{16} = \dfrac{(142\frac{7}{18})x-153}{81}+\dfrac{92x-65}{64}.$

5. $\dfrac{18x-19}{7}+\dfrac{135x+62\frac{1}{2}}{65} = \dfrac{27x+14}{13}+\dfrac{(106\frac{5}{13})x-144}{42}.$

6. $\dfrac{33-19x}{15}-\dfrac{41+27x}{28}+\dfrac{164+(107\frac{1}{17})x}{112}-\dfrac{164\frac{13}{28}-95x}{75} = 0.$

7. $\dfrac{18-41x}{9}-\dfrac{17-16x}{8}+\dfrac{9\frac{16}{21}-10x}{5}-\dfrac{14-32x}{7} = 0.$

8. $\dfrac{98x-73}{21} = \dfrac{14x-9}{3}-\dfrac{13x-16}{15x-9}.$

9. $\dfrac{95x-159}{35}=\dfrac{19x-29}{7}-\dfrac{17x-47}{23x-59}$

10. $\dfrac{91x-21}{56}+\dfrac{24x-93}{35x-138}=\dfrac{13x+9}{8}$

11. $\dfrac{117x-26}{135}+\dfrac{16x-77}{23x-110}=\dfrac{13x+4}{15}+\dfrac{3\frac{4}{5}}{27}$

12. $\dfrac{6x-7\frac{1}{4}}{13-2x}+2x+\dfrac{1+16x}{24}=4\tfrac{5}{12}-\dfrac{12\frac{5}{8}-8x}{3}.$

13. $\dfrac{2x+8\frac{1}{2}}{9}-\dfrac{13x-2}{17x-32}+\dfrac{x}{3}=\dfrac{7x}{12}-\dfrac{x+16}{36}.$

14. $\dfrac{41-35x}{105}-\dfrac{7-2x^2}{14(x-1)}=\dfrac{1+3x}{21}-\dfrac{2x-2\frac{1}{5}}{6}$

15. $\dfrac{1}{x-1}-\dfrac{2}{x+7}=\dfrac{1}{7(x-1)}$

16. $\dfrac{2}{5(3x+4)}+\dfrac{4}{2x+3}=\dfrac{6}{3x+4}$

17. $\dfrac{3}{3x-5}-\dfrac{6}{7(4x-7)}=\dfrac{7}{9(3x-5)}+\dfrac{2}{4x-7}.$

18. $\dfrac{11}{12(14x-19)}+\dfrac{7}{9(13x-14)}=\dfrac{3}{14x-19}-\dfrac{2}{13x-14}.$

19. $\dfrac{50}{3x-1}+\dfrac{37-\frac{1}{3}x}{12x-1}=\dfrac{35}{12x-1}+\dfrac{49-\frac{1}{12}x}{3x-1}$

20. $\dfrac{(1\frac{3}{7})x+19\frac{13}{17}}{2x+5}-\dfrac{\frac{7}{5}x+8}{x+8}=\dfrac{20\frac{13}{17}-(1\frac{4}{7})x}{2x+5}+\dfrac{(1\frac{4}{9})x-9}{2(x+8)}.$

21. $\dfrac{(9\frac{12}{25})x-32}{4x+7}+\dfrac{65x+4\frac{14}{23}}{8x+29}=\dfrac{75x+5\frac{14}{23}}{8x+29}+\dfrac{(4\frac{12}{27})x-29}{4x+7}$

22. $\dfrac{1}{x-1}+\dfrac{2}{x-2}=\dfrac{3}{x-3}$ 23. $\dfrac{3}{4x+1}+\dfrac{4}{4x+5}=\dfrac{7}{4x+3}.$

24. $\dfrac{15}{3x+11}-\dfrac{8}{3x+17}=\dfrac{7}{3x+5}.$

25. $\dfrac{6}{5x+7}-\dfrac{4}{5x+13}=\dfrac{9}{5x+13}-\dfrac{7}{5x+19}.$

26. $\dfrac{8}{2x+17}-\dfrac{12}{2x+25}=\dfrac{5}{2x+25}-\dfrac{9}{2x+33}$.

27. $\dfrac{5}{3-4x}+\dfrac{9}{4x+13}-\dfrac{4}{4x+5}=0$

28. $\dfrac{6}{5-6x}+\dfrac{13}{6x+19}=\dfrac{7}{6x+7}$. 29. $\dfrac{7}{4-5x}-\dfrac{8}{17-5x}=\dfrac{1}{5x+9}$

30 $\dfrac{9}{3-7x}+\dfrac{1}{7x+15}=\dfrac{8}{12-7x}$.

31. $\dfrac{2}{2x-5}+\dfrac{1}{x-3}=\dfrac{6}{3x-1}$. 32. $\dfrac{10}{2x-5}+\dfrac{1}{x+5}=\dfrac{18}{3x-5}$.

33. $\dfrac{9}{3x-4}+\dfrac{20}{4x+1}=\dfrac{8}{x+7}$. 34. $\dfrac{10}{5x-9}+\dfrac{14}{2x+9}=\dfrac{9}{x+8}$

35. $\dfrac{12}{3x-8}=\dfrac{20}{4x-13}-\dfrac{1}{x+9}$. 36. $\dfrac{a+b}{x-c}=\dfrac{a}{x-a}+\dfrac{b}{x-b}$

37. $\dfrac{a^2}{ax-b}+\dfrac{b^2}{bx-a}=\dfrac{a+b}{x+c}$. 38. $\dfrac{m(x+a)}{x+b}+\dfrac{n(x+b)}{x+a}=m+n$.

39. $\dfrac{b-c}{x+a}+\dfrac{a-b}{x+b}=\dfrac{a-c}{x+c}$.

40 $\dfrac{2a-3b}{x-a+b}-\dfrac{2b-3a}{x+a-b}=\dfrac{5(a-b)}{x+a+b}$

41. $\dfrac{1}{x-6a}+\dfrac{2}{x+3a}+\dfrac{3}{x-2a}=\dfrac{6}{x-a}$.

42. $\dfrac{1}{x+1}+\dfrac{4}{2x-1}+\dfrac{9}{3x-1}=\dfrac{36}{6x-1}$.

6. Solution of fractional equations facilitated by the division of each numerator by its denominator.

Example 1. Solve $\dfrac{x+1}{x-1}+\dfrac{x+2}{x-2}=\dfrac{22x+30}{11x-18}$.

We have $\dfrac{(x-1)+2}{x-1}+\dfrac{(x-2)+4}{x-2}=\dfrac{2(11x-18)+66}{11x-18}$,

$$\text{or,}\quad \left\{1+\frac{2}{x-1}\right\}+\left\{1+\frac{4}{x-2}\right\}=2+\frac{66}{11x-18},$$

$$\text{or,}\quad \frac{2}{x-1}+\frac{4}{x-2}=\frac{66}{11x-18}.$$

Hence, by transposition,

$$\frac{2}{x-1}-\frac{22}{11x-18}=\frac{44}{11x-18}-\frac{4}{x-2}$$

or, $$\frac{-14}{(x-1)(11x-18)}=\frac{-16}{(11x-18)(x-2)}.$$

Therefore $$\frac{7}{x-1}=\frac{8}{x-2}$$

or, $7x-14=8x-8$,

$\therefore \quad x=-6$

Example 2. Solve $\frac{4x^2+7}{2x-1}+\frac{6x^2-8x+11}{3x-1}=\frac{4x^2+3x+6}{x+1}$.

We have $$\frac{(4x^2-1)+8}{2x-1}+\frac{2x(3x-1)-2(3x-1)+9}{3x-1}$$
$$=\frac{4x(x+1)-(x+1)+7}{x+1}$$

or, $$\left\{2x+1+\frac{8}{2x-1}\right\}+\left\{2x-2+\frac{9}{3x-1}\right\}$$
$$=4x-1+\frac{7}{x+1}.$$

Hence, $$\frac{8}{2x-1}+\frac{9}{3x-1}=\frac{7}{x+1}$$

For the subsequent part of the solution the student is referred to example 5 worked out on page 204

Example 3. Solve $\frac{7x-55}{x-8}+\frac{2x-17}{x-9}=\frac{6x-71}{x-12}+\frac{3x-14}{x-5}$.

We have

$$\frac{7(x-8)+1}{x-8}+\frac{2(x-9)+1}{x-9}=\frac{6(x-12)+1}{x-12}+\frac{3(x-5)+1}{x-5}.$$

or, $$\left(7+\frac{1}{x-8}\right)+\left(2+\frac{1}{x-9}\right)=\left(6+\frac{1}{x-12}\right)+\left(3+\frac{1}{x-5}\right);$$

$$\therefore \quad \frac{1}{x-8}+\frac{1}{x-9}=\frac{1}{x-12}-\frac{1}{x-5}$$

Hence, by transposition,

$$\frac{1}{x-8}-\frac{1}{x-5}=\frac{1}{x-12}-\frac{1}{x-9}.$$

or, $$\frac{3}{(x-8)(x-5)}=\frac{3}{(x-12)(x-9)},$$

$\therefore$ $$(x-8)(x-5)=(x-12)(x-9)$$

or, $$x^2-13x+40=x^2-21x+108$$

$$\therefore \quad 8x=68, \quad \text{or,} \quad x=8\tfrac{1}{2}.$$

Exercise (65).

Solve the following equations —

1. $\dfrac{2x-1}{x-1}+\dfrac{3x-4}{x-2}=\dfrac{5x-12}{x-3}.$

2. $\dfrac{2x+7}{x+2}+\dfrac{4x+29}{x+6}-\dfrac{6x-10}{x-3}=0.$

3. $\dfrac{25x-40}{5x-6}-\dfrac{7x+9}{x+2}+\dfrac{6x-1}{3x+4}=0.$

4. $\dfrac{12x+5}{4x+3}+\dfrac{15x+11}{3x+4}=\dfrac{8x+44}{x+6}.$

5. $\dfrac{15x-13}{3x-5}+\dfrac{12x+59}{3x+5}=\dfrac{9x+80}{x+7}.$

6. $\dfrac{42x-37}{6x-1}+\dfrac{20x+13}{5x+12}=\dfrac{11x+76}{x+8}.$

7. $\dfrac{15x-7}{5x-4}+\dfrac{4x+3}{4x-3}=\dfrac{8x+1}{2x-1}$

8. $\dfrac{4x-7}{4x+5}+\dfrac{15x+11}{5x+7}=\dfrac{12x+1}{3x+4}.$

9. $\dfrac{4x^3+4x^2+8x+1}{2x^2+2x+3}=\dfrac{2x^2+2x+1}{x+1}.$

10. $\dfrac{12x^3+16x^2+29x-1}{3x^2+4x+8}=\dfrac{4x^2+20x-1}{x+5}$

11. $\dfrac{x^2-x+1}{x-1}+\dfrac{x^2-2x+1}{x-2}=2x+\dfrac{2}{x-3}$

12. $\dfrac{x^2+3}{x-1}+\dfrac{x^2-x+1}{x-2}=\dfrac{2x^2-4x+1}{x-3}$

13. $\dfrac{2x^2-3x+7}{2x-1}+\dfrac{6x^2+2x+21}{3x+1}=\dfrac{3x^2+8x+7}{x+3}$.

14. $\dfrac{3+2x}{1+2x}-\dfrac{5+2x}{7+2x}=1-\dfrac{4x^2-2}{7+16x+4x^2}$.

15. $\dfrac{2x-3}{x-2}+\dfrac{3x-20}{x-7}=\dfrac{x-3}{x-4}+\dfrac{4x-19}{x-5}$.

16. $\dfrac{3x-8}{x-3}+\dfrac{4x-35}{x-9}=\dfrac{2x-9}{x-5}+\dfrac{5x-34}{x-7}$.

17. $\dfrac{3x-18}{x-4}+\dfrac{4x-41}{x-10}=\dfrac{2x-13}{x-6}+\dfrac{5x-41}{x-8}$.

18. $\dfrac{4x+21}{x+5}+\dfrac{5x-69}{x-14}=\dfrac{3x-5}{x-2}+\dfrac{6x-41}{x-7}$.

19. $\dfrac{5-6x}{3x-1}+\dfrac{2x+7}{x+3}=\dfrac{31-12x}{3x-7}+\dfrac{4x+21}{x+5}$.

20. $\dfrac{x^2+3x+3}{x+2}+\dfrac{x^2-15}{x-4}=\dfrac{x^2+7x+11}{x+5}+\dfrac{x^2-4x-20}{x-7}$.

21. $\dfrac{2x+11}{x+5}-\dfrac{9x-9}{3x-4}=\dfrac{4x+13}{x+3}-\dfrac{15x-47}{3x-10}$.

22. $\dfrac{x-2}{x-3}+\dfrac{x-3}{x-4}=\dfrac{x-1}{x-2}+\dfrac{x-4}{x-5}$. (C U Entr Paper, 1887)

7. Miscellaneous Examples.

Example 1. Solve $\dfrac{3abc}{a+b}+\dfrac{a^2b^2}{(a+b)^3}+\dfrac{(2a+b)b^2x}{a(a+b)^2}=3cx+\dfrac{bx}{a}$.

By transposition we have

$$\frac{ab}{a+b}\left\{3c+\frac{ab}{(a+b)^2}\right\}=x\left\{3c+\frac{b}{a}-\frac{(2a+b)b^2}{a(a+b)^2}\right\}$$
$$=x\left\{3c+\frac{b}{a}\left[1-\frac{(2a+b)b}{(a+b)^2}\right]\right\}$$
$$=x\left\{3c+\frac{b}{a}\cdot\frac{a^2}{(a+b)^2}\right\}$$
$$=x\left\{3c+\frac{ab}{(a+b)^2}\right\}.$$

Therefore $x=\dfrac{ab}{a+b}$.

Example 2 Solve $\dfrac{ax^2+bx+c}{px^2+qx+r}=\dfrac{ax+b}{px+q}$.

We have $$\frac{x(ax+b)+c}{x(px+q)+r}=\frac{ax+b}{px+q}.$$

Hence, putting m for $ax+b$ and n for $px+q$,

we have $\dfrac{mx+c}{nx+r}=\dfrac{m}{n}$,

$$\therefore\quad mnx+cn=mnx+rm,$$

$$\therefore\quad cn=rm$$

or, $$c(px+q)=r(ax+b),$$

$$\therefore\quad x(cp-ar)=br-cq,$$

$$\therefore\quad x=\frac{br-cq}{cp-ar}.$$

Example 3. Solve $(x-2a)^3+(x-2b)^3=2(x-a-b)^3$.

By transposition, we have

$$(x-2a)^3-(x-a-b)^3=(x-a-b)^3-(x-2b)^3.$$

Putting X for $x-2a$, Y for $x-2b$, and Z for $x-a-b$,

we have $$X^3-Z^3=Z^3-Y^3,$$

or, $(X-Z)(X^2+XZ+Z^2)=(Z-Y)(Z^2+ZY+Y^2)$.

But $X-Z=Z-Y$, because each of them $=b-a$.

$$\therefore\quad X^2+XZ+Z^2=Z^2+ZY+Y^2.$$

Hence, by transposition,

$$X^2-Y^2=Z(Y-X).$$

Removing the common factor $X-Y$, which $=2b-2a$,

we have $$X+Y=-Z,$$

i.e., $$(x-2a)+(x-2b)=-(x-a-b).$$

Hence, $3x=3(a+b)$, and $\therefore\quad x=a+b$.

Example 4. Solve $\dfrac{x+a}{x+b}=\left(\dfrac{2x+a+c}{2x+b+c}\right)^2$.

Since $\dfrac{x+a}{x+b}=\dfrac{(x+b)+(a-b)}{x+b}=1+\dfrac{a-b}{x+b}$,

and $\frac{2x+a+c}{2x+b+c} = \frac{(2x+b+c)+(a-b)}{2x+b+c} = 1+\frac{a-b}{2a+b+c}$,

we have $1+\frac{a-b}{x+b} = \left\{1+\frac{a-b}{2x+b+c}\right\}^2$

$$= 1+\frac{2(a-b)}{2x+b+c}+\frac{(a-b)^2}{(2x+b+c)^2}.$$

Hence, transposing and dividing by $a-b$, we have

$$\frac{1}{x+b}-\frac{2}{2x+b+c} = \frac{a-b}{(2x+b+c)^2}$$

or, $$\frac{c-b}{(x+b)(2x+b+c)} = \frac{a-b}{(2x+b+c)^2},$$

$$\therefore \quad \frac{c-b}{x+b} = \frac{a-b}{2x+b+c},$$

$$\therefore \quad 2x(c-b)+(c^2-b^2) = x(a-b)+b(a-b),$$

$$x(a+b-2c) = c^2-ab,$$

$$\therefore \quad x = \frac{c^2-ab}{a+b-2c}.$$

Example 5. Solve

$$\frac{4x}{3}-\frac{125x^2-5}{(5x-1)(x+5)} = 5x-\frac{5}{3}\,\frac{3x^2}{x+5}-\frac{95-4x}{3}.$$

Since $\frac{125x^2-5}{(5x-1)(x+5)} = \frac{5(25x^2-1)}{(5x-1)(x+5)} = \frac{5(5x+1)}{x+5}$

and $\frac{5}{3}\,\frac{3x^2-1}{x+5} = \frac{\frac{5}{3}(3x^2-1)}{x+5} = \frac{5x^2-\frac{5}{3}}{x+5}$,

we have $\frac{4x}{3}-\frac{5(5x+1)}{x+5} = 5x-\frac{5x^2-\frac{5}{3}}{x+5}-\frac{95}{3}+\frac{4x}{3}.$

Hence, transposing and dividing by 5, we have

$$\frac{y^2-\frac{1}{3}-(5x+1)}{x+5} = x-6\tfrac{1}{3}.$$

Hence, $x^2-5x-1\frac{1}{3} = x^2-(1\frac{1}{3})x-31\frac{2}{3}.$

$$\therefore \quad (3\tfrac{2}{3})x = 30\tfrac{1}{3},$$

$$\therefore \quad x = \tfrac{91}{11} = 8\tfrac{3}{11}.$$

Exercise (66).

Solve the following equations :—

1. $\dfrac{2x}{x-4}+\dfrac{7x-3}{x+1}=9.$ 2. $\dfrac{x+4a+b}{x+a+b}+\dfrac{4x+a+2b}{x+a-b}=5.$

3. $\dfrac{3x+5}{x+1}=\dfrac{4x+8}{3x+3}+\dfrac{10x+1}{6x+3}.$ 4. $\dfrac{6x+8}{2x+1}-\dfrac{2x+38}{x+12}=1.$

5. $\dfrac{x+18}{x-2}-\dfrac{27-3x}{3x-19}=2.$ 6. $\dfrac{x-b}{x-a}-\dfrac{x-a}{x-b}=\dfrac{2(a-b)}{x-(a+b)}.$

7. $\dfrac{x+2a}{2b-x}+\dfrac{x-2a}{2b+x}+\dfrac{4ab}{x^2-4b^2}=0.$

8. $\dfrac{(x-a)(x-b)}{x-a-b}=\dfrac{(x-c)(x-d)}{x-c-d}.$

9. $\dfrac{1}{x^2+3x+2}+\dfrac{2x}{x^2+4x+3}+\dfrac{1}{x^2+5x+6}=14-\dfrac{60+4x}{x+3}.$

10. $\dfrac{a+x}{a^2+ax+x^2}+\dfrac{a-x}{a^2-ax+x^2}=\dfrac{3a}{x(a^4+a^2x^2+x^4)}.$

11. $\dfrac{x}{x-2}+\dfrac{x-9}{x-7}=\dfrac{x+1}{x-1}+\dfrac{x-8}{x-6}.$

12. $\dfrac{1}{(x+a)^2-b^2}+\dfrac{1}{(x+b)^2-a^2}=\dfrac{1}{x^2-(a+b)^2}+\dfrac{1}{x^2-(a-b)^2}.$

13. $\dfrac{3x^2+5x+8}{5x^2+6x+12}=\dfrac{3x+5}{5x+6}.$

14. $\dfrac{58x^2+87x+7}{87x^2+145x+11}=\dfrac{2x+3}{3x+5}.$

15. $\dfrac{a^2(a-2b)}{b(a-b)^2}x+\dfrac{2abc}{a-b}-\dfrac{ax}{b}=2cx-\dfrac{a^2b^2}{(a-b)^3}.$

16. $(x-23)^3+(x-27)^3=2(x-25)^3.$

17. $\dfrac{4x-17}{9}-\dfrac{3\frac{2}{3}-22x}{33}=x-\dfrac{6}{x}\left(1-\dfrac{x^2}{54}\right).$

18. $\left(\dfrac{x-2a}{x+2b}\right)^2=\dfrac{x-2a-2b}{x+2a+2b}.$ 19. $\dfrac{x+19}{x+10}=\left(\dfrac{2x+33}{2x+24}\right)^2.$

20. $\left(\dfrac{x-a}{x+b}\right)^3=\dfrac{x-2a-b}{x+a+2b}.$

21. $\dfrac{3x}{2}-\dfrac{8(x^2-9)}{(3x-1)(x+3)}=3x-\dfrac{3}{2}\cdot\dfrac{2x^2-1}{x+3}-\dfrac{57-3x}{2}.$

8. A Simple equation cannot have more than one root. Every simple equation is ultimately reducible to the form $ax=b$ Let this equation, if possible, have two different roots α and β.

Then we must have $\left.\begin{aligned} a\alpha &= b \\ \text{and also } a\beta &= b \end{aligned}\right\}$

Hence, by subtraction,

$$a(\alpha-\beta) = 0.$$

But this is impossible because a is not zero and by supposition $\alpha-\beta$ also is not zero.

Thus a simple equation cannot have more than one root

CHAPTER XIII.

PROBLEMS LEADING TO SIMPLE EQUATIONS.

Symbolical Expression. The chief difficulty in solving an algebraical problem lies in expressing correctly the conditions of the problem by means of symbols The student should, therefore, be first of all introduced to this art before the solution of any problem is presented to him The following examples will serve as illustrations.

Example 1. If a man earns x rupees per month, how many four-anna pieces will he earn in half a month?

Since 1 rupee $=$ 4 four-anna pieces,

$\therefore$ x rupees $= 4x$ four-anna pieces

Clearly therefore the man earns $4x$ four-anna pieces per month.

Hence, the number of four-anna pieces earned in half a month $= \frac{1}{2}$ of $4x = 2x$.

Example 2 If an insect creeps up a pole x inches per minute, how many feet will it rise in y hours?

Since 1 inch $= \frac{1}{12}$th of a foot,

$\therefore$ x inches $= \frac{x}{12}$ths of a foot

Hence, in 1 minute the insect creeps up $\frac{x}{12}$ths ft.;

$\therefore$ in 60 minutes " " " " $\frac{x}{12} \times 60$ft.

Thus in 1 hour the insect creeps up $5x$ ft.

Therefore in y hours it rises $(5x \times y)$ ft.

Thus the required number of feet $= 5xy$.

Example 3. If a man travels at the rate of x miles per hour, in what time will he finish a journey of 10 miles?

Since x miles is travelled in 1 hour,

$\therefore$ 1 mile " " " $\frac{1}{x}$th of an hour;

$\therefore$ 10 miles is travelled in $\frac{10}{x}$hours.

Example 4. The digits of a number beginning from the left are x and y. How would you represent the number?

If the digits be 4 and 5, the number $= 10 \times 4 + 5$;
if the digits be 5 and 7, the number $= 10 \times 5 + 7$,
if the digits be 8 and 4, the number $= 10 \times 8 + 4$;
and so on.

Hence, it is quite clear that when x and y stand for the digits, the number is to be represented by $10x + y$.

Exercise (67).

1. The sum of two numbers is 15, if one of the numbers be x, what is the other?

2. The difference of two numbers is 20, if x be the greater, what is the other?

3. The difference of two numbers is 25; if x be the smaller, what is the greater?

4. What is the excess of 25 over y?

5. What is the defect of $2x$ from y?

6. If x be one factor of 21, what is the other factor?

7. What number is less than 100 by $3x$?

8. What number taken from $4x$ gives $3y$ as a remainder?

9. If a man travels x hours at the rate of y miles an hour, how many miles does he travel?

10 If a man travels at the rate of y miles per hour, in what time will he finish a journey of x miles?

11. A man is x years of age, how old will he be 20 years hence? How old was he 3 years ago?

12. In x days a man travels 60 miles, what is his rate per day?

13. If a train travells 30 miles in x hours, how many feet does it travel in one second?

14. If I spend x annas a week, how many rupees do I save out of a yearly income of $5x$ rupees?

15. Write down 5 consecutive numbers of which x is the middle one

16. Write down the sum of 3 consecutive numbers of which the middle one is x

17. What is the odd number next after $2m+1$?

18. What is the even number next before $2x$?

19. If x men take 10 days to do a work, in what time will y men do it?

20 A room is a yards long and b feet wide, what is the measure of the area of the floor in square feet?

21. In the last question find the number of square units in the area when the unit of length is 4 feet

22. How many miles can a person walk in 20 minutes, if he walks x miles in y hours?

23. In what time will a person walk 16 miles, if he walks x miles in a hours?

24. What is the present age of a man who was $(x-5)$ years old 20 years ago? What will be his age 30 years hence?

25. If the digits of a number beginning from the right are x and y, what is the number?

26. If x, y, z be the digits of a number beginning from the left, what is the number?

27. In the preceding question, if the digits be inverted, how would you represent the new number?

2. Easy Problems. We shall now work out some problems which will fairly introduce the beginner to the subject of the present chapter. The unknown quantity will invariably be represented by x.

Example 1. Divide 34 into two parts whose difference is 8.

Let x denote the larger part.

Then $34-x$ denotes the smaller part.

Hence, by the question,

$$x-(34-x)=8,$$

$$\text{or,}\qquad 2x-34=8;$$

$$\therefore\ 2x=42,\qquad \therefore\ x=21.$$

Thus the larger part is 21 and the smaller part is 13.

Example 2. What number is that of which the *third* part exceeds the *fifth* part by 4?

Let x represent the required number.

Then, by the given condition,

$$\frac{x}{3}-\frac{x}{5}=4;$$

$$\therefore\ 5x-3x=60,$$

$$\text{or, } 2x=60,\qquad \therefore\ x=30.$$

Example 3. A starts upon a walk at the rate of 4 miles an hour, and after 15 minutes B starts at the rate of $4\frac{3}{4}$ miles an hour. When and where will he overtake A?

Let x = number of hours in which B overtakes A.

The distance travelled by B in x hours $=4\frac{3}{4}\times x$ miles $=\frac{19}{4}x$ miles.

And since A started a quarter of an hour earlier, the distance travelled by him when he is overtaken $=(x+\frac{1}{4})\times 4$ miles $=(4x+1)$ miles.

Thus the same distance is denoted by $\frac{19}{4}x$ as well as by $4x+1$

$$\therefore \quad \tfrac{19}{4}x = 4x+1,$$
$$\therefore \quad 19x = 16x+4,$$
$$\therefore \quad 3x = 4, \quad \therefore \quad x = \tfrac{4}{3}.$$

Hence, the time in which B overtakes A $= \frac{4}{3}$ hours $= 1$ hour 20 minutes

Hence, also, the distance of the place where B overtakes A from the place of starting $= \frac{4}{3} \times 4\frac{3}{4}$ miles $= \frac{19}{3}$ miles $= 6\frac{1}{3}$ miles

Example 4. Two persons started at the same time from A One rode on horse back at the rate of $7\frac{1}{2}$ miles an hour and arrived at B, 30 minutes later than the other who travelled the same distance by train at the rate of 30 miles an hour Find the distance between A and B

(Calcutta University Entrance Paper, 1873)

Let x be the distance in miles between A and B Then the time taken by the first man to travel the distance $= \dfrac{x}{7\frac{1}{2}}$ hours $= \dfrac{2x}{15}$ hours, and the time taken by the other $=$ $\dfrac{x}{30}$ hours.

But the time taken by the former is half an hour more than that taken by the latter

Hence, $\quad \dfrac{2x}{15} = \dfrac{x}{30} + \dfrac{1}{2},$

or, $\quad 4x = x+15,$

$$\therefore \quad 3x = 15, \quad \therefore \quad x = 5.$$

Thus, the distance between A and B $= 5$ miles

Example 5. A person being asked his age, replied, 'Ten years ago I was 5 times as old as my son, but 20 years hence I shall be only twice as old as he' What is his age ?

Let the present age of the person be x years.

Then 10 years ago his age was $(x-10)$ years and ∴ that of his son was $\frac{1}{5}(x-10)$ years.

Hence the present age of his son $= \{\frac{1}{5}(x-10)+10\}$ years, and ∴ the son's age 20 years hence will be $\{\frac{1}{5}(x-10)+30\}$

years; and the age of the person 20 years hence will evidently be $(x+20)$ years

Hence, by the second condition of the problem, we must have

$$x+20 = 2\{\tfrac{1}{5}(x-10)+30\}$$
$$= \tfrac{2}{5}(x-10)+60,$$
$$\therefore \quad 5x+100 = 2x-20+300,$$
$$\therefore \quad 3x = 180, \quad \therefore \quad x = 60.$$

Note Fractions might have been avoided by assuming the present age of the person to be $5x$ years The student can easily proceed on this assumption.

Example 6. A and B have the same income. A lays by a *fifth* of his, but B, by spending annually £80 more than A, at the end of 4 years finds himself £220 in debt What was their income?

Let £x be the income of each.

Then A spends £$\tfrac{4}{5}x$ annually. Hence, B spends annually £$(\tfrac{4}{5}x+80)$.

But spending at this rate B contracts a debt of £220 in 4 years, or a debt of £55 per year His annual income therefore falls short of his annual expenses by £55.

Hence, we must have

$$x = (\tfrac{4}{5}x+80)-55,$$
$$\therefore \quad \tfrac{1}{5}x = 25, \quad \therefore \quad x = 125.$$

Thus A and B had each an income of £ 125.

Example 7. A market woman bought a certain number of eggs at 2 a penny, and as many at 3 a penny, and sold them at the rate of 5 for two pence, losing *4d* by her bargain What number of eggs did she buy?

Let x = the number of eggs bought.

Then, since one half of them were bought at 2 a penny, and the other half at 3 a penny, the whole cost in buying the eggs

$$= \left(\frac{x}{2}\cdot\frac{1}{2}+\frac{x}{2}\cdot\frac{1}{3}\right)\text{pence} = \left(\frac{x}{4}+\frac{x}{6}\right)\text{pence}$$

By selling the eggs at 5 for two pence, the amount realised $= x\times\tfrac{2}{5}$ pence.

Hence, by the question,

$$\frac{2x}{5} = \left\{\frac{x}{4}+\frac{x}{6}\right\}-4,$$

or, $24x = 15x+10x-240,\quad \therefore\quad x = 240$

Thus, altogether 240 eggs were bought

Example 8 There is a number consisting of two digits, the digit in the unit's place is twice that in the ten's place, and if 2 be subtracted from the sum of the digits, the difference is equal to $\frac{1}{6}$th of the number Find the number.

Let $x =$ the digit in the ten's place

Then $2x =$ „ „ „ „ unit's „ .

Clearly therefore the number $= 10x+2x$.

(See example 4 worked out in the last article)

Hence, by the second condition of the problem.

$$\left\{x+2x\right\}-2 = \frac{10x+2x}{6},$$

whence $18x-12 = 12x,$

$6x = 12,\quad \therefore\quad x = 2.$

Hence, the required number $= 24$

Exercise (68).

1. Find two numbers whose sum is 50, and whose difference is 30

2. Find a number such that it is equal to five times its defect from 96

3. Find a number which being multiplied by 8, the product will be greater than half the number by 90

4 What number is that from which if you subtract 40, the difference will be one-third of the original number ?

5. What number is that of which the excess over 35 is less by 22 than its defect from 67 ?

6. Four times the excess of a number over 16 is equal to the defect of the number from 416 find the number

7. Find 3 consecutive numbers whose sum shall be 129.

8. Find a number which when multiplied by 7 is as much above 132 as it was originally below it

9. Divide 90 into two parts such that three times one of the parts together with four times the other may be equal to 335

10. The sum of two numbers is 39 and *one-fifth* of one of them is equal to *one-eighth* of the other. Find them.

11. Find a number whose *fourth* part exceeds its *ninth* part by 5.

12. Find a number whose *sixth* part exceeds its *eighth* part by 3

13. Divide 21 into two parts, so that ten times one of them may exceed nine times the other by 1

14. A house and a garden cost £850 and the price of the garden = $\frac{5}{12}$ths of the price of the house, find the price of each

15. Divide £420 among two persons, so that for every shilling one receives, the other may receive half a crown.

16. Two shepherds, owning a flock of sheep, agree to divide its value A takes 72 sheep, while B takes 92 sheep and pays A £35 Find the value of a sheep

17. The ages of two men differ by 10 years, and 15 years ago the elder was just twice as old as the younger; find the ages of the men

18. The length of a field is twice its breadth, another field, which is 50 yds longer and 10 yds broader, contains 6800 square yds. more than the former, find the size of each

19. If a train, which travels at the rate of 35 miles an hour, start one quarter of an hour after a luggage train, and overtake it in ten minutes, find the speed of the luggage train

20. A person has just 5 hours at his disposal, how far may he ride in a coach which travels 8 miles an hour so as to return home in time, walking back at the rate of 2 miles an hour

21 A father's age is three times that of his son, and in 10 years it will be twice as great, how old are they?

22. The length of a room exceeds its breadth by 3 feet; if the length had been increased by 3 feet, and the breadth

diminished by 2 feet, the area would not have been altered, find the dimensions

23. A and B begin to play with equal sums, and when B has lost five-elevenths of what he had to begin with, A has gained £6 more than half of what B has left; what had they at first?

24. The ages of a father and his son together are 80 years, and if the age of the son be doubled, it will exceed the father's age by 10 years Find the age of each.

25. A person distributed £5 among 36 persons, old men and widows, giving 3*s* each to the men and 2*s*. 6*d*. each to the women How many were there of each?

26. There are two places 154 miles distant from each other, from which two persons A and B set out at the same instant with a design to meet on the road, A travelling at the rate of 3 miles in 2 hours, and B at the rate of 5 miles in 4 hours How long and how far did each travel before they met?

27. A labourer was engaged for 36 days, upon the condition that he should receive 2*s* 6*d* for every day he worked, but should pay 1*s* 6*d* for every day he was idle. At the end of the time he received 58*s*. How many days did he work?

28. A person bought a picture at a certain price and paid the same price for the frame, if the frame had cost £1 less and the picture 15*s* more, the price of the frame would have been only half that of the picture Find the cost of the picture.

29. A post is a fourth of its length in the mud, a third of its length in the water and 10 feet above the water, what is its length?

30. A labourer is engaged for 30 days, on condition that he receives 2*s* 6*d* for each day he works and loses 1*s*. for each day he is idle, he receives £2. 7*s* in all How many days does he work, and how many days is he idle?

31. A can do a piece of work in 9 days, B in twice that time, C can only do $\frac{3}{4}$ as much as A in a day; how long would A, B and C, working together, require to do the same piece of work? (C U Entr Paper, 1876.)

32. Two sums of money are together equal to £54. 12s. and there are as many pounds in the one as shillings in the other. What are the sums? (C U Entrance Paper, 1885)

33. A certain sum is to be divided among A, B and C. A is to have £30 less than half, B is to have £10 less than the third part, and C is to have £8 more than the fourth part, of the sum. What does each receive?

34. A farmer wishing to purchase a number of sheep, found that if they cost him £2 2s a head, he would not have money enough by £1. 8s; but if they cost him £2 a head, he would then have £2 more than he required Find the number of sheep, and the money which he had

35 Two coaches start at the same time from York and London, a distance of 200 miles, travelling, one at $9\frac{1}{2}$ miles an hour, the other at $9\frac{1}{4}$ Where will they meet? and in what time from starting?

36. I bought a certain number of apples at three a penny, and five-sixths of that number at four a penny; by selling them at sixteen for six pence I gained $3\frac{1}{2}d$.; how many apples did I buy?

37. A number consists of two digits; the sum of the digits is 5, and if the left digit be increased by 1 it will be equal to $\frac{1}{5}$th of the number. Find the number.

38. A number consists of two digits, the digit in the ten's place exceeds that in the unit's place by 5, and if 5 times the sum of the digits be subtracted from the number, the digits will be inverted Find the number.

39. There is a number, the sum of whose digits is 5, and if 10 times the digit in the place of tens be added to 4 time the digit in the place of units, the number will be inverted What is the number?

40. Divide the number 39 into four parts, such that if the first be increased by 1, the second diminished by 2, the third multiplied by 3, and the fourth divided by 4, the results will all be equal

41. Divide 60 into 4 parts, such that if the first be diminished by 3, the second increased by 11, the third multiplied by 4, and the fourth divided by 2, the results will all be equal

42. Divide the number 116 into four such parts that if the first be increased by 5, the second diminished by 4,

the third multiplied by 3, and the fourth divided by 2, the result in each case shall be the same

3. More difficult problems.

Example 1. At what time between 1 o'clock and 2 o'clock is there exactly one minute-division between the two hands of a clock?

Suppose it is x minutes past one when the hands are one minute-division apart from each other

Then, at the required instant the minute-hand is at a distance of x minute-division from the 12 o'clock mark and since the minute-hand moves twelve times as fast as the hour-hand, the hour-hand moves over $\frac{x}{12}$ths of a minute-division whilst the minute-hand moves over x minute divisions, therefore at the required instant the hour-hand is at a distance of $\left(5+\frac{x}{12}\right)$ minute-division from the 12 o' clock mark

Hence, as the minute-hand is at the required instant one minute-division apart from the hour-hand, we must have $x = \left(5+\frac{x}{12}\right) \pm 1$

The upper sign being taken when the *minute-hand is* ahead of the hour-hand, and the lower when behind it.

$$\tfrac{11}{12}x = 5 \pm 1 = 6 \text{ or } 4,$$

$$\therefore \quad x = \tfrac{72}{11} \quad = 6\tfrac{6}{11}, \text{ or } = \tfrac{48}{11} = 4\tfrac{4}{11}$$

Thus the hands are 1 minute-division apart at $4\frac{4}{11}$ or $6\frac{6}{11}$ minutes past one

Example 2. The distance from a place P to another place Q is $3\frac{1}{2}$ miles Two persons, A and B, start together from P to go to Q, the former by carriage which travels at the rate of 6 miles an hour, the latter walking at the rate of 3 miles an hour If A remains at Q for 15 minutes, and then returns by the carriage to P, find where he will meet B. (Calcutta University Entrance Paper, 1882)

Let x miles be the distance of the place of meeting from P

Then, during the time that B travels x miles, A finishes the journey, remains at Q for 15 minutes, and then travels back $(3\frac{1}{2}-x)$ miles

Now, the time in which A does all these

$$= \left(\frac{3\frac{1}{2}}{6}+\frac{1}{4}+\frac{3\frac{1}{2}-x}{6}\right) \text{ hours},$$

and the time in which B travels x miles $= \dfrac{x}{3}$ hours;

$$\therefore \quad \frac{3\frac{1}{2}}{6}+\frac{1}{4}+\frac{3\frac{1}{2}-x}{6} = \frac{x}{3}$$

$$\text{or,} \qquad 7+3+(7-2x) = 4x,$$

$$\therefore \quad 6x = 17, \quad \therefore \quad x = 2\tfrac{5}{6}.$$

Thus A will meet B at a distance of $2\frac{5}{6}$ miles from P.

Example 3. A Landlord let his farm for £10 a year in money, and a corn-rent When corn sold at 10*s*. a bushel he received at the rate of 10 shillings an acre for his land; but when it sold at 13*s* 6*d* a bushel, 13 shillings an acre. Of how many bushels did the corn-rent consist ?

Let $x =$ the number of bushels the corn-rent consisted of.

Then, when corn sold at 10*s* a bushel, the annual income was £$10+10x$ shillings or $(200+10x)$ shillings; hence, as the income in this case was at the rate of 10*s*. an acre, the number of acres must evidently be $\dfrac{200+10x}{10}$, or, $20+x$

In the second case (*i e*, when corn sold at 13*s*. 6*d*. a bushel) the annual income amounted to £$10+(13\frac{1}{2})x$ shillings, or, $\dfrac{400+27x}{9}$ shillings, but now the income was at the rate of 13*s* an acre. Hence the number of acres must also be equal to $\dfrac{400+27x}{26}$.

$$\text{Hence,} \quad 20+x = \frac{400+27x}{26},$$

$$\text{or,} \qquad 520+26x = 400+27x, \qquad \therefore \quad x = 120.$$

Thus the corn-rent consisted of 120 bushels.

Example 4. A hare is eighty of her own leaps before a greyhound; she takes three leaps for every two that he takes, but he covers as much ground in one leap as she

does in two How many leaps will the hare have taken before she is caught ?

Let $3x$ = the number of leaps the hare takes.

Then $2x$ = the number of leaps the greyhound takes in the same time.

The distance of the place where the hare is caught from the first position of the greyhound $= (80+3x)$ leaps of the hare, and is also $= 2x$ leaps of the greyhound.

But, 1 leap of the greyhound being equal to 2 leaps of the hare, $2x$ leaps of the greyhound $= 4x$ leaps of the hare

$$\therefore \quad 80+3x = 4x, \qquad \therefore \quad x = 80$$

Hence, the number of leaps which the hare takes before she is caught $= 3 \times 80 = 240$

Example 5. A banker has two kinds of money, silver and gold, and a pieces of silver or b pieces of gold, make up the same sum s A person comes and wishes to be paid the sum s with c pieces of money, how many of each must the banker give him ?

Let x = the number of silver pieces required,

then $c-x$ = " " " gold ",

$$\left.\begin{array}{l} \text{The value of one piece of silver} = \dfrac{s}{a} \\ \text{and that of one piece of gold} = \dfrac{s}{b} \end{array}\right\}$$

Hence, since by supposition x pieces of silver and $(c-x)$ pieces of gold are together equal in value to s, we must have

$$s = x.\frac{s}{a}+(c-x)\frac{s}{b},$$

$$\therefore \quad 1 = \frac{x}{a}+\frac{c-x}{b}$$

$$\text{or, } x\left(\frac{1}{b}-\frac{1}{a}\right) = \frac{c}{b}-1,$$

$$\therefore \quad x = \frac{b(c-b)}{a-b},$$

$$\text{and} \quad \therefore \quad c-x = c-\frac{a(c-b)}{a-b} = \frac{b(a-c)}{a-b}.$$

Thus, $\dfrac{a(c-b)}{a-b}$ pieces of silver and $\dfrac{b(a-c)}{a-b}$ pieces of gold will be required.

Example 6. AB is a railway 220 miles long, and three trains (P, Q, R) travel upon it at the rate of 25, 20 and 30 miles per hour respectively, P and Q leave A at 7 A.M. and 8-15 A.M. respectively, and R leaves B at 10-30 A.M. When and where will P be equidistant from Q and R?

A Q P R B

Let P, Q, R, as in the figure, be the respective positions of the trains at the instant when P is equally distant from Q and R.

Let this happen x hours after R has left B, *i.e.*, x hours after 10-30 A.M.

Then, since P left A $3\frac{1}{2}$ hours before 10-30 A.M., it has evidently been travelling for $(3\frac{1}{2}+x)$ hours up to the instant in question.

Similarly, Q has been travelling for $(2\frac{1}{4}+x)$ hours up to that instant.

Hence, clearly $AP = (3\frac{1}{2}+x).25$ miles,

and $AQ = (2\frac{1}{4}+x).20$ miles;

also $BR = 30x$ miles.

Hence, $PQ = AP-AQ$

$= \{(3\frac{1}{2}+x).25-(2\frac{1}{4}+x).20\}$ miles,

and $PR = AB-AP-BR$

$= \{220-(3\frac{1}{2}+x).25-30x\}$ miles.

But $PQ = PR$.

$\therefore\ (3\frac{1}{2}+x).25-(2\frac{1}{4}+x).20$

$= 220-(3\frac{1}{2}+x).25-30x,$

$\therefore\ 50(3\frac{1}{2}+x)-(2\frac{1}{4}+x).20 = 220-30x,$

$\therefore\ 60x = 220-175+45 = 90,$

$\therefore\ x = 1\frac{1}{2}.$

Thus P will be equally distant from Q and R at $1\frac{1}{2}$ hours after 10-30 A.M., *i.e.*, at 12 A.M.

Also as P left A at 7 A.M., its distance form A at that instant will be 5×25 or 125 miles.

Example 7 Two passengers have together 5 cwt. of luggage and are charged for the excess above the weight allowed 5*s* 2*d* and 9*s*, 10*d*, respectively but if the luggage had all belonged to one of them he would have been charged 19*s* 2*d* How much luggage is each passenger allowed to carry free of charge? And how much luggage had each passenger? (Calcutta University Entrance Paper, 1877)

Let x cwt = weight of luggage that each passenger is allowed to carry free of charge

Then (5*s* 2*d*.)+(9*s* 10*d*) = charge for $(5-2x)$ cwt,

$$\therefore \quad \frac{15\times 12}{5-2x}d = \text{charge for 1 cwt}$$

Also, 19*s*. 2*d*. = charge for $(5-x)$ cwt,

$$\therefore \quad \frac{230}{5-x}d = \text{charge for 1 cwt}$$

$$\text{Hence,} \quad \frac{15\times 12}{5-2x} = \frac{230}{5-x},$$

$$\therefore \quad 18(5-x) = 23(5-2x)$$

$$\text{or,} \quad 28x = 115-90 = 25, \therefore x = \tfrac{25}{28};$$

i e, weight of luggage allowed $= \frac{25}{28}$ cwt. $= \frac{25}{28}\times 4\times 28$ lbs. = 100 lbs

Now charge for 1 cwt

$$= \frac{230}{5-x}d = \frac{230}{5-\frac{25}{28}}d = \frac{230\times 28}{5\times 23}d = 56d$$

And since charge for excess luggage of the first passenger = 5*s* 2*d*. = 62*d*, and charge for excess luggage of the second passenger = 9*s* 10*d* = 118*d*.;

$\therefore$ weight of excess luggage of the first passenger $= \frac{62}{56}$ cwt. $= \frac{62}{56}\times 4\times 28$ lbs = 124 lbs.;

and weight of excess luggage of the second passenger $= \frac{118}{56}$ cwt $= \frac{118}{56}\times 4\times 28$ lbs = 236 lbs

Hence, whole luggage of the first passenger

= (100+124) lbs. = 224 lbs;

and whole luggage of the second passenger

= (100+236) lbs = 336 lbs

Example 8. A person buys some tea at 3 shillings a pound and some at 5 shillings a pound; he wishes to mix them, so that by selling the mixture at 3s. 8*d.* a pound, he may gain 10 per cent on each pound sold. Find how many pounds of the inferior tea he must mix with each pound of the superior.

Suppose x lbs. of the inferior tea are mixed with each pound of the superior,

the price of x lbs of the inferior tea and one pound of the superior $= (3x+5)$ shillings;

$\therefore$ the average cost per pound $= \dfrac{3x+5}{x+1}$ shillings.

But by selling the mixture at $3\frac{2}{3}s$ a pound, he *gains* 10 *per cent* on each pound, *i e.*, realises 110*s* for every 100*s.* or $\frac{11}{10}$s. for every shilling.

Hence, $\qquad 3\frac{2}{3}s = \frac{11}{10}$ of the cost per pound;

$$\therefore \quad 3\tfrac{2}{3} = \frac{11}{10}\times\frac{3x+5}{x+1},$$

$$\text{or,} \quad \frac{11}{3} = \frac{11}{10}\times\frac{3x+5}{x+1},$$

$$\therefore \quad 10(x+1) = 3(3x+5),$$

$$\therefore \quad x = 5$$

Thus, 5 pounds of the inferior tea must be mixed with each pound of the superior.

Example 9. An officer can form his men into a hollow square 5 deep, and also into a hollow square 6 deep, but the front in the latter formation contains 4 men fewer than in the former; find the number of men. (C U Entr Paper, 1887)

[A number of men are said to be arranged in a *solid* square when they are arranged in parallel rows and the number of rows is equal to the number of men in each row. The accompanying diagram, in which A_1, B_1, C_1, &c represent men, will give the student a correct notion of such arrangements

$$
\begin{array}{cccccccc}
A_1 & B_1 & C_1 & D_1 & E_1 & F_1 & G_1 & H_1 \\
A_2 & B_2 & C_2 & D_2 & E_2 & F_2 & G_2 & H_2 \\
A_3 & B_3 & C_3 — & D_3 — & E_3 — & F_3 & G_3 & H_3 \\
A_4 & B_4 & C_4 & D_4 — & E_4 & F_4 & G_4 & H_4 \\
A_5 & B_5 & C_5 & D_5 — & E_5 & F_5 & G_5 & H_5 \\
A_6 & B_6 & C_6 — & D_6 — & E_6 — & F_6 & G_6 & H_6 \\
A_7 & B_7 & C_7 & D_7 & E_7 & F_7 & G_7 & H_7 \\
A_8 & B_8 & C_8 & D_8 & E_8 & F_8 & G_8 & H_8
\end{array}
$$

The diagram represents an arrangement in which there are 8 rows, each containing 8 men. This is a *solid* square. If the square $C_3\ F_3\ F_6\ C_6$ be removed from inside, the remainder will be a *hollow* square *two deep* having 8 men in the front rank; if, however, the square $D_4\ E_4\ E_5\ D_5$ be removed, the remainder will be a *hollow* square *three deep* having the same 8 men in the front rank.

Hence the number of men in a *hollow* square two deep having x men in the front rank $= x^2-(x-4)^2$; in one three deep $= x^2-(x-6)^2$, and so on; thus the number of men in a hollow square n deep having x men in the front row $= x^2-(x-2n)^2$.]

Let $x =$ the number of men in the front row of the first arrangement.

Then $x-4 =$ the number of men in the front row of the second arrangement.

Hence, the number of men in the first square

$$= x^2-(x-10)^2 \quad . \qquad (1)$$

and the number of men in the second square

$$= (x-4)^2-\{(x-4)-12\}^2.$$

But the men that form the first square are exactly those that form the second;

$$\therefore\quad x^2-(x-10)^2 = (x-4)^2-\{(x-4)-12\}^2$$

$$\text{or,}\qquad 20x-100 = 24(x-4)-144,$$

$$\therefore\quad 4x = 144+96-100 = 140,$$

$$\therefore\quad x = 35.$$

Hence, from (1) the total number of men

$$= (35)^2-(25)^2 = 60\times 10 = 600.$$

Exercise (69).

1. Find the time between 3 and 4 o'clock, when the two hands of a watch are coincident

2. At what time are the hands of a watch together between 5 and 6 o'clock ? (C U Entr Paper, 1886)

3. Find the respective times between 7 and 8 o'clock when the hour and minute hands of a watch are, first, exactly opposite to each other ; second, at right angles to each other , third, coincident.

4. What is the *first* hour after 6 o'clock at which the two hands of a watch are (i) directly opposite, and (ii) at right angles, to each other ?

5. Two men set out at the same time to walk, one from A to B, and the other from B to A, a distance of a miles The former walks at the rate of p miles and the latter at the rate of q miles an hour , at what distance from A will they meet ?

6. Two persons walk at the rate of 5 and 6 miles an hour respectively They set out to meet each other from two places 22 miles apart. Having passed each other once, find the place of their *second* meeting, supposing them to continue their journey between the two places Also find the time when the second meeting takes place.

7. A man rides one-third of the distance from A to B at the rate of a miles per hour and the remainder at the rate of $2c$ miles per hour. If he had travelled at a uniform rate of $3c$ miles per hour he could have ridden from A to B and back again in the same time

Prove that $\frac{2}{c} = \frac{1}{a} + \frac{1}{b}$. (C U Entr Paper 1889)

8. A and B start to run a race At the end of 5 minutes, when A has run 900 yards and has outstripped B by 75 yards, he falls , but though he loses ground by the accident, and for the rest of the course makes 20 yards a minute less than before, he comes in only half a minute behind B. How long did the race last ?

9. A person sets out to walk from a certain town ; but when he has accomplished a quarter of his journey, he finds that if he continues at the same pace he will have gone only $\frac{5}{6}$ths of the whole distance when he ought to be

at his destination. He therefore increases his speed by a mile an hour, and arrives just in time Find the rate of walking

10 A tenant hired his farm for £80 a year in money and a corn-rent in rice When rice sold at £1. 5s a bushel, he paid at the rate of £1. 15s. an acre for his land ; when it sold at £1 10s a bushel, he paid at the rate of £2 an acre Find the number of bushels of rice in the rent

11. A footman who contracted for £8 a year and a livery suit, was turned away at the end of 7 months and received only £2. 3s 4d and his livery What was its value ?

12 A hare, 50 of her leaps before a greyhound, takes 4 leaps to the greyhound's three , but two of the greyhound's leaps are as much as three of the hare's How many must the greyhound take to catch the hare ?

13. A greyhound spying a hare at the distance of 60 of his own leaps from him, pursues her, making 4 leaps for every 5 leaps of the hare , but he passes over as much ground in 3 leaps as the hare does in 4 How many leaps did each make during the whole course ?

14. The St John's boat is ahead of the Caius by a distance equivalent to 33 strokes of the former The Johnians pull 4 strokes to 3 strokes of the Caius, but 2 of the latter are equivalent to 3 of the former How many strokes must the Caius take to bump the St John's boat ?

15 A and B find a purse with shillings in it A takes out two shillings and one-sixth of what remains , then B takes out three shillings and one-sixth of what remains , and then they find that they have taken out equal shares How many shillings were in the purse, and how many did each take ?

16 A ship sails with a supply of biscuit for 60 days at a daily allowance of 1 pound a head after being at sea 20 days she encounters a storm in which 5 men are washed overboard and damage sustained, that will cause a delay of 24 days, and it is found that each man's allowance must be reduced to $\frac{5}{7}$ths of a pound. Find the original number of the crew.

17. If 19 lbs of gold weigh 18 lbs. in water, and 10 lbs of silver weigh 9 lbs. in water, find the quantity of gold

and silver in a mass of gold and silver weighing 106 lbs. in air and 99lbs. in water.

18. A person rows from Cambridge to Ely, a distance of 20 miles and back again in 10 hours, the stream flowing uniformly in the same direction all the time, and he finds that he can row 2 miles against the stream in the same time that he rows 3 miles with it. Find the time of his going and returning.

19. A person passed $\frac{1}{6}$th of his age in childhood, $\frac{1}{12}$th in youth, $\frac{1}{7}$th $+5$ years in matrimony, he had then a son, whom he survived 4 years, and who reached only one-half the age of his father. Find the son's age when he died

20. There are two bars of metal, the first containing 14 *oz.* of silver and 6 of tin, the second containing 8 of silver and 12 of tin; how much must be taken from each to form a bar of 20 *oz.* containing equal weights of silver and tin?

21. Divide £607 1*s* 8*d* into two sums, such that the simple interest of the greater sum for two years, at $3\frac{1}{2}$ per cent shall exceed that of the less for $2\frac{1}{2}$ years, at $3\frac{1}{4}$ per cent by £18 16*s*

22 To remove four articles of furniture, I required for the 1st article two coolies for the 2nd three, for the third four, and for the 4th five After giving the first set of men one group of pice and one pice more, to the 2nd set an equal group and four pice more, to the 3rd an equal group and five pice more, and to the 4th an equal group and nine pice more, I found that each man of the 3rd and 4th sets had received the same number of pice How many pice were there in each group, how many pice did each man receive, and how many pice did I distribute?

23. Fifteen current guineas should weigh 4 ounces; but a parcel of light gold being weighed and counted, was found to contain 9 more guineas than was supposed from the weight, and a part of the whole exceeding the half by 10 guineas and a half, was found to be $1\frac{1}{3}$ *oz* deficient in weight. What was the number of guineas in the parcel?

24. A silversmith received in payment for a certain weight of wrought plate, the price of which was £10, the same weight of unwrought plate and £3. 15*s* besides At another time he exchanged 12 *oz* of wrought plate of

the same workmanship as before for 8 *oz* of unwrought (for which he allowed the same price as before), and £2 16*s* in money What was the price of wrought plate per ounce, and the weight of the first sold ?

25 Two passengers are charged for excess of luggage 2*s*. 30*d* and 7*s* 6*d* respectively, had the luggage all belonged to one of them, he would have been charged for excess 14*s* 6*d* how much would they have been charged if none had been allowed free ?

26. How many bundles of hay, at Rs 5 per thousand, must a *ghaswala* mix with 5600 bundles at Rs 6 per thousand, in order that he may gain 20 per cent by selling the whole at 11 as per hundred ? (C U Entr Paper, 1875)

27. A boy buys a certain number of oranges at 3 for 2*d* and one-third of that number at 2 for 1*d*., at what price must he sell them to get 20 per cent profit, if his profit be 5*s* 4*d*, find the number bought. (C U Entr Paper, 1885)

28. From each of a number of foreign gold coins a person filed a fifth part, and had passed two-thirds of them, when the rest were seized as light coins except one, with which the man decamped, having lost upon the whole half as much as he had gained before How many coins were there at first ?

29. Find a number of three digits, each greater by unity than that which follows it, so that its excess above one-fourth of the number formed by inverting the digits shall be 36 times the sum of the digits

30. A number of troops being formed into a solid square it was found there were 60 over, but when formed into a column with 5 men more in front than before and 3 less in depth, there was just one man wanting to complete it Find the number.

31. An officer can form the men of his regiment into a hollow square 10 deep The number of men in the regiment is 2800. Find the number of men in the front of the hollow square

32. A company of men is formed into a hollow square 4 deep and also into a hollow square 8 deep, the front in the latter formation contains 19 men fewer than that in the former formation, find the number of men

33. A detachment from an army was marching in regular column with 5 men more in depth than in front; but upon the enemy coming in sight, the front was increased by 845 men; and by this movement the detachment was drawn up in five lines. Find the number of men in the detachment.

CHAPTER XIV

SIMULTANEOUS EQUATIONS OF THE FIRST DEGREE

1. **Introductory remarks.** The equation $x-y=2$, in which x and y are both unknown, evidently admits of an infinite number of solutions; for any pair of numbers whose difference is 2 will satisfy it. [For instance, the equation will be satisfied if $x=3$, $y=1$; if $x=4$, $y=2$; if $x=5$, $y=3$; if $x=6$, $y=4$; and so on.] If however x and y be such that they must *also* satisfy the equation $x+y=8$, then of the different pairs of numbers whose difference is 2 we shall have to reject all excepting that of which the sum is 8. Thus the two equations,

$$\left.\begin{aligned} x-y &= 2 \\ x+y &= 8 \end{aligned}\right\}$$

will both be satisfied by the *same* values of x and y *only when* $x=5$ and $y=3$.

Again, it may be seen that the three equations,

$$\left.\begin{aligned} x+y+z &= 6 \\ x-y+z &= 4 \\ x+y-z &= 2 \end{aligned}\right\}$$

will be satisfied by the *same* values of x, y, z *only when* $x=3$, $y=1$, $z=2$. The equations may be *individually* satisfied by innumerable sets of values of the unknown quantities, but there is *only one* set which will satisfy them *all*.

Two or more equations (like those just referred to) which are *all* satisfied by the same values of the unknown quantities involved in them are called *simultaneous equations*. They are said to be *simple* or of the first degree when each

unknown quantity occurs only in the first power, and the product of the unknown quantities does not occur.

We shall consider first of all simultaneous equations involving two unknown quantities, and later on, those that involve more than two There are three general methods for solving such equations and we shall treat them successively in the next three articles

2. First Method. From either equation find one of the unknown quantities in terms of the other and substitute the value thus found in the other equation

Example 1 Solve $\left.\begin{aligned} 5x-24y &= 16 \\ 4x-y &= 31 \end{aligned}\right\}$

From the second equation, we have

$$y = 4x-31, \quad (1)$$

Substituting this value of y in the 1st equation, we have

$$5x-24(4x-31) = 16$$

or, $$5x-96x+744 = 16$$

$$\therefore \quad -91x = -728, \quad \therefore \ x = 8.$$

Hence, from (1), $y = 4\times 8-21 = 1$

Thus we have $x = 8$, and $y = 1$

Note The student is recommended to verify for his own satisfaction that these values of x and y do really satisfy *both* of the given equations

Example 2. Solve

$$\frac{3x-5y}{3}+3 = \frac{2x+y}{5}, \quad 8-\frac{x-2y}{4} = \frac{x}{2}+\frac{y}{3}.$$

Multiplying both sides of the first equation by 10,

we have $$5(3x-5y)+30 = 2(2x+y),$$

or, $$15x-25y+30 = 4x+2y,$$

$$11x = 27y-30 \quad (1)$$

Multiplying both sides of the second equation by 12,

we have $$96-3(x-2y) = 6x+4y,$$

or, $$96-3x+6y = 6x+4y,$$

$$\therefore \quad 2y-9x+96 = 0 \quad (2)$$

From (1), we have $x = \dfrac{27y-30}{11}$ (3)

Substituting this value of x in (2), we have

$$2y-\frac{9(27y-30)}{11}+96=0;$$

$$\therefore \quad 22y-9(27y-30)+1056=0,$$

or, $$22y-243y+270+1056=0,$$

$$\therefore \quad 221y=1326, \quad \therefore \quad y=6.$$

Hence, from (3), $x=\dfrac{27\times6-30}{11}=\dfrac{132}{11}=12.$

Thus we have $x=12, \quad y=6.$

Exercise (70).

Solve the following equations :—

1. $\left.\begin{aligned} x+4y &= 14 \\ 7x-3y &= 5 \end{aligned}\right\}.$

2. $\left.\begin{aligned} 5x-8y &= 9 \\ 13x+7y &= 79 \end{aligned}\right\}.$

3. $\left.\begin{aligned} 2x+3y &= 32 \\ 11y-9x &= 3 \end{aligned}\right\}.$

4. $\left.\begin{aligned} 9x-4y &= 8 \\ 13x+7y &= 101 \end{aligned}\right\}.$

5. $\left.\begin{aligned} x+ay &= b \\ ax-by &= c \end{aligned}\right\}.$

6. $\left.\begin{aligned} 2x-\tfrac{1}{5}(y-3) &= 4 \\ 3y+\tfrac{1}{3}(x-2) &= 9 \end{aligned}\right\}.$

7. $\left.\begin{aligned} \tfrac{1}{2}(x+y) &= \tfrac{1}{3}(2x+4) \\ \tfrac{1}{3}(x-y) &= \tfrac{1}{2}(x-24) \end{aligned}\right\}.$

8. $\left.\begin{aligned} \tfrac{1}{3}(x-y) &= \tfrac{1}{4}(y-1) \\ \tfrac{1}{7}(4x-5x) &= x-7 \end{aligned}\right\}.$

(C U Entrance Paper, 1872)

9 $\left.\begin{aligned} \tfrac{1}{2}(3x-2y)-3 &= \tfrac{1}{4}(2x-y) \\ \tfrac{1}{2}(5x-4y)-3 &= \tfrac{1}{3}(4x-3y) \end{aligned}\right\}.$

10. $\left.\begin{aligned} \tfrac{1}{6}(2x+3y)+\tfrac{1}{3}x &= 8 \\ \tfrac{1}{2}(7y-3x)-y &= 11 \end{aligned}\right\}$

3. Second Method. From each equation find the value of the same unknown quantity in terms of the other and equate the values thus found

Example 1. Solve $\left.\begin{aligned} 6x-5y &= 11 \\ 2x+3y &= 27 \end{aligned}\right\}.$

From the 1st equation, we have

$$5y=6x-11,$$

$$\therefore \quad y=\frac{6y-11}{5}. \quad \ldots \quad \ldots \quad (1)$$

From the 2nd equation, we have

$$3y = 27-2x,$$

$$y = \frac{27-2x}{3}. \qquad (2)$$

Hence, from (1) and (2), we have

$$\frac{6x-11}{5} = \frac{27-2x}{3},$$

$$\therefore\ 3(6x-11) = 5(27-2x)$$

$$\text{or,}\quad 18x-33 = 135-10x,$$

$$\therefore\quad 28x = 168, \therefore\ x = 6.$$

Hence, from (1), $y = \dfrac{6\times 6-11}{5} = 5.$

Thus we have $x = 6, \qquad y = 5$

Example 2. Solve $\left.\begin{aligned} \frac{7+x}{5}-\frac{2x-y}{4} &= 3y-5 \\ \frac{5y-7}{2}+\frac{4x-3}{6} &= 18-5x \end{aligned}\right\}$

(Calcutta University Entrance Paper, 1880)

Multiplying both sides of the 1st equation by 20,

$$4(7+x)-5(2x-y) = 20(3y-5)$$

$$\text{or,}\quad 28-6x+5y = 60y-100,$$

$$\therefore\quad 55y+6x = 128. \qquad (1)$$

Multiplying both sides of the 2nd equation by 6,

$$3(5y-7)+(4x-3) = 6(18-5x)$$

$$\text{or,}\quad 15y+4x-24 = 108-30x,$$

$$\therefore\quad 15y+34x = 132 \qquad (2)$$

From (1), $\qquad y = \dfrac{128-6x}{55} \qquad (3)$

From (2), $\qquad y = \dfrac{132-34x}{15}. \qquad (4)$

Hence, from (3) and (4), we have

$$\frac{128-6x}{55} = \frac{132-34x}{15}$$

or $\qquad \dfrac{64-3x}{11} = \dfrac{66-17x}{3}$ [Multiplying both sides by [illegible]]

$\therefore \qquad 3(64-3x) = 11(66-17x)$

or, $\qquad 192-9x = 726-187x,$

$\therefore \qquad 178x = 534, \quad \therefore \quad x = 3$

Hence, from (3), $y = \dfrac{128-18}{55} = \dfrac{110}{55} = 2$

Thus we have $\qquad x = 3, \qquad y = 2$

Exercise (71).

Solve the following equations —

1. $\left.\begin{array}{ll} 5x-3y & = 9 \\ 5y+2x & = 16 \end{array}\right\}$ 2. $\left.\begin{array}{ll} 3y-4x & = 1 \\ 3x+4y & = 18 \end{array}\right\}$

3. $\left.\begin{array}{ll} 3x-7y & = 7 \\ 11x+5y & = 87 \end{array}\right\}$ 4. $\left.\begin{array}{ll} y(3+x) & = x(7+y) \\ 4x+9 & = 5y-14 \end{array}\right\}$

5. $\left.\begin{array}{ll} 32x-25y & = 28 \\ 14x+15y & = 116 \end{array}\right\}$ 6 $\left.\begin{array}{l} \frac{1}{7}(3x+y) = \frac{1}{5}(2x+y+1) \\ 8-\frac{1}{5}(x-y) = 6 \end{array}\right\}$

7. $\left.\begin{array}{ll} \frac{11}{12}(5x-6y)+3x & = 4y-2 \\ \frac{1}{6}(5x+6y)-\frac{1}{4}(3x-2y) & = 2y-2 \end{array}\right\}$

8. $\left.\begin{array}{ll} 2x-\frac{1}{4}(y+3) & = 7+\frac{1}{5}(3y-2x) \\ 4y+\frac{1}{3}(x-2) & = 26\frac{1}{2}-\frac{1}{2}(2y+1) \end{array}\right\}$

9 $\left.\begin{array}{ll} 2x-\frac{1}{3}(2y-1) & = 3\frac{5}{24}+\frac{1}{4}(3x-2y) \\ 4y-\frac{1}{4}(5-2x) & = 6 \;-\frac{1}{5}(3 \;-2y) \end{array}\right\}$ (C U Entr Paper, 1873)

10. $\left.\begin{array}{ll} 6x-\dfrac{2y-x}{23-x} & = 20-\dfrac{59-12x}{2} \\[2ex] 3y+\dfrac{y-3}{x-18} & = 20=\dfrac{43-9y}{3} \end{array}\right\}$

4. **Third Method.** "Multiply the equations by such numbers as will make the co-efficient of one of the unknown quantities the same in the two resulting equations ; then by addition or subtraction we can form an equation containing only the other unknown quantity "

Example 1. Solve $\left.\begin{aligned} 3x-4y &= 5 \\ 5x+2y &= 17 \end{aligned}\right\}$

Multiplying the 2nd equation by 2, we have

$$\left.\begin{aligned} 10x+4y &= 34 \\ 3x-4y &= 5 \end{aligned}\right\}$$

and the first equation is

Hence, by addition, $13x = 39$, $\therefore$ $x = 3$.

Subtracting this value of x in the 1st equation,

we have $4y = 9-5 = 4$, $\therefore$ $y = 1$.

Thus we have $x = 3$, $y = 1$.

Example 2. Solve $\left.\begin{aligned} 5x+9y &= 89 \\ 2x-17y &= 15 \end{aligned}\right\}$

Multiplying the 1st equation by 2, and the 2nd by 5,

we have $\left.\begin{aligned} 10x+18y &= 178 \\ 10x-85y &= 75 \end{aligned}\right\}$
and

Hence, by subtraction,

$$103y = 103, \quad \therefore \quad y = 1.$$

Subtracting this value of y in the 2nd equation,

we have $2x = 15+17 = 32$, $x = 16$.

Thus we have $x = 16$, $y = 1$.

Note We might as well have multiplied the 1st equation by 17 and the 2nd equation by 9 and added the two resulting equations; this would have given us the value of x But we have preferred the other alternative because the co-efficients of x being smaller, the required multiplications have been more easily effected

Example 3. Solve $\left.\begin{aligned} 23x-24y &= 21 \\ 25x-16y &= 43 \end{aligned}\right\}$

Multiplying the 1st equation by 2, and the 2nd by 3,

we have $\left.\begin{aligned} 46x-48y &= 42 \\ 75x-48y &= 129 \end{aligned}\right\}$
and

Hence, by subtraction,

$$29x = 87, \quad \therefore \quad x = 3.$$

Substituting this value of x in the 2nd equation,

we have $16y = 75-43 = 32$, $\therefore$ $y = 2$.

Thus we have $x = 3$, $y = 2$

Note It may be noticed that the co-efficient of y in each of the resulting equations is the *least common multiple* of 24 and 16 and this is all that is required The process would have been unnecessarily tedious if the 1st equation were multiplied by 16 and the 2nd by 24

Example 4. Solve
$$\left.\begin{aligned}\frac{x-2}{2}-\frac{x+y}{14}&=\frac{x-y-1}{8}-\frac{y+12}{4}\\ \frac{x+7}{3}+\frac{y-5}{10}&=1-x-\frac{5(y+1)}{7}\end{aligned}\right\}$$

(C U Entrance Paper, 1882)

From the 1st equation, we have

$$\frac{7(x-2)-(x+y)}{14}=\frac{(x-y-1)-2(y+12)}{8},$$

or, $$\frac{6x-y-14}{7}=\frac{x-3y-25}{4},$$

or, $$24x-4y-56=7x-21y-175,$$

or, $$17x+17y=-119,$$

or, $$x+y=-7. \qquad \ldots \qquad (1)$$

From the 2nd equation, we have

$$\frac{10(x+7)+3(y-5)}{30}=\frac{7(1-x)-5(y+1)}{7},$$

or, $$\frac{10x+3y+55}{30}=\frac{2-7x-5y}{7},$$

or, $$70x+21y+385=60-210x-150y,$$

or, $$280x+171y=-325. \qquad \ldots \qquad (2)$$

Multiplying (1) by 171, we have

$$\left.\begin{aligned}171x+171y&=-1197,\\ \text{also,}\quad 280x+171y&=-\ \ 325.\end{aligned}\right\}$$

Hence, by subtraction,

$$-109x=872, \qquad \therefore\ x=8.$$

Substituting this value of x in (1),

we have $y=-7-8=-15$

Thus we have $x=\ 8,\ y=-15.$

Example 5. Solve
$$\left.\begin{aligned}\frac{2}{x}+\frac{3}{y}&=1\\ \frac{7}{x}+\frac{4}{y}&=1\tfrac{7}{8}\end{aligned}\right\}$$

Multiplying the 1st equation by 4, and the 2nd by 3 we have

$$\frac{8}{x}+\frac{12}{y}=4 \text{ and } \frac{21}{x}+\frac{12}{y}=\frac{45}{8}.$$

Hence, by subtraction,

$$\frac{13}{x}=\frac{13}{8}, \qquad \therefore \qquad x=8.$$

Substituting this value of x in the 1st equation,

we have $$\frac{3}{y}=1-\frac{1}{4}=\frac{3}{4}, \qquad \therefore \qquad y=4.$$

Thus we have $x=8,\ y=4$

Exercise (72).

Solve the following equations —

1. $\left.\begin{aligned}7x-5y&=11\\3x+2y&=13\end{aligned}\right\}$

2. $\left.\begin{aligned}13x+6y&=58\\5x-11y&=9\end{aligned}\right\}$

3. $\left.\begin{aligned}8x-9y&=20\\7x-10y&=9\end{aligned}\right\}$

4. $\left.\begin{aligned}25x-14y&=8\\12x+7y&=45\end{aligned}\right\}$

5. $\left.\begin{aligned}12x+11y&=70\\8x-7y&=18\end{aligned}\right\}$

6. $\left.\begin{aligned}13x-14y&=22\\17x-21y&=18\end{aligned}\right\}$

7. $\left.\begin{aligned}28x-15y&=41\\21x+13y&=55\end{aligned}\right\}$

8. $\left.\begin{aligned}19x+24y&=34\\23x+36y&=62\end{aligned}\right\}$

9. $\left.\begin{aligned}47x-56y&=123\\25x+84y&=293\end{aligned}\right\}$

10. $\left.\begin{aligned}51x-16y&=3\\68x+23y&=137\end{aligned}\right\}$

11. $\left.\begin{aligned}52x-9y&=34\\39x+14y&=67\end{aligned}\right\}$

12. $\left.\begin{aligned}12x+85y&=-49\\19x-34y&=91\end{aligned}\right\}$

13. $\left.\begin{aligned} 65x-14y &= 9 \\ 91x-15y &= 31 \end{aligned}\right\}$

14. $\left.\begin{aligned} 15x+46y &= 17 \\ 13x+69y &= 78 \end{aligned}\right\}$

15. $\left.\begin{aligned} 14x+81y &= 53 \\ 17x+135y &= 101 \end{aligned}\right\}$

16. $\left.\begin{aligned} 5x+11y &= 146 \\ 11x+5y &= 110 \end{aligned}\right\}$

17. $\left.\begin{aligned} ax+by &= c \\ a^2x+b^2y &= c^2 \end{aligned}\right\}$

(C U Entrance Paper, 1876)

18. $\left.\begin{aligned} \frac{x+y}{2}+\frac{3x-5y}{4} &= 2 \\ \frac{x}{14}+\frac{y}{18} &= 1 \end{aligned}\right\}$

(C U Entr Paper, 1876)

19. $\left.\begin{aligned} \frac{4x+5y}{40} &= x-y \\ \frac{2x-y}{3}+2y &= \frac{1}{2} \end{aligned}\right\}$

20. $\left.\begin{aligned} \frac{4x-3y-7}{5} &= \frac{3x}{10}-\frac{2y}{15}-\frac{5}{6} \\ \frac{y-1}{3}+\frac{x}{2}-\frac{3y}{20} &= \frac{y-x}{15}+\frac{x}{6}+\frac{11}{10} \end{aligned}\right\}$

21. $\left.\begin{aligned} \frac{6x+9}{4}+\frac{3x+5y}{4x-6} &= 3\tfrac{1}{4}+\frac{3x+4}{2} \\ \frac{8y+7}{10}+\frac{6x-3y}{2y-8} &= 4+\frac{4y-9}{5} \end{aligned}\right\}$

22. $\left.\begin{aligned} \frac{3x-5y}{3}-\frac{2x-8y-33}{12} &= \frac{y}{2}+\frac{x}{3}+\frac{1}{4} \\ 3\tfrac{1}{2}\left(\frac{x}{7}+\frac{y}{4}+1\tfrac{1}{3}\right) &= 3\tfrac{1}{3}\left(4x-\frac{y}{8}-24\right) \end{aligned}\right\}$

23. $\left.\begin{aligned} 2{\cdot}4x+{\cdot}32y-\frac{{\cdot}18x-{\cdot}025}{{\cdot}25} &= {\cdot}8x+\frac{5{\cdot}2+{\cdot}01y}{5} \\ \frac{2y+5}{1{\cdot}5} &= \frac{49x-7}{4{\cdot}2} \end{aligned}\right\}$

24. $\left.\begin{aligned} \frac{4}{x}+\frac{10}{y} &= 2 \\ \frac{3}{x}+\frac{2}{y} &= \frac{19}{20} \end{aligned}\right\}$ (C U Entrance Paper, 1879)

25. $\left.\begin{aligned} \frac{2}{x}+\frac{3}{y} &= 2 \\ \frac{5}{x}+\frac{10}{y} &= 5\tfrac{5}{6} \end{aligned}\right\}$ (C U Entr. Paper, 1887)

26. $\left.\begin{aligned} \frac{a}{x}+\frac{b}{y} &= m \\ \frac{b}{x}+\frac{a}{y} &= n \end{aligned}\right\}$

27. $\left.\begin{aligned} \frac{1}{3x}+\frac{1}{5y} &= 1 \\ \frac{1}{5x}+\frac{1}{3y} &= 1\tfrac{2}{15} \end{aligned}\right\}$

28. $\left.\begin{aligned} \frac{3}{y}-\frac{1}{x} &= 1 \\ \frac{2}{5x}+\frac{5}{2y} &= 7 \end{aligned}\right\}$

29. $\left.\begin{aligned} \frac{x}{4}+\frac{2}{y} &= 2 \\ \frac{2x}{5}+\frac{3}{2y} &= 2\tfrac{7}{20} \end{aligned}\right\}$

30. $\left.\begin{aligned} \frac{1}{5x}+\frac{y}{9} &= 5 \\ \frac{1}{3x}+\frac{y}{2} &= 14 \end{aligned}\right\}$

5. Method of Cross Multiplication

If $a_1x+b_1y+c_1z = 0$, and $a_2x+b_2y+c_2z = 0$, † to prove that

$$\frac{x}{b_1c_2-b_2c_1} = \frac{y}{c_1a_2-c_2a_1} = \frac{z}{a_1b_2-a_2b_1}.$$

Multiplying the 1st equation by c_2, and the 2nd by c_1, we have

$$a_1c_2x+b_1c_2y+c_1c_2z = 0,$$

and $$a_2c_1x+b_2c_1y+c_2c_1z = 0.$$

Hence, by subtraction,

$$(c_1a_2-c_2a_1)x+(b_2c_1-b_1c_2)y = 0,$$

$$\therefore \quad (c_1a_2-c_2a_1)x = (b_1c_2-b_2c_1)y,$$

$$\therefore \quad \frac{x}{b_1c_2-b_2c_1} = \frac{y}{c_1a_2-c_2a_1}. \quad . \quad (1)$$

Again, multiplying the 1st equation by a_2 and the 2nd by a_1, we have

$$a_1a_2x+b_1a_2y+c_1a_2z = 0,$$

and $$a_2a_1x+b_2a_1y+c_2a_1z = 0.$$

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† It is necessary to point out to the student the notation here used The letter a_1 is as different from a_2 as c is from d, or as any letter of the alphabet from any other, a similar remark applies to the pairs of letters (b_1, b_2) and (c_1, c_2) But it is very convenient as an aid to memory to use the same letter with different suffixes to denote corresponding co-efficients in different equations, thus whilst a_1 denotes the co efficient of x in the 1st equation, a_2 denotes the co efficient of x in the 2nd equation, and precisely a similar meaning is attached to the letters b_1, b_2 and c_1, c_2 Sometimes however letters with accents serve the same purpose, thus if a, b, c denote the co-efficients of x, y, z in one equation, the corresponding co-efficients in a second equation are denoted by a', b', c', in a third equation by a'', b'', c'', and so on

Hence, by subtraction,

$$(a_1b_2-a_2b_1)y+(c_2a_1-c_1a_2)z=0.$$

$$\therefore \quad (a_1b_2-a_2b_1)y=(c_1a_2-c_2a_1)z,$$

$$\therefore \quad \frac{y}{c_1a_2-c_2a_1}=\frac{z}{a_1b_2-a_2b_1}. \quad \ldots \quad (2)$$

Hence, from (1) and (2),

$$\frac{x}{b_1c_2-b_2c_1}=\frac{y}{c_1a_2-c_2a_1}=\frac{z}{a_1b_2-a_2b_1}.$$

Note This result can be easily remembered, writing down the equation one above the other,

$$\left.\begin{array}{l} a_1x+b_1y+c_1z=0 \\ a_2x+b_2y+c_2z=0 \end{array}\right\}, \text{ we find that}$$

(i) The quantity under x = co-efficient of y in the 1st equation $\times$ co-efficient of z in the second *minus* co-efficient of y in the 2nd $\times$ co-efficient of z in the 1st,

(ii) The quantity under y = co-efficient of z in the 1st equation $\times$ co-efficient of x in the second *minus* co-efficient of z in the 2nd $\times$ co-efficient of x in the 1st,

(iii) The quantity under z = co-efficient of x in the 1st equation $\times$ co-efficient of y in the second *minus* co-efficient of x in the 2nd $\times$ co-efficient of y in the 1st

Cor. In the above equations if we put $z=1$, we have

$$\frac{x}{b_1c_2-b_2c_1}=\frac{x}{c_1a_2-c_2a_1}=\frac{1}{a_1b_2-a_2b_1},$$

which gives the solution of the equations

$$\left.\begin{array}{r} a_1x+b_1y+c_1=0 \\ \text{and } a_2x+b_2y+c_2=0 \end{array}\right\}$$

Note The above results should be thoroughly committed to memory as ready applications of them will enable the student to solve with neatness not only simple equations involving *two* unknown quantities, but also a certain class of equations involving *three* unknown quantities The following examples are intended for illustration —

Example 1. Solve $\left.\begin{array}{l} 3x-5y+9=0 \\ 5x-3y-1=0 \end{array}\right\}$

Here $a_1=3,\ b_1=-5,\ c_1=9;$
$a_2=5,\ b_2=-3,\ c_2=1.$

Hence, we must have

$$\frac{x}{(-5)(-1)-(-3)\,9} = \frac{y}{9\times 5-(-1)\,3} = \frac{1}{3\,(-3)-5\,(-5)},$$

$$\text{or, } \frac{x}{5+27} = \frac{y}{45+3} = \frac{1}{-9+25},$$

$$\text{or, } \frac{x}{32} = \frac{y}{48} = \frac{1}{16}.$$

$$\therefore \quad x = \frac{32}{16} = 2, \text{ and } y = \frac{48}{16} = 3.$$

Thus we have $x = 2$, and $y = 3$

Example 2. Solve $\left.\begin{aligned} -7x+8y &= 9 \;..\; (1) \\ 5x-4y &= -3 \;..\; (2) \end{aligned}\right\}$

From (1), $\left.\begin{aligned} -7x+8y-9 &= 0 \\ 5x-4y+3 &= 0 \end{aligned}\right\}$

Hence, $$\frac{x}{8\times 3-(-4)(-9)} = \frac{y}{(-9)\,5-3\,(-7)}$$
$$= \frac{1}{(-7)(-4)-5\times 8},$$

$$\text{or, } \frac{x}{24-36} = \frac{y}{-45+21} = \frac{1}{28-40},$$

$$\text{or, } \frac{x}{-12} = \frac{y}{-24} = \frac{1}{-12},$$

$$\therefore \quad x = \frac{-12}{-12} = 1, \text{ and } y = \frac{-24}{-12} = 2$$

Thus we have $x = 1$, and $y = 2$.

Example 3. Solve

$$\left.\begin{aligned} (x+7)(y-3)+7 &= (y+3)(x-1)+5 \;.\; (1) \\ 5x-11y+35 &= 0 \;\;...\;\;..\;\; (2) \end{aligned}\right\}$$

(C U Entrance Paper, 1888)

From (1) $xy+7y-3x-14 = xy+3x-y+2,$

$$\therefore \quad 6x-8y+16 = 0,$$

$$\left.\begin{aligned} \therefore \quad 3x-4y+8 &= 0, \\ \text{also} \quad 5x-11y+35 &= 0 \end{aligned}\right\}$$

Hence, $$\frac{x}{(-4).35-(-11)8} = \frac{y}{8\times5-35\times3}$$
$$= \frac{1}{3(-11)-5(-4)},$$

or, $$\frac{x}{-140+88} = \frac{y}{40-105} = \frac{1}{-33+20},$$

or, $$\frac{x}{-52} = \frac{y}{-65} = \frac{1}{-13}.$$

Hence $x = 4$, and $y = 5$.

Example 4. Solve
$$\left.\begin{aligned} 2x-3y+4z &= 0 \;..\; (1)\\ 7x+2y-6z &= 0 \;...\; (2)\\ 4x+3y+\;z &= 37 \;...\; (3)\end{aligned}\right\}$$

From (1) and (2), we have

$$\frac{x}{(-3)(-6)-2\times4} = \frac{y}{4\times7-(-6)2} = \frac{z}{2\times1-7.(-3)},$$

or, $$\frac{x}{10} = \frac{y}{40} = \frac{z}{25},$$

or, $$\frac{x}{2} = \frac{y}{8} = \frac{z}{5}.$$

Now, let k denote the common value of these fractions, which is at present unknown.

Then, we have $\frac{x}{2} = \frac{y}{8} = \frac{z}{5} = k$.

$\therefore \quad x = 2k, y = 8k, \; z = 5k \;.\; .. \;....\; ..(a)$

Substituting the values of x, y, z in (3), we have

$k(8+24+5) = 37,$

or, $\quad 37k = 37, \quad \therefore \; k = 1$

Hence, from (a), $x = 2, \quad y = 8, \quad z = 5$

Example 5. Solve
$$\begin{aligned} x+6y &= 5z \;\;.\;\; (1)\\ 7x+\;z &= 6y \;..\; (2)\\ 5x+6y-4z &= 24 \;..\; (3)\end{aligned}$$

From (1), $\quad \left.\begin{aligned} x+6y-5z &= 0\\ \end{aligned}\right.$
From (2), $\quad 7x-6y+z = 0$ $\Big\}$

Hence, $$\frac{x}{6\times 1-(-6).(-5)}=\frac{y}{(-5)\,7-1\times 1}=\frac{z}{1.(-6)-7\times 6}$$

or, $$\frac{x}{6-30}=\frac{y}{-35-1}=\frac{z}{-6-42},$$

or, $$\frac{x}{-24}=\frac{y}{-36}=\frac{z}{-48},$$

$$\therefore\ \frac{x}{2}=\frac{y}{3}=\frac{z}{4}.$$ [Multiplying each fraction by -12]

Supposing each of these fractions $=k$, we have

$x=2k,\ y=3k,\ z=4k$ . . . . . (A)

Substituting these values of x, y, z in (3), we have

$k(10+18-16)=24$

or, $12k=24$, $\therefore$ $k=2$.

Hence, from (A), $x=4$, $y=6$, $z=8$.

Exercise (73).

Solve the following equations —

1. $\left.\begin{aligned}2x+3y-8&=0\\3x-4y+5&=0\end{aligned}\right\}$

2. $\left.\begin{aligned}2x-5y+9&=0\\5x+2y-16&=0\end{aligned}\right\}$

3. $\left.\begin{aligned}4x-5y+8&=0\\2x-3y+6&=0\end{aligned}\right\}$

4. $\left.\begin{aligned}-3x+2y+2&=0\\5x-3y-5&=0\end{aligned}\right\}$

5. $\left.\begin{aligned}6x-7y+12&=0\\-7x+4y+11&=0\end{aligned}\right\}$

6. $\left.\begin{aligned}7x-8y&=-14\\5x-3y&=9\end{aligned}\right\}$

7. $\left.\begin{aligned}-6x+5y+2&=0\\13x-9y&=19\end{aligned}\right\}$

8. $\left.\begin{aligned}-7x+5y+11&=0\\8x-5y&=19\end{aligned}\right\}$

9. $\left.\begin{aligned}4x-11y+6&=0\\9x-13y&=10\end{aligned}\right\}$

10. $\left.\begin{aligned}8x-7y&=19\\10x-9y&=23\end{aligned}\right\}$

11. $\left.\begin{aligned}-12x+17y+16&=0\\9x-13y&=11\end{aligned}\right\}$

12. $\left.\begin{aligned}14x-11y+18&=0\\11x-7y+1&=0\end{aligned}\right\}$

13. $\left.\begin{aligned} 17x-7y &= 52 \\ 3x &= 2y \end{aligned}\right\}$ **14.** $\left.\begin{aligned} 9x+5y &= 124 \\ 7x &= 3y \end{aligned}\right\}$

[From the 2nd equation $\frac{x}{2}=\frac{y}{3}=k$ (suppose)].

15. $\left.\begin{aligned} 15x+7y &= 246 \\ 9x &= 4y \end{aligned}\right\}$ **16.** $\left.\begin{aligned} 9x &= 8y \\ 10x+23y-287 &= 0 \end{aligned}\right\}$

17. $\left.\begin{aligned} 4x-3y &= 0 \\ 7x-11y+92 &= 0 \end{aligned}\right\}$ **18.** $\left.\begin{aligned} 4x-7y &= 0 \\ 10x-9y-102 &= 0 \end{aligned}\right\}$

19. $\left.\begin{aligned} 13x-12y+15 &= 0 \\ 8x-7y &= 0 \end{aligned}\right\}$ **20.** $\left.\begin{aligned} 11x-10y+82 &= 0 \\ 14x-9y &= 0 \end{aligned}\right\}$

21. $\left.\begin{aligned} \tfrac{1}{3}(x+y)+\tfrac{1}{4}(x-y) &= 59 \\ 5x-33y &= 0 \end{aligned}\right\}$ **22.** $\left.\begin{aligned} \frac{4x+5y}{40} &= x-y \\ \frac{2x-y}{3}+2y &= 20 \end{aligned}\right\}$

23. $\left.\begin{aligned} y(3+x) &= x(7+y) \\ 4x+9 &= 5y-14 \end{aligned}\right\}$ **24.** $\left.\begin{aligned} \frac{4y-6}{x+y} &= 2 \\ \frac{8x-5}{y-x} &= 9 \end{aligned}\right\}$

25. $\left.\begin{aligned} (x+5)(y+7) &= (x+1)(y-9)+112 \\ 2x+10 &= 3y+1 \end{aligned}\right\}$

26. $\left.\begin{aligned} 4x-5y+2z &= 0 \\ 2x-7y+4z &= 0 \\ x+y+z &= 0 \end{aligned}\right\}$ **27.** $\left.\begin{aligned} 5x+6y+8z &= 0 \\ 3x+4y+6z &= 0 \\ x+5y+16z &= 8 \end{aligned}\right\}$

28. $\left.\begin{aligned} 2x-7y+11z &= 0 \\ 6x-8y+7z &= 0 \\ 3x+4y+5z &= 35 \end{aligned}\right\}$ **29.** $\left.\begin{aligned} 7x+3y-8z &= 0 \\ 5x-7y+8z &= 0 \\ 3x+5y+7z &= 64 \end{aligned}\right\}$

30. $\left.\begin{aligned} x-2y+z &= 0 \\ 6x-8y+3z &= 0 \\ 2x+3y+5z &= 36 \end{aligned}\right\}$ **31.** $\left.\begin{aligned} 2(4x+9y) &= 7(2y+z) \\ 7(x+2y) &= 8(y+z) \\ 3x+4y+5x &= 38 \end{aligned}\right\}$

(C U Ent. Paper, 1887)

32. $\left.\begin{aligned} 4(x+y) &= 3(2z-y) \\ 5(x-2y) &= 3(2y-3z) \\ 6(x-2)+7(y-3)+8(z-4) &= 67 \end{aligned}\right\}$

33. $\left.\begin{aligned} 5x = 2y,\ 7y &= 5z \\ 4x+5y+6z &= 150 \end{aligned}\right\}$ **34.** $\left.\begin{aligned} 15x = 10y &= 6z \\ 7x+8y+9z &= 332 \end{aligned}\right\}$

35. $$\left.\begin{aligned} 4x-13y+8z &= 0 \\ 7x+6y-9z &= 0 \\ \frac{5}{x}+\frac{8}{y}+\frac{15}{z} &= 6\tfrac{2}{3} \end{aligned}\right\}$$

6. Equations of the form $a_1x+b_1y+c_1z=d_1$, $a_2x+b_2y+c_2z=d_2$, $a_3x+b_3y+c_3z=d_3$.

Multiply the first equation by c_2 and the 2nd by c_1; then by subtraction, we have

$$(a_1c_2-a_2c_1)x+(b_1c_2-b_2c_1)\,y = d_1c_2-d_2c_1 \quad (1)$$

Similarly, multiplying the first equation by c_3 and the 3rd by c_1, we have

$$(a_1c_3-a_3c_1)x+(b_1c_3-b_3c_1)y = d_1c_3-d_3c_1 \quad \ldots \; (2)$$

Now, from (1) and (2), the values of x and y can be at once found by cross multiplication Then substituting the values of x and y thus found in any of the given equations, the value of z will be obtained.

Otherwise :—

Multiply the first equation by d_2 and the 2nd by d_1; then by subtraction, we have

$$(a_1d_2-a_2d_1)x+(b_1d_2-b_2d_1)y+(c_1d_2-c_2d_1)\,z = 0 \quad \ldots \; (\alpha)$$

Similarly, multiplying the first equation by d_3 and the 3rd by d_1, we have

$$(a_1d_3-a_3d_1)x+(b_1d_3-b_3d_1)y+(c_1d_3-c_3d_1)z = 0 \quad . \; (\beta)$$

Now, evidently (α) and (β) together with any one of the given equations form a group which can be easily solved by the method illustrated in the last article.

Example 1. Solve $$\left.\begin{aligned} 4x-3y+2z &= 40 \quad (1) \\ 5x+9y-7z &= 47 \quad (2) \\ 9x+8y-3z &= 97 \quad . \; (3) \end{aligned}\right\}$$

Multiplying (1) by 7 and (2) by 2, we have

$$\left.\begin{aligned} 28x-21y+14z &= 280, \\ \text{and} \qquad 10x+18y-14z &= 94 \end{aligned}\right\}$$

Hence, by addition, $38x-3y = 374$. (4)

Again, multiplying (1) by 3 and (3) by 2, we have

$$\left.\begin{aligned} 12x-\ 9y+6z &= 120 \\ \text{and} \qquad 18x+16y-6z &= 194 \end{aligned}\right\}$$

Hence, by addition, $30x+7y = 314 \quad \ldots \quad (5)$

Now, from (4) and (5), we have

$$\left.\begin{aligned} 38x-3y-374 &= 0 \\ \text{and} \qquad 30x+7y-314 &= 0 \end{aligned}\right\}$$

Hence, $\dfrac{x}{3\times 314-7.(-374)} = \dfrac{y}{(-374)\,30-(314).38}$

$$= \frac{1}{38\times 7-30\,(-3)},$$

or, $\dfrac{x}{942+2618} = \dfrac{y}{-11220+11932} = \dfrac{1}{266+90},$

or, $\dfrac{x}{3560} = \dfrac{y}{712} = \dfrac{1}{356}.$

Therefore $x = 10$, and $y = 2$

Substituting these values of x and y in (1), we have

$$40-6+2z = 40, \text{ whence } z = 3.$$

Thus we have $x = 10,\ y = 2,\ z = 3$

Example 2. Solve

$$\left.\begin{aligned} 2x-\ 4y+\ 9z &= 28 \quad .. \quad (1) \\ 7x+\ 3y-\ 5z &= \ 3 \quad .. \quad (2) \\ 9x+10y-11z &= \ 4 \quad\quad\ (3) \end{aligned}\right\}$$

Multiplying (1) by 3, and (2) by 4, we have

$$\left.\begin{aligned} 6x-12y+27z &= 84 \\ \text{and} \qquad 28x+12y-20z &= 12 \end{aligned}\right\}$$

Hence, by addition, $34x+\ 7z = 96 \quad .. \quad (4)$

Again, multiplying (2) by 10, and (3) by 3, we have

$$\left.\begin{aligned} 70x+30y-50z &= 30 \\ \text{and} \qquad 27x+30y-33z &= 12 \end{aligned}\right\}$$

Hence, by subtraction, $43x-17z = 18 \quad \ldots \quad (5)$

Now, from (4) and (5), we have

$$\left.\begin{aligned} 34x+\ 7z-96 &= 0 \\ \text{and} \qquad 43x-17z-18 &= 0 \end{aligned}\right\}$$

Hence, $\dfrac{x}{7.(-18)-(-17)(-96)} = \dfrac{z}{(-96)43-(-18)34}$

$$= \frac{1}{34(-17)-43\times 7},$$

or, $\dfrac{x}{-126-1632} = \dfrac{z}{-4128+612} = \dfrac{1}{-578-301},$

or, $\dfrac{x}{-1758} = \dfrac{z}{-3516} = \dfrac{1}{-879}.$

Therefore $x = \dfrac{-1758}{-879} = 2$, and $= \dfrac{-3516}{-879} = 4$

Substituting these values of x and z in (2), we have

$$14+3y-20 = 3,$$

whence $3y = 9$, and $\therefore\ y = 3.$

Thus we have $x = 2,\ y = 3,\ z = 4$

Example 3. Solve
$$\left.\begin{array}{rl} 12x+\ 9y-\ 7z = 2 & \ldots (1) \\ 8x-\ 26y+9z = 1 & \ldots (2) \\ 23x+21y-15z = 4 & \quad (3) \end{array}\right\}.$$

Multiplying (2) by 2, we have

$$16x-52y+18z = 2,$$

also, $12x+\ 9y-\ 7z = 2 \quad \ldots \ldots (1)$

Hence, by subtraction, $4x-61y+25z = 0 \quad \ldots \ldots (4)$

Again, multiplying (1) by 2, we have

$$24x+18y-14z = 4$$

also, $23x+21y-15z = 4 \quad \ldots \ldots (3)$

Hence, by subtraction, $x-3y+\ z = 0 \quad \ldots \ldots (5)$

Now, since we have $4x-61y+25z = 0 \quad \ldots \ldots (4)$

and $x-\ 3y+\ z = 0. \quad \ldots \ldots (5)$

therefore, by cross multiplication,

$$\frac{x}{-61+75} = \frac{x}{25-4} = \frac{z}{-12+61},$$

or, $\dfrac{x}{14} = \dfrac{y}{21} = \dfrac{z}{49}$, or, $\dfrac{x}{2} = \dfrac{y}{3} = \dfrac{z}{7}.$

Supposing each of these fractions $= k$, we have

$$x = 2k,\ y = 3k,\ z = 7k.$$

Hence, from (1), $k(24+27-49) = 2$,

or, $2k = 2$,

$\therefore\ k = 1$.

Therefore, $x = 2,\ y = 3,\ z = 7$.

Exercise (74).

Solve the following equations :—

1. $\left.\begin{array}{rl} 2x-3y+5z = & 11 \\ 5x+2y-7z = & -12 \\ -4x+3y+z = & 5 \end{array}\right\}$

2. $\left.\begin{array}{rl} 3x+2y+5z = & 32 \\ 2x+5y+3z = & 31 \\ 5x+3y+2z = & 27 \end{array}\right\}$

3. $\left.\begin{array}{rl} x+y-z = & 1 \\ 8x+3y-6z = & 1 \\ 3z-4x-y = & 1 \end{array}\right\}$

4. $\left.\begin{array}{rl} 2x+3y+4z = & 29 \\ 3x+2y+5z = & 32 \\ 4x+3y+2z = & 25 \end{array}\right\}$

5. $\left.\begin{array}{rl} 2x+3y+4z = & 16 \\ 3x+2y-5z = & 8 \\ 5x-6y+3z = & 6 \end{array}\right\}$

6. $\left.\begin{array}{rl} 4x-3y+2z = & 8 \\ 3x-4y+5z = & 6 \\ -6x+5y+7z = & -1 \end{array}\right\}$

7. $\left.\begin{array}{rl} 8x-7y-5z = & 1 \\ -7x+5y+6z = & -1 \\ 12x-8y-11z = & 2 \end{array}\right\}$

8. $\left.\begin{array}{rl} x+5y-4z = & 5 \\ 3x-2y+2z = & 14 \\ -10x+8y+z = & 6 \end{array}\right\}$

9. $\left.\begin{array}{rl} 2x+4y+5z = & 49 \\ 3x+5y+6z = & 64 \\ 4x+3y+4z = & 55 \end{array}\right\}$

10. $\left.\begin{array}{rl} x+3y+5z = & 10 \\ 3x+5y+7z = & 14 \\ 5x+7y+8z = & 15 \end{array}\right\}$

11. $\left.\begin{array}{rl} 12x+8y-11z = & -8 \\ 11x-13y-z = & 2 \\ 8x+17y-12z = & -2 \end{array}\right\}$

12. $\left.\begin{array}{rl} 5x-4y+9z = & 19 \\ 7x+6y-12z = & 16 \\ -9x+8y+15z = & -13 \end{array}\right\}$

13. $\left.\begin{array}{rl} x-y-z = & -15 \\ y+x+2z = & 40 \\ 4z-5x-6y = & -150 \end{array}\right\}$

(C. U. Entrance Paper, 1886.)

14. $\left.\begin{array}{rl} 2(x-y) = & 3z-2 \\ x-3z = & 3y-1 \\ 2x+3z = & 4(1-y) \end{array}\right\}$

15. $\left.\begin{array}{rl} 3x+2y-z = & 20 \\ 2x+3y+6z = & 70 \\ x-y+6z = & 41 \end{array}\right\}$

16. $\begin{array}{rl} 4(y-x) = & 5z-29 \\ 3z+4x = & 6y+ \\ z-3y = & 14- \end{array}$

17. $\left.\begin{aligned} 5x+2y+z &= 30 \\ \tfrac{1}{2}x+\tfrac{4}{5}y-\tfrac{1}{10}z &= 4 \\ 2x+5y+10z &= 129 \end{aligned}\right\}$

18. $\left.\begin{aligned} \tfrac{1}{2}x+\tfrac{1}{3}y &= 12-\tfrac{1}{6}z \\ \tfrac{1}{2}y+\tfrac{1}{3}z-\tfrac{1}{6}x &= 8 \\ \tfrac{1}{3}x+\tfrac{1}{3}z &= 10 \end{aligned}\right\}$

19. $\left.\begin{aligned} \frac{1}{x}+\frac{5}{y}-\frac{4}{z} &= \frac{1}{12} \\ \frac{3}{x}-\frac{4}{y}+\frac{5}{z} &= \frac{19}{24} \\ -\frac{4}{x}+\frac{5}{y}+\frac{6}{z} &= \frac{1}{2} \end{aligned}\right\}$

20. $\left.\begin{aligned} \frac{3}{x}-\frac{4}{5y}+\frac{1}{z} &= 7\tfrac{3}{5} \\ \frac{1}{3x}+\frac{1}{2y}+\frac{2}{z} &= 10\tfrac{1}{6} \\ \frac{4}{5x}-\frac{1}{2y}+\frac{4}{z} &= 16\tfrac{1}{10} \end{aligned}\right\}$

21. $\left.\begin{aligned} 5x+8y &= 65 \\ 2y-z &= 11 \\ 3x+4z &= 57 \end{aligned}\right\}$

22. $\left.\begin{aligned} \frac{2}{x}+\frac{1}{y} &= \frac{3}{2} \\ \frac{3}{z}-\frac{2}{y} &= 2 \\ \frac{1}{x}+\frac{1}{z} &= \frac{4}{3} \end{aligned}\right\}$

23. $\left.\begin{aligned} ay+bx &= c \\ cx+az &= b \\ bz+cy &= a \end{aligned}\right\}$

24. $\left.\begin{aligned} 3x+4y-11 &= 0 \\ 5y-6z &= -8 \\ 7z-8x-13 &= 0 \end{aligned}\right\}$

(Calcutta University Entrance Paper, 1877)

25. $\left.\begin{aligned} 3y+x-2 &= 0 \\ 3z-4y &= x+15 \\ 2x+7z &= 7 \end{aligned}\right\}$ (C U Entrance Paper, 1883)

7. Miscellaneous Examples.

Example 1. Solve $\dfrac{a}{x}+\dfrac{b}{y}=1, \dfrac{b}{y}+\dfrac{c}{z}=1, \dfrac{c}{z}+\dfrac{a}{x}=1.$

Adding together the given equations, we have

$$2\left(\frac{a}{x}+\frac{b}{y}+\frac{c}{z}\right)=3,$$

$$\text{or,}\quad \frac{a}{x}+\frac{b}{y}+\frac{c}{z}=\frac{3}{2} \quad \ldots \quad (\alpha)$$

Subtracting the 2nd equation from (α), we have

$$\frac{a}{x}=\frac{1}{2}, \quad \therefore \quad x=2a.$$

Similarly, we have $y=2b$, and $z=2c$.

Example 2 Solve

(i) $\dfrac{xy}{x+y}=1$, (ii) $\dfrac{xz}{x+z}=2$, (iii) $\dfrac{yz}{y+z}=3$.

From (i) we have $\dfrac{x+y}{xy}=1$, or, $\dfrac{1}{y}+\dfrac{1}{x}=1$ .. (4)

" (ii) " " $\dfrac{x+z}{xz}=\frac{1}{2}$, or, $\dfrac{1}{z}+\dfrac{1}{x}=\frac{1}{2}$ .. (5)

" (iii) " " $\dfrac{y+z}{yz}=\frac{1}{3}$, or, $\dfrac{1}{z}+\dfrac{1}{y}=\frac{1}{3}$ (6)

From (4), (5) and (6), by addition, we have

$$2\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)=1+\frac{1}{2}+\frac{1}{3}=\frac{11}{6},$$

$$\therefore\quad \frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{11}{12}. \qquad \ldots (7)$$

Subtracting (6) from (7),

$$\frac{1}{x}=\frac{11}{12}-\frac{1}{3}=\frac{7}{12},\quad \therefore\quad x=\frac{12}{7}.$$

Subtracting (5) from (7),

$$\frac{1}{y}=\frac{11}{12}-\frac{1}{2}=\frac{5}{12},\quad \therefore\quad y=\frac{12}{5}.$$

Subtracting (4) from (7),

$$\frac{1}{z}=\frac{11}{12}-1=-\frac{1}{12},\quad \therefore\quad z=-12.$$

Example 3. Solve

$$xyz=a(yz-zx-xy)=b(zx-xy-yz)=c(xy-yz-zx).$$

Since $xyz=a(yz-zx-xy)$ we have

$$\frac{1}{a}=\frac{1}{x}-\frac{1}{y}-\frac{1}{z} \ldots (1)$$ [Dividing both sides by $a\times xyz$.]

Similarly, we have

$$\frac{1}{b}=\frac{1}{y}-\frac{1}{z}-\frac{1}{x} \ldots (2)$$

and $$\frac{1}{c}=\frac{1}{z}-\frac{1}{x}-\frac{1}{y} \ldots (3)$$

Adding together (2) and (3), we have

$$-\frac{2}{x} = \frac{1}{b}+\frac{1}{c} = \frac{b+c}{bc}, \quad \therefore \quad x = \frac{-2bc}{b+c}.$$

Similarly,

$$-\frac{2}{y} = \frac{1}{c}+\frac{1}{a} = \frac{a+c}{ac}, \quad \therefore \quad y = \frac{-2ca}{c+a};$$

and $$-\frac{2}{z} = \frac{1}{a}+\frac{1}{b} = \frac{a+b}{ab}, \quad \therefore \quad z = \frac{-2ab}{a+b}.$$

Example 4. Solve
$$\left.\begin{aligned} x+y+z &= 0 \\ (b+c)x+(c+a)y+(a+b)z &= 0 \\ bcx+cay+abz &= 1 \end{aligned}\right\}$$

Since $$\left.\begin{aligned}(b+c)x+(c+a)y+(a+b)z &= 0\\ x+\quad y+\quad z &= 0\end{aligned}\right\},$$
and

therefore, by cross multiplication,

$$\frac{x}{(c+a)-(a+b)} = \frac{y}{(a+b)-(b+c)} = \frac{z}{(b+c)-(c+a)},$$

or, $$\frac{x}{c-b} = \frac{y}{a-c} = \frac{z}{b-a}$$

Supposing each of these fractions $= k$, we have

$$x = k(c-b), y = k(a-c), z = k(b-a)$$

Substituting these values of x, y, z in the 3rd equation we have $k\{bc(c-b)+ca(a-c)+ab(b-a)\} = 1.$

But $bc(c-b)+ca(a-c)+ab(b-a)$
$$= bc(c-b)+a^2(c-b)-a(c^2-b^2)$$
$$= (c-b)\{bc+a^2-a(c+b)\}$$
$$= (c-b)(a-c)(a-b)$$

Thus, $$k(c-b)(a-c)(a-b) = 1,$$

$$\therefore \quad k = \frac{1}{(c-b)(a-c)(a-b)}.$$

Hence,
$$x = k(c-b) = \frac{1}{(a-c)(a-b)},$$
$$y = k(a-c) = \frac{1}{(c-b)(a-b)},$$
$$z = k(b-a) = \frac{1}{(c-b)(c-a)}.$$

Exercise (75).

Solve the following equations —

1. $\dfrac{x}{a}+\dfrac{y}{b}=1,\quad \dfrac{x}{a}+\dfrac{z}{c}=1,\quad \dfrac{y}{b}+\dfrac{z}{c}=1.$

2. $\dfrac{1}{x}+\dfrac{1}{y}=a,\quad \dfrac{1}{x}+\dfrac{1}{z}=b,\quad \dfrac{1}{y}+\dfrac{1}{z}=c$

3. $\dfrac{yz}{y+z}=a,\quad \dfrac{zx}{z+x}=b,\quad \dfrac{xy}{x+y}=c$

4. $\left.\begin{aligned} axy &= c(bx+ay) \\ bxy &= c(ax-by) \end{aligned}\right\}$

5. $3xy=4(x+y),\quad 2xz=3(x+z),\quad 5yz=12(y+z)$

6. $y+z=4,\ z+x=6,\ x+y=8$

7. $y+z-x=6,\quad z+x-y=10,\quad x+y-z=14.$

8. $\left.\begin{aligned} x-4y+z &= -10 \\ y-4z+x &= -15 \\ z-4x+y &= -35 \end{aligned}\right\}$

9. $\left.\begin{aligned} y+z-7x+16 &= 0 \\ z+x-7y+24 &= 0 \\ x+y-7z+40 &= 0 \end{aligned}\right\}$

10. $\left.\begin{aligned} a^2x+b^2y &= 2ab(a+b) \\ b(2a+b)x+a(a+2b)y &= a^3+a^2b+ab^2+b^3 \end{aligned}\right\}$

11. $\left.\begin{aligned} x+y+z &= A \\ ax+by+cz &= 0 \\ a^2x+b^2y+c^2z &= 0 \end{aligned}\right\}$

12. $\left.\begin{aligned} x+y+z &= 0 \\ (a+b)x+(a+c)y+(b+c)z &= 0 \\ abx+acy+bcz &= 1 \end{aligned}\right\}$

13. $\left.\begin{aligned} x+y+z &= 0 \\ \frac{x}{a}+\frac{y}{b}+\frac{z}{c} &= 0 \\ \frac{x}{a^2}+\frac{y}{b^2}+\frac{z}{c^2} &= 1 \end{aligned}\right\}$

14. $\left.\begin{aligned} x-ay+a^2z &= a^3 \\ x-by+b^2z &= b^3 \\ x-cy+c^2z &= c^3 \end{aligned}\right\}$

15. $\left.\begin{aligned} ax+by+cz &= 0 \\ (b+c)x+(c+a)y+(a+b)z &= 0 \\ a^2x+b^2y+c^2z &= a^2(b-c)+b^2(c-a)+c^2(a-b) \end{aligned}\right\}$

16. Find the condition that the three equations,

$a_1x+b_1y+c_1=0$, $a_2x+b_2y+c_2=0$, $a_3x+b_3y+c_3=0$, may be consistent

17. Find the value of a so that the four equations,

$2x-3y+5z = 18$, $3x-y+4z = 20$, $4x+2y-z = 5$, $(a+1)x+(a+2)y+(a+3)z = 76$, may by consistent.

18. $\left.\begin{aligned} 3w-2y &= 2 \\ 5x-7z &= 11 \\ 2x+3y &= 52 \\ 4y+3z &= 41 \end{aligned}\right\}$

19. $\left.\begin{aligned} 9x-2z+w &= 41 \\ 7y-5z-t &= 12 \\ 4y-3x+2w &= 5 \\ 3y-4w+3t &= 7 \\ 7z-5w &= 11 \end{aligned}\right\}$

20. $\left.\begin{aligned} x+y+z &= ab+bc+ca \\ \frac{x}{ab}+\frac{y}{bc}+\frac{z}{ca} &= 3 \\ (c-b)x+(a-b)y+(c-a)z &= 2abc-ab^2-b^2c+ac^2-a^2c \end{aligned}\right\}$

CHAPTER XV

PROBLEMS PRODUCING SIMPLE EQUATIONS WITH MORE THAN ONE UNKNOWN QUANTITY

Easy Problems

Example 1. A and B each had a number of mangoes. A said to B, "If you give me 30 of your mangoes my number will be *twice* yours". B replied, "If *you* give me 10, my number will be *thrice* yours". How many had each?

Let $x =$ the number of mangoes A had,

and $y =$ " " " " B ".

Then, in accordance with what A said we must have the equation

$$x+30 = 2(y-30) \qquad (1)$$

and in accordance with B's reply we must have the equation

$$y+10 = 3(x-10) \quad \ldots \quad (2)$$

From (2), $3x - y = 40$

or, $6x - 2y = 80 \quad .. \quad (3)$

and from (1), $x - 2y = -90 \quad .. \quad (4)$

Hence, by subtraction, $5x = 170$, $\therefore x = 34$.

Substituting the value of x in (4), we have

$$2y = 34 + 90 = 124, \quad \therefore \quad y = 62.$$

Thus A had 34 mangoes, and B had 62.

Example 2 A certain fraction becomes 2 when 7 is added to its numerator, and 1 when 2 is subtracted from the denominator What is the fraction ?

Let $\frac{x}{y}$ represent the fraction

Then we have $\frac{x+7}{y} = 2 \quad . \quad (1)$

and $\frac{x}{y-2} = 1 \quad . \quad (2)$

From (1), $x + 7 = 2y$, $\therefore \left.\begin{array}{l} x = 2y-7 \\ x = y-2 \end{array}\right\}$

From (2),

Therefore, $2y - 7 = y - 2$, whence $y = 5$.

Hence, $x = 5 - 2 = 3$.

Example 3. Two men and 7 boys can do in 4 days a piece of work which would be done in 3 days by 4 men and 4 boys. How long would it take one man or one boy to do it ?

Let $x =$ the number of days in which one man would do the work,

and $y =$ the number of days in which one boy would do it.

Then in one day a man does $\frac{1}{x}$ th of the work and a boy does $\frac{1}{y}$th of it

Hence, since 2 men and 7 boys do $\frac{1}{4}$th of the work in one day, we must have

$$\frac{2}{x} + \frac{7}{y} = \frac{1}{4} \quad \ldots \ldots (1)$$

Again, since 4 men and 4 boys do $\frac{1}{3}$rd of the work in one day, we must have

$$\frac{4}{x}+\frac{4}{y}=\frac{1}{3}. \quad . \quad . \ (2)$$

Multiplying (1) by 2, and subtracting (2) from the resulting equation, we have

$$\frac{10}{y}=\frac{1}{2}-\frac{1}{3}=\frac{1}{6}, \quad \therefore \quad y=60.$$

Hence from (2), $\frac{4}{x}=\frac{1}{3}-\frac{1}{15}=\frac{4}{15}, \quad \therefore \quad x=15$

Thus one man would do the work in 15 days and one boy in 60 days

Example 4 Two plugs are opened in the bottom of a cistern containing 192 gallons of water, after 3 hours one of them becomes stopped, and the cistern is emptied by the other in 11 hours, had 6 hours elapsed before the stoppage, it would have only required 6 hours more to empty it. How many gallons will each plug-hole discharge in an hour, supposing the discharge uniform?

Let x, y be the numbers of gallons of water which the plugs can respectively discharge in an hour

In the first case, the first plug remains opened for 3 hours and the second for $3+11$ or 14 hours

Hence, $$3x+14y=192 \quad \ldots \quad (1)$$

In the second case, the first plug remains opened for 6 hours, and the second for $6+6$ or 12 hours.

Hence, $$6x+12y=192. \quad \ldots \quad (2)$$

Multiplying (1) by 2 and subtracting (2) from the resulting equation, we have

$$16y=2\times 192-192$$
$$=192, \qquad \therefore \quad y=12.$$

Hence, from (2), $6x=192-144=48, \quad \therefore \quad x=8.$

Thus the plug-holes respectively discharge 8 and 12 gallons in an hour.

Example 5. The dimensions of a rectangular court are such that if the length were increased by 3 yards, and the

breadth diminished by the same, its area would be diminished by 18 square yards, and if its length were increased by 3 yards, and its breadth increased by the same, its area would be increased by 60 square yards; find the dimensions (Calcutta University Entrance Paper, 1888.)

Let x yards = length of the court,

and y yards = its breadth.

Then from the first condition of the problem, we have

$$(x+3)(y-3) = xy-18 \quad \ldots \quad (1)$$

and from the second condition,

$$(x+3)(y+3) = xy+60 \quad \ldots \quad (2)$$

From (1), $3y-3x = -9$,

or, $y-x = -3$. (3)

From (2), $3y+3x = 51$,

or, $y+x = 17$ (4)

From (3) and (4), by addition,

$$2y = 14, \quad \therefore \quad y = 7,$$

and by subtraction,

$$2x = 20, \quad \therefore \quad x = 10$$

Thus the length of the court is 10 yards, and the breadth is 7 yards.

Example 6. There is a certain number, to the sum of whose digits if you add 7, the result will be three times the left-hand digit; and if from the number itself you subtract 18, the digits will be inverted. Find the number.

Let x and y be the left and right-hand digits respectively; then the required number is represented by $10x+y$, and the number with inverted digits $= 10y+x$

Hence, by the conditions of the problem,

$$x+y+7 = 3x \quad \ldots \quad (1)$$

and $$(10x+y)-18 = 10y+x \quad \ldots \quad (2)$$

From (1), $2x-y = 7$ ... (3)

and from (2), $9x-9y = 18$, or $x-y = 2$ ... (4)

Subtracting (4) from (3), we have

$$x = 7-2 = 5$$

Hence, from (4), $y = 5-2 = 3$

Thus, the required number is 53

Example 7 A and B play at bowls, and A bets B three shillings to two upon every game, after a certain number of games it appears that A has won three shillings but if A had bet five shillings to two and lost one game more out of the same number, he would have lost thirty shillings. How many games did each win?

Let x = number of games that A won,

and y = „ „ „ „ B „

Then the total number of games played is evidently $x+y$

Now since A receives from B, 2*s* for every game that he wins and gives B, 3*s* for every game that he loses (*i e*, for every game that B wins), his total gain must be equal to $(2x-3y)$ shillings

Hence, $$2x-3y = 3 \qquad (1)$$

According to the other condition, A would have gained $2(x-1)$ shillings, and lost $5(y+1)$ shillings, and therefore his total loss would have been $[5(y+1)-2(x-1)]$ shillings

Hence $$5(y+1)-2(x-1) = 30,$$

or, $$5y-2x = 28 \qquad (2)$$

From (1) and (2), by addition,

$$2y = 26, \qquad y = 13$$

Hence from (1), $x = \dfrac{3+39}{2} = 21.$

Thus A won 21 games and B won 13 games

Exercise (76).

1. What fraction is that whose numerator being doubled and denominator increased by 7, the value becomes $\frac{2}{3}$; but the denominator being doubled, and the numerator increased by 2, the value becomes $\frac{3}{5}$?

2. Find two numbers such that if the first be added to 5 times the second, the sum is 52, and if the second be added to 8 times the first the sum is 65.

3. Find two numbers such that five times the greater exceeds four times the less by 22, and three times the greater together with seven times the less is 32

4. What numbers are those whose difference is 45, and the quotient of the greater by the less is 4?

5 There are two numbers such that one-fourth of the greater added to one-third of the less is 11, and if one-fifth of the less be taken from one-eighth of the greater, the remainder is nothing, find the numbers.

6. A certain fraction becomes $\frac{1}{2}$ when 1 is subtracted from its denominator, and 1 when 7 is added to its numerator. What is the fraction?

7. What fraction is that which, if 1 be added to the numerator, becomes 1, and if 1 be added to the denominator becames $\frac{1}{2}$? (Calcutta University Entrance Paper, 1862.)

8. A certain fraction becomes $\frac{1}{2}$ when its numerator is increased by unity, and $\frac{1}{3}$ when its denominator is increased by unity What is the fraction?

9. A and B have 39 rupees between them, but if A were to lose two-thirds of his money, and B three-fourths of his, they would then have only 11 rupees How much has each?

10 Two numbers are such that if 7 be added to the less the sum is twice the greater, and if 4 be added to the greater the sum is 3 times the less Find the numbers

11. Two persons 27 miles apart, setting out at the same time, meet together in 9 hours if they walk in the same direction, but in 3 hours if they walk in opposite directions, find their rates of walking

12. A banker was asked to pay £10 in sovereign and half-crowns, so that the number of the latter should be exactly twice that of the former How must he do it?

13. A man and a boy can do in 15 days a piece of work which would be done in 2 days by 7 men and 9 boys How long would it take one man to do it?

14. A rectangle is of the same area as another which is 6 yards longer and 4 yards narrower, it is also of the same

area as a third, which is 8 yards longer and 5 yards narrower. What is its area ?

15. If 15 lbs of tea and 17 lbs. of coffee together cost £3 5*s* 6*d*, and 25 lbs. of tea and 13 lbs. of coffee together cost £4 6s. 2d., find the price of each per pound

16. A takes 3 hours longer than B to walk 30 miles ; but if he doubles his pace he takes 2 hours less time than B , find their rates of walking

17. Says Charles to William, "If you give me 10 of your marbles, I shall then have just *twice* as many as you" - but says William to Charles, "If you give me 10 of yours, I shall then have *three times* as many as you" How many had each ?

18 Rs 1100 are so divided among A, B and C, that if A were to give B Rs 200, B would then have twice as much as A, and three times as much as C How many rupees did A, B and C each receive originally ?

(Calcutta University Entrance Paper, 1872)

19. If a certain number be divided by the sum of its two digits the quotient is 6 and the remainder is 3 If the digits be inverted and the resulting number be divided by the sum of the digits, the quotient is 4 and the remainder 9 Find the number.

20. Find that number of 2 figures, to which, if the number formed by changing the places of the digits, be added, the sum is 121 , and if the smaller number be subtracted from the larger, the remainder is 9

21. A bill of 25 guineas was paid with crowns and half-guineas , and twice the number of half-guineas exceeded 3 times that of the crowns by 17 , how many were there of each ?

22. A person sells to one person 9 horses and 7 cows for £300 , and to another, at the same prices, 6 horses and 13 cows for the same sum , what was the price of each ?

23. A and B received £5 17*s* for their wages, A having been employed for 15 and B for 14 days , and A received for working four days 11*s* more than B did for three days ; what were their daily wages ?

24. A and B can do a piece of work in 16 days , they work together for 4 days, when A leaves, and B finishes it

in 36 days more, in what time would each do the work separately?

25. If the numerator of a fraction is increased by 2 and the denominator by 1, it becomes equal to $\frac{5}{8}$, and if the numerator and denominator are each diminished by 1, it becomes equal to $\frac{1}{2}$, find the fraction

26. A traveller walks a certain distance, had he gone half a mile an hour faster, he would have walked it in four-fifths of the time, had he gone half a mile an hour slower, he would have been $2\frac{1}{2}$ hours longer on the road Find the distance

27. A certain number between 10 and 100 is eight times the sum of its digits, and if 45 be subtracted from it the digits will be reversed find the number

28. A and B lay a wager of 10*s*, if A loses, he will have twenty-five shillings less than twice as much as B will then have, but if B loses, he will have five-seventeenths of what A will then have, find how much money each of them has

29. A farmer wishing to purchase a number of sheep found that if they cost him £2 2*s* a head, he would not have money enough by £1 8*s*, but if they cost him £2 a head, he would then have £2 more than he required find the number of sheep, and the money which he had

30. There is a number consisting of two digits the number is equal to three times the sum of its digits, and if it be multiplied by three, the result will be equal to the square of the sum of its digits. Find the number

More Difficult Problems.

Example 1. A cask P contains 12 gallons of wine and 18 gallons of water, and another cask Q contains 9 gallons of wine and 3 gallons of water How many gallons must be drawn from each cask so as to produce by their mixture 7 gallons of wine and 7 gallons of water?

Out of 30 gallons of the mixture of wine and water in P, there are 12 gallons of wine, hence $\frac{12}{30}$ or $\frac{2}{5}$ths of the mixture consists of wine, and $\frac{3}{5}$ths water

Hence, for every gallon drawn from P there are taken out $\frac{2}{5}$ths of a gallon of wine and $\frac{3}{5}$ ths of a gallon of water.

Similarly, for every gallon drawn from Q there are taken out $\frac{3}{4}$ths of a gallon of wine and $\frac{1}{4}$th of a gallon of water

Let $x =$ the number of gallons to be drawn from P,
and $y =$ „ „ „ „ „ „ „ „ Q

Then, since x gallons from P contain $\frac{2}{5}x$ gallons of wine and $\frac{3}{5}x$ gallons of water, and y gallons from Q contain $\frac{3}{4}y$ gallons of wine and $\frac{1}{4}y$ gallons of water, in the new mixture there are $(\frac{2}{5}x+\frac{3}{4}y)$ gallons of wine and $(\frac{3}{5}x+\frac{1}{4}y)$ gallons of water.

Hence, by the conditions of the problem,

$$\left.\begin{aligned} \tfrac{2}{5}x+\tfrac{3}{4}y &= 7 \quad \ldots \quad (1) \\ \text{and} \quad \tfrac{3}{5}x+\tfrac{1}{4}y &= 7 \quad \ldots \quad (2) \end{aligned}\right\}$$

Multiplying (2) by 3, and subtracting (1) from the resulting equation, we have

$$\tfrac{7}{5}x = 14, \qquad x = 10$$

Hence, from (2), $\quad y = 4(7-\frac{3}{5}\times 10) = 4$

Thus 10 gallons must be drawn from P, and 4 gallons from Q

Example 2 The fore-wheel of a carriage makes 6 revolutions more than the hind-wheel in going 120 yards, if the circumference of the fore-wheel be increased by one-fourth of its present size, and the circumference of the hind-wheel by one-fifth of its present size, the *six* will be changed to *four* Required the circumference of each wheel

Let x yards be the circumference of the fore-wheel,
and „ y „ „ „ „ „ „ hind-wheel

Then the numbers of revolutions made by the wheels in going 120 yards are respectively $\dfrac{120}{x}$ and $\dfrac{120}{y}$

When the circumference of the fore-wheel is increased by one-fourth, and that of the hind-wheel by one-fifth, the circumferences respectively become

$$\left(x+\frac{x}{4}\right) \text{ and } \left(y+\frac{y}{5}\right) \text{ yards, or, } \frac{5x}{4} \text{ and } \frac{6y}{5} \text{ yards.}$$

Therefore, the numbers of revolutions made by the wheels are respectively

$$120 \div \frac{5x}{4} \text{ and } 120 \div \frac{6y}{5}, \text{ or, } \frac{96}{x} \text{ and } \frac{100}{y}.$$

Hence, from the conditions of the problem,

$$\left.\begin{aligned} \frac{120}{x} &= \frac{120}{y}+6 \quad \dots \quad \dots \quad (1) \\ \text{and} \qquad \frac{96}{x} &= \frac{100}{y}+4 \quad \dots \quad \dots \quad (2) \end{aligned}\right\}$$

Multiplying (1) by 5 and (2) by 6, we have

$$\frac{600}{x} = \frac{600}{y}+30,$$

and
$$\frac{576}{x} = \frac{600}{y}+24,$$

$\therefore$ by subtraction $\dfrac{24}{x} = 6$, $\therefore$ $x = 4$.

Hence from (1), $\dfrac{120}{y} = \dfrac{120}{4}-6 = 24$, $\therefore$ $y = 5$.

Thus the circumferences of the wheels are respectively 4 and 5 yards

Example 3 A pound of tea and three pounds of sugar cost six shillings, but if sugar were to rise 50 per cent, and tea 10 per cent, they would cost seven shillings Find the price of tea and sugar

Let x shillings be the price of a pound of tea, and y shillings the price of a pound of sugar,

then we must have

$$x+3y = 6 \quad \dots \quad (1)$$

When the price of tea *rises* 10 *per cent*, the price of a pound of tea becomes $\left(x+\dfrac{x}{10}\right)$, or $\dfrac{11}{10}x$, shillings, and the price of sugar *rising* 50 *per cent.*, the price of a pound of sugar becomes $\left(y+\dfrac{y}{2}\right)$, or $\dfrac{3y}{2}$, shillings

Hence, $\dfrac{11}{10}x+3\,\dfrac{3y}{2} = 7 \quad \dots \quad \dots \quad (2)$

From (2), $\frac{11}{5}x+9y = 14,$

and from (1), $3x+9y = 18,$

$$\therefore \quad (3-\tfrac{11}{5})x = 4,$$

$$\text{or,} \quad \frac{4x}{5} = 4, \quad \therefore \quad x = 5.$$

Hence from (1), $\qquad y = \dfrac{6-5}{3} \qquad = \dfrac{1}{3}.$

Thus the price of a pound of tea $= 5s$, and that of pound of sugar $= \frac{1}{3}s = 4d$.

Example 4 A certain sum of money is to be divide among a certain number of men, if there were 3 men les each man would have £150 more, but if there were 6 me more, each man would have £120 less Find the sum c money and the number of men

Let $x =$ the sum of money in pounds,

and $y =$ the number of men

Therefore, each man gets £ $\dfrac{x}{y}$, if there were 3 me less each would get £ $\dfrac{x}{y-3}$; and if there were 6 men moı each would get £$\dfrac{x}{y+6}$.

Hence, from the conditions of the problem,

$$\frac{x}{y-3} = \frac{x}{y}+150 \quad \ldots \quad . \quad (1)$$

$$\text{and } \frac{x}{y+6} = \frac{x}{y}-120 \quad . \quad (2)$$

From (1), $\qquad 150 = x\left(\dfrac{1}{y-3}-\dfrac{1}{y}\right)$

$$= \frac{3x}{y^2-3y}, \quad \therefore \quad x = 50(y^2-3y).$$

From (2), $\ldots \quad 120 = x\left(\dfrac{1}{y}-\dfrac{1}{y+6}\right)$

$$= \frac{6x}{y^2+6y}, \quad \therefore \quad x = 20(y^2+6y).$$

Hence, $50(y^2-3y) = 20(y^2+6y)$,

or, $30y^2 = (150+120)y = 270y$, $\therefore \quad y = 9$.

$\therefore \; x = 20(81+54) = 20\times 135 \quad = 2700$

Thus there are 9 men and a sum of £2700

Example 5. A man has to travel a certain distance When he has travelled 40 miles, he increases his speed 2 miles per hour If he had travelled with his increased speed, during the whole of his journey, he would have arrived 40 minutes earlier, but if he had continued at his original speed, he would have arrived 20 minutes later. How far had he to travel ?

Let $x =$ the number of miles the man had to travel; and suppose his original speed was y miles an hour.

Hence, the time actually taken to complete the journey

$$= \left(\frac{40}{y}+\frac{x-40}{y+2}\right) \text{ hours} = \frac{80+xy}{y(y+2)} \text{ hours}$$

The time he would have taken if he had travelled at the increased speed during the whole of his journey $= \dfrac{x}{y+2}$ hours, and the time he would have taken if he had travelled all the way at his original speed $= \dfrac{x}{y}$ hours.

Hence, from the conditions of the problem,

$$\therefore \quad \frac{x}{y+2} = \frac{80+xy}{y(y+2)} - \frac{2}{3} \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

$$\text{and} \quad \frac{x}{y} = \frac{80+xy}{y(y+2)} + \frac{1}{3} \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

Subtracting (1) from (2),

$$x\left(\frac{1}{y} - \frac{1}{y+2}\right) = 1$$

$$\text{or,} \quad 2x = y(y+2), \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

Also from (2),

$$3x(y+2) = 3(80+xy)+y(y+2),$$

$$\text{or,} \quad 6x-240 = y(y+2). \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

Hence from (3) and (4),

$$6x-240 = 2x,$$

or, $\quad 4x = 240, \quad \therefore \quad x = 60.$

Thus the man had to travel 60 miles

Example 6. If there were no accidents, it would take half as long to travel the distance from A to B by rail road as by coach, but three hours being allowed for accidental stoppages by the former, the coach will travel the distance all but fifteen miles in the same time, if the distance were two-thirds as great as it is, and the same time allowed for railway stoppages the coach would take exactly the same time. Required the distance

Let x miles be the distance from A to B.

Suppose the coach travels at the rate of y miles an hour, then evidently, the rate of the train is $2y$ miles an hour

The time in which the train can travel the distance *plus* 3 hours = the time in which the coach travels only $(x-15)$ miles

Hence, $\dfrac{x}{2y}+3 = \dfrac{x-15}{y}$ ... ... (1).

Also, from the second condition of the problem,

$$\frac{\frac{2}{3}x}{2y}+3 = \frac{\frac{2}{3}x}{y}, \qquad \text{or,} \quad \frac{x}{3y}+3 = \frac{2x}{3y} \quad \ldots \quad (2)$$

From (2), $\dfrac{x}{3y} = 3,$ or, $x = 9y$ ... (3).

From (1), $x+6y = 2x-30,$ or, $6y = x-30$. (4).

Hence, from (3) and (4), $6y = 9y-30,$

whence $\quad y = 10,$

and $\quad x = 9 \times 10 = 90$

Thus the required distance = 90 miles.

Example 7. A boat goes up stream 30 miles and down stream 44 miles in 10 hours, it also goes up stream 40 miles and down stream 55 miles in 13 hours, find the rate of the stream and of the boat. (C U Entr Paper, 1880)

Suppose the boat would travel x miles per hour if there were no current, and that the current flows at the rate of y miles per hour

Then it is clear that *with the current* the boat travels $(x+y)$ miles per hour, and *against the current* $(x-y)$ miles per hour.

Hence, the time taken to travel 30 miles up stream $= \dfrac{30}{x-y}$ hours, and the time taken to travel 44 miles down stream $= \dfrac{44}{x+y}$ hours and by the 1st condition of the problem, we must have

$$\frac{30}{x-y}+\frac{44}{x+y} = 10 \qquad (1)$$

Similarly by the 2nd condition, we have

$$\frac{40}{x-y}+\frac{55}{x+y} = 13 \qquad . \quad . \qquad (2)$$

Multiplying (1) by 4, and (2) by 3, we have

$$\frac{120}{x-y}+\frac{176}{x+y} = 40$$

and

$$\frac{120}{x-y}+\frac{165}{x+y} = 39$$

Therefore, by subtraction,

$$\frac{11}{x+y} = 1, \quad \therefore \quad x+y = 11.$$

Hence, from (1) $\dfrac{30}{x-y} = 10-4 = 6,$

$$\therefore \quad x-y = 5$$

Thus we have $\left.\begin{aligned} x+y &= 11 \\ \text{and} \quad x-y &= 5 \end{aligned}\right\}$

Hence, by addition, $2x = 16, \ \therefore \ x = 8$
and by subtraction, $2y = 6, \ \therefore \ y = 3$

Thus the rates of the stream and the boat are respectively 3 miles and 8 miles per hour.

Example 8 A challenged B to ride a bicycle race of 1040 yards He first gave B 120 yards start, but lost by 5 seconds , he then gave B 5 seconds start, and won by 120 feet. How long does each take to ride the distance ?

(Calcutta University Entrance Paper, 1881)

Let the times which A and B take to ride the distance be x seconds and y seconds respectively

Then, the times they take to travel one yard are respectively $\frac{x}{1040}$ and $\frac{y}{1040}$ seconds

Let PQ represent the given distance, and let PR, SQ on it respectively represent 120 yards and 120 feet (or, 40 yards)

R S

P Q

In the first race B is at R, and A at P when they start, but B reaches Q 5 seconds earlier than A , therefore, the time taken by B to travel RQ = $(x-5)$ seconds

$$\text{Hence,} \quad x-5 = (1040-120)\times\frac{y}{1040}$$

$$= \left(1-\frac{3}{26}\right)y = \frac{23}{26}y \quad . \quad . \quad (1)$$

In the second race B starts from P 5 seconds earlier than A, but arrives at S when A arrives at Q , therefore the time taken by B to travel PS = $(x+5)$ seconds

$$\text{Hence,} \quad x+5 = (1040-40)\times\frac{y}{1040}$$

$$= \left(1-\frac{1}{26}\right)y = \frac{25}{26}y \quad . \quad . \quad (2)$$

Subtracting (1) from (2), we have

$$\tfrac{2}{26}y = 10, \quad \therefore \quad y = 130.$$

$$\text{Hence, from (1),} \quad x = 5+\tfrac{23}{26}\times 130$$

$$= 5+115 = 120$$

Thus the times required by A and B to ride the distance are respectively 2 minutes and 2 minutes 10 seconds.

Exercise (77).

1. There is a certain number consisting of 3 digits which is equal to 25 times the sum of the digits, and if 198 be added to the number, the digits will be reversed; also the sum of the extreme digits exceeds the middle digit by unity, find the number.

2. A shop-keeper, on account of bad book-keeping know neither the weight nor the prime cost of a certain article which he purchased He only recollects, that if he had sold the whole at 30*s* per lb, he would have gained £5 by it, and if he had sold it at 22*s* per lb, he would have lost £15 by it What was the weight and prime cost of the article?

3. Two persons, A and B, played cards After a certain number of games, A had won half as much as he had at first and found that if he had 15 shillings more, he would have had just three times as much as B But B afterwards won 10 shillings back, and he had then twice as much as A. What had each at first?

4 A and B can do a piece of work together in 12 days, which B working for 15 days and C for 30 would together complete; in 10 days they would finish it, working all three together, in what time could they separately do it?

5. A has twice as many pennies as shillings, B, who has 8*d* more than A, has twice as many shillings as pennies; together they have one more penny than they have shillings How much has each?

6. Two persons A and B could finish a work in m days they worked together n days when A was called off, and B finished it in p days. In what time could each do it?

7. A, B, C compare their fortunes, A says to B, 'give me Rs 700 of your money, and I shall have twice as much as you retain'; B says to C, 'give me Rs 1400, and I shall have thrice as much as you have remaining', C says to A, 'give me Rs 420, and then I shall have five times as much as you retain' How much has each?

8. A man walks 35 miles partly at the rate of 4 miles an hour, and partly at 5, if he had walked at 5 miles an hour when he walked at 4, and *vice versa*, he would have

covered two miles more in the same time Find the time he was walking

9. A train travelled a certain distance at a uniform rate. Had the speed been 6 miles an hour more, the journey would have occupied 4 hours less, and had the speed been 6 miles an hour less, the journey would have occupied 6 hours more Find the distance

10. Two vessels contain mixtures of wine and water, in one there is three times as much wine as water, in the other five times as much water as wine Find how much must be drawn off from each to fill a third vessel which holds seven gallons, in order that its contents may be half wine and half water

11. A number consists of 3 digits whose sum is 10. The middle digit is equal to the sum of the other two, and the number will be increased by 99 if its digits be reversed Find the number

12. A man has one pound's worth of silver in half-crowns, shillings and six-pences, and he has in all 20 coins If he changed the six pences for pennies, and the shillings for six-pences, he would have 73 coins. How many coins of each kind has he ?

13. A sum of money is divided equally among a certain number of persons, if there had been four more, each would have received a shilling less than he did, if there had been 5 fewer, each would have received two shillings more than he did, find the number of persons and what each received.

14. There is a cistern, into which water is admitted by three cocks, two of which are exactly of the same dimensions When they are all open, five-twelfths of the cistern is filled in four hours, and if one of the equal cocks is stopped, seven-ninths of the cistern is filled in ten hours and forty minutes In how many hours would each cock fill the cistern ?

15. A person exchanged 12 bushels of wheat for 8 bushels of barley, and £2 16*s*, offering at the same time to sell a certain quantity of wheat for an equal quantity of barley, and £3 15*s* in money, or for £10 in money. Required the prices of the wheat and barley per bushel

16. A wine-merchant has two sorts of wine, one sort worth 2 shillings a quart, and the other worth 3*s*. 4*d* a

quart, from these he wants to make a mixture of 100 quarts worth 2*s* 4*d* a quart How many quarts must he take from each sort?

17. The rent of a farm is paid in certain fixed numbers of quarters of wheat and barley, when wheat is at 55*s*. and barley at 33*s*. per quarter, the portions of rent by wheat and barley are equal to one another; but when wheat is at 65*s* and barley at 41*s* per quarter, the rent is increased by £7. What is the corn-rent?

18 A train 60 yards long passed another train 72 yards long which was travelling in the same direction on a parallel line of rails, in 12 seconds Had the slower train been travelling half as fast again, it would have been passed in 24 seconds. Find the rates at which the trains were travelling.

19 A farmer with 28 bushels of barley at 2*s* 4*d*. a bushel would mix rye at 3*s* per bushel, and wheat at 4*s* per bushel, so that the whole mixture may consist of 100 bushels and be worth 3*s* 4*d* per bushel How many bushel, of rye, and how many of wheat must he mix with the barley?

20. A person has £27. 6*s*. in guineas and crown-pieces, out of which he pays a debt of £14 17*s*, and finds that he has exactly as many guineas left as he has paid away crowns, and as many crowns as he has paid away guineas How many of each had he at first and how many of each had he left?

21. A waterman finds that he can row with the tide from A to B, a distance of 18 miles, in an hour and a half, and that to return from B to A against the same tide, though he rows back along the shore where stream is only three-fifths as strong as in the middle, takes him just two hours and a quarter Find the rate at which the tide runs in the middle where it is strongest

22. A and B run a mile First A gives B a start of 44 yards, and beats him by 51 seconds, at the second heat A gives B a start of 1 minute 15 seconds, and is beaten by 88 yards. Find the times in which A and B can run a mile separately

23. A and B run a race round a two-mile course In the first heat B reaches the winning post 2 minutes before A. In the second heat A increases his speed 2 miles an

hour, and B diminishes his by the same quantity, and A then arrives at the winning post 2 minutes before B. Find at what rate each ran in the first heat.

24. A railway train running from London to Cambridge meets on the way with an accident, which causes it to diminish its speed to $\frac{1}{n}$th of what it was before, and it is in consequence a hours late. If the accident had happened b miles nearer Cambridge, the train would have been c hours late. Find the rate of the train before the accident occurred.

25. A railway train after travelling for one hour, meets with an accident, which delays it one hour, after which it proceeds at three-fifths of its former rate, and arrives at the terminus three hours behind time; had the accident occured 50 miles further on, the train would have arrived 1 hour 20 minutes sooner. Required the length of the journey.

Miscellaneous Exercises (3).

I.

1. Multiply a^2+ax+x^2 by a^2-ax+x^2.

2. If $a^2+b^2=1=c^2+d^2$, show that

$$(ac-bd)^2+(ad+bc)^2=1.$$

3. Resolve $x^3+y^3+3xy-1$ into elementary factors.

4. If $2s=a+b+c$, show that

$$\frac{2bc+(b^2+c^2-a^2)}{2bc-(b^2+c^2+a^2)}=\frac{s(s-a)}{(s-b)(s-c)}.$$

5. Reduce $\dfrac{x^6}{x^2-1}-\dfrac{x^4}{x^2+1}-\dfrac{1}{x^2-1}+\dfrac{1}{x^2+1}$.

6. Solve $\dfrac{x+4a+b}{x+a+b}+\dfrac{4x+a+2b}{x+a-b}=5$.

7. A and B have the same income. A lays by a fifth of his, but B, by spending annually £80 more than A, at the end of 4 years finds himself £220 in debt. What was their income?

8. A certain sum of money is to be divided among a certain number of men; if there were 3 men less, each man would have £150 more; but if there were 6 men more each man would have £120 less. Find the sum of money and the number of men

II.

1. Simplify $\dfrac{1}{m^2+m+1}+\dfrac{2m}{m^4+m^2+1}$.

2. Show that

$$(x+y)^3-(y+z)^3+(z-x)^3 = 3(x+y)(y+z)(x-z).$$

3. If $x+\dfrac{1}{x} = 2(a+m)$, $x-\dfrac{1}{x} = 2b$, $y+\dfrac{1}{y} = 2(c+n)$,

$$y-\frac{1}{y} = 2d, \text{ find the value of } xy+\frac{1}{xy}.$$

4. Solve $\dfrac{1}{x-1}+\dfrac{2}{x-2}=\dfrac{3}{x-3}$.

5. Solve $ax+1 = by+1 = ay+bx$.

6. Solve

$$\left.\begin{aligned} x+y+z &= 1 \\ 2x+3y+z &= 4 \\ 4x+9y+z &= 16 \end{aligned}\right\}.$$

7. One pipe can fill a cistern in a hours, another can do it in b hours, in what time could the two running together fill it? And if a third pipe could empty the cistern in c hours, how long would it take to do this if the first two were running at the same time?

8. What fraction is that which becomes equal to $\frac{3}{4}$ when its numerator is increased by 6, and equal to $\frac{1}{2}$ when its denominator is diminished by 2?

III.

1. Divide $x+y^{\frac{1}{2}}+z^{\frac{1}{3}}-3x^{\frac{1}{3}}y^{\frac{1}{6}}z^{\frac{1}{9}}$ by $x^{\frac{1}{3}}+y^{\frac{1}{6}}+z^{\frac{1}{9}}$.

2. Find the H C F of $a^2x^3+a^5-2abx^3+b^2x^3+a^3b^2-2a^4b$ and $2a^2x^4-5a^4x^2+3a^6-2b^2x^4+5a^2b^2x^2-3a^4b^2$.

3. Simplify

$$\frac{1}{bc(b-a)(c-a)}+\frac{1}{ca(c-b)(a-b)}+\frac{1}{ab(a-c)(b-c)}.$$

4. Find the value of $\frac{x}{y}+\frac{x-1}{y+1}$,

$$\text{when } x=\frac{a}{a+b} \text{ and } y=\frac{b}{a-b}.$$

5. Given $\frac{4x+3}{9}+\frac{7x-29}{5x-12}=\frac{8x+19}{18}$, find the value of x.

6. Solve the equations $\left.\begin{aligned}\frac{3}{x}+\frac{2}{y}&=13\\ \frac{7}{x}+\frac{3}{y}&=27\end{aligned}\right\}$.

7. At what time between 5 and 6 will the hands of a clock coincide ?

8. A number of three digits has 5 in the units' place, and the middle figure is half the sum of the other two ; if 108 be added to the number, the hundreds' figure will take the place of the units', and the units' the place of the tens'. Find the number

IV.

1. Show that $(a^2+ab\sqrt{2}+b^2)(a^2-ab\sqrt{2}+b^2)=a^4+b^4$.

2. Find the H. C. F. of —

(i) $x^3-(a+p)x^2+(q+ap)x-aq$ and $x^3+ax^2-3a^2x+a^3$;

(ii) $x^3-y^3-z^3-3xyz$ and $x^2-2xy+y^2-2xz+2yz+z^2$.

3. Reduce to its lowest terms

$$\frac{a^2+b^2+c^2+2ab+2bc+2ca}{a^2-b^2-c^2-2bc},$$

and simplify $\dfrac{\frac{a}{a+b}+\frac{b}{a-b}}{\frac{a}{a-b}-\frac{b}{a+b}}$.

4. Simplify $\left(\frac{a+b}{a-b}+\frac{a-b}{a+b}\right)\div\left(\frac{a^{\frac{1}{2}}+b^{\frac{1}{2}}}{a^{\frac{1}{2}}-b^{\frac{1}{2}}}-\frac{a^{\frac{1}{2}}-b^{\frac{1}{2}}}{a^{\frac{1}{2}}+b^{\frac{1}{2}}}\right)$.

5. Solve the equations —

(i) $(7+x)(8-x)-\frac{7x}{3}=17x+1-x^2$;

(ii) $ax+y=x+by=\frac{1}{2}(x+y)+1$.

6. Show that $8(a+b+c)^3-(a+b)^3-(b+c)^3-(c+a)^3$
$=3(2a+b+c)(a+2b+c)(a+b+2c)$.

7. A person looking at his watch between 10 and 11 mistook the hands, and imagined the time to be 54 minutes earlier than it really was. What time was it?

8. An income of £196 is derived from two sums invested, one at 4 per cent., the other at 7 per cent; if the interest on the former had been 5 per cent., and on the latter 6 per cent., the income derived would have been £212. Find the sums invested.

V.

1. Resolve $15x^2-41x+14$ into elementary factors; and show that $(2b^2+a^2-ac)(2b^2+c^2-ac)-b^2(a+c)^2$
$=(b^2-ac)\{4b^2+(a-c)^2\}$.

2. There are two quantities, a, b, of which the L. C. M. is x, and the G. C. M. is y,

if $x+y=ma+\frac{b}{m}$, prove that $x^3+y^3=m^3a^3+\frac{b^3}{m^3}$.

3. Simplify

$$\left\{\frac{y^2-yz+z^2}{x}+\frac{x^2}{y+z}-\frac{3}{\frac{1}{y}+\frac{1}{z}}\right\}\cdot\frac{\frac{2}{y}+\frac{2}{z}}{\frac{1}{yz}+\frac{1}{zx}+\frac{1}{xy}}+(x+y+z)^2.$$

4. Find the H. C. F. of

$x^3+(5m-3)x^2+3m(2m-5)x-18m^2$ and
$x^3+(m-3)x^2-m(2m+3)x+6m^2$.

5. Find the L. C. M. of

$x^3-ax^2+a^2x-a^3$, $x^3+ax^2+a^2x+a^3$ and
$x^3+ax^2-a^2x-a^3$.

6. Solve the equations :—

(i) $\frac{3}{7}(6x-7)+\frac{1-7x}{6} = x$;

(ii) $\left.\begin{array}{r} 3x+2y+5z = 1 \\ 5x+3y-2z = 2 \\ 2x-5y-3z = 7 \end{array}\right\}$

7. A man buys a certain number of eggs at two a penny, four times as many at 5*d* a dozen, five times as many at 8*d* a score and sells them at 3*s* 8*d* a hundred, gaining by the transaction 3*s*. 6*d*. How many eggs did he buy ?

8. A and B ran a race which lasted 5 minutes. B had a start of 20 yards, but A ran 3 yards while B was running 2, and won by 30 yards Find the length of the course and the speed of each.

VI.

2. Simplify the following expressions —

(i) $\left(\frac{a}{b}+\frac{b}{a}\right)^4-2\left(\frac{a^2}{b^2}-\frac{b^2}{a^2}\right)^2+\left(\frac{a}{b}-\frac{b}{a}\right)^4$,

(ii) $\frac{x^6-y^6}{x^3+2x^2y+2xy^2+y^3}$.

2. Resolve into elementary factors :—

(i) $x^5-y^5-(x-y)^5$, (ii) $a^3+b^3+c^3-3abc$, and find the condition that $x^3+(p+q)x+a$ may be divisible by $x+p+q$

3. Reduce to its simplest form

$$\frac{x-y}{x+y}\times\frac{x^2-xy+y^2}{x^2+xy+y^2}\times\frac{x^3-y^3}{x^3+y^3}-\frac{(x-y)^4}{x^4-y^4}.$$

4. Find the L C. M of

$x^5+x^4+x^3+x^2+x+1$ and $x^5-x^4+x^3-x^2+x-1$

5. Given $\frac{x-1}{4}-\frac{2(x+1)}{9}+\frac{5(x-5)}{12}-4 = \frac{x+1}{18}$, find the value of x.

6. Solve $5y-9z = 18$, $4x-3z = 14$, $7x-6y = 14$

7 Two men leave two places A and B, distant d miles from each other and travel a and b miles a day respectively

in the same direction AB; what is their distance apart at the end of t days, and after what time will they come together?

Explain the result (1) when $a = b$, (2) when $a = b$ and $d = 0$.

8. At what time between 6 and 7 o'clock are the hands of a clock at right angles?

VII.

1. Multiply $a + b^{\frac{2}{3}} + c^{\frac{1}{2}} - b^{\frac{1}{3}}c^{\frac{1}{4}} - c^{\frac{1}{4}}a^{\frac{1}{2}} - a^{\frac{1}{2}}b^{\frac{1}{3}}$ by $a^{\frac{1}{2}} + b^{\frac{1}{3}} + c^{\frac{1}{4}}$;

and divide $\frac{x^4}{3} - \frac{11x^3}{12} + \frac{41x^2}{8} - \frac{23x}{4} + 6$ by $\frac{2x^2}{3} - \frac{5x}{6} + 1$.

2. Simplify $\left(1 + \frac{x^2 + y^2 - z^2}{2xy}\right) - \left(1 + \frac{y^2 + z^2 - x^2}{2yz}\right)$.

3. Reduce to its lowest terms $\frac{x^4 - x^3 - x + 1}{x^4 - 2x^3 - x^2 - 2x + 1}$.

4. Find the H C F of

$10x^3 - 54x^2 + 87x - 45$ and $5x^4 - 36x^3 + 87x^2 - 90x + 54$.

5. Prove that

$$\frac{a^2 + bx}{(a+b)(a-x)} + \frac{b^2 + ax}{(b+a)(b-x)} + \frac{x^2 + ab}{(x-a)(x-b)} = \frac{2ab}{(a-x)(b-x)}.$$

6. Solve

$$\left.\begin{aligned} x - y^{\frac{1}{2}} + z^{\frac{1}{3}} &= 2 \\ 4x + 6y^{\frac{1}{2}} + 5z^{\frac{1}{3}} &= 31 \\ 5x - 11y^{\frac{1}{2}} + 18z^{\frac{1}{3}} &= 22 \end{aligned}\right\}$$

7. There is a waggon with a mechanical contrivance by which the difference of the number of revolutions of the wheels on a journey is noted The circumference of the fore-wheel is a feet, and of the hind-wheel b feet; what is the distance gone over, when the fore-wheel has r revolutions more than the hind-wheel?

8. A and B together can do a piece of work in a certain time If they each did one-half of the work separately, A would have to work one day less, B two days more than before. Find the time in which A and B together can do the work

VIII.

1. Reduce to its simplest form

$$\left\{\frac{a^4-y^4}{a^2-2ay+y^2}-\frac{a^2+ay}{a-y}\right\}\times\left\{\frac{a^5-a^3y^2}{a^3+y^3}-\frac{a^4-2a^3y+a^2y^2}{a^2-ay+y^2}\right\}.$$

2. Find the H. C. F. of

$6x^4+5ax^3+3a^2x^2+13a^3x-3a^4$ and

$3x^3+16ax^2+15a^2x-4a^3$

3. Find the L C M of

$2x^2-x-1,\ 2x^2+3x+1,\ x^2-1.$

4. Simplify $\dfrac{a^2+bc+ca+ab}{a^2+2bc+2ca+ab}\times\dfrac{a^3+8c^3}{a^4+a^2c^2+6ac^3+4c^4}.$

5. If $ap=bq=cr$, show that

$$\frac{p^2}{qr}+\frac{q^2}{rp}+\frac{r^2}{pq}=\frac{bc}{a^2}+\frac{ca}{b^2}+\frac{ab}{c^2}.$$

6. If $x=\dfrac{a-b}{m-c},\ y=\dfrac{b-c}{m-a},\ z=\dfrac{c-a}{m-b},$

find the value of $x+y+z+xyz$

7. Given $ax+by=c^2$ and $\dfrac{a}{b+y}-\dfrac{b}{a+x}=0,$

find x and y.

8. A certain sum of money had to be divided equally among 15 persons, if £2 more had been available for division, each person would have received 5 per cent. more. What was the sum ?

IX.

1. If $\dfrac{a}{b}+\dfrac{c}{d}=\dfrac{b}{a}+\dfrac{d}{c}$, prove that

$$\frac{a^3}{b^3}+\frac{c^3}{d^3}=\frac{b^3}{a^3}+\frac{d^3}{c^3}.$$

2. Find the value of

$$\frac{(x-a)(x-b)}{(x-a-b)^2}, \text{ when } x = \frac{a^2+ab+b^2}{a+b}.$$

3. Reduce to its simplest form

$$\left\{\frac{2x}{x+y}-\frac{x^2}{x^2-y^2}+\frac{2y}{x-y}\right\}\times\left(\frac{1}{x}+\frac{1}{y}\right)\div\left\{\frac{3}{x-y}-\frac{2}{x}+\frac{1}{y}\right\}.$$

4. Simplify

$$\frac{x^4+x^2y^2+y^4}{x^2+y^2}\times\frac{x^2+y(2x+y)}{x^3-y^3}-\frac{x^3+y^3}{x^2-y(2x-y)}.$$

5. If $x = \dfrac{2ac}{a+c}$, show that the value of $\dfrac{(x-a)^2+(x-c)^2}{a^2+c^2}$

$+\dfrac{4ac}{(a+c)^2}$ is the same for all values of a and c.

6. Solve

$$\left.\begin{aligned} x+2y+3z &= 14 \\ 2x-3y+4z &= 8 \\ 3x+4y-5z &= -4 \end{aligned}\right\}.$$

7. A market woman bought apples at three for a penny and as many more at four for a penny; and thinking to make her money again, she sold them at seven for $2d$. She lost, however, $3d$ by the business. How much did she sell them for?

8. A, B and C run a mile race at uniform speed; A wins by 160 yards, B comes in second, beating C by $76\frac{12}{23}$ yards in distance and by $\frac{1}{4}$ minute in time. What is the pace of each?

X

1. Find the H. C. F. of

$a^{x+4}-a^xb^4+a^4b^y-b^{y+4}$ and $a^{x+2}-a^xb^2+a^2b^y-b^{y+2}$, and the L. C. M. of x^3-3x-2 and x^3+2x^2-x-2.

2. Simplify $\left\{\dfrac{a+b}{2(a-b)}-\dfrac{a-b}{2(a+b)}+\dfrac{2b^2}{a^2-b^2}\right\}\dfrac{a-b}{b}.$

3. Show that the two expressions

$(a^2-a+1)(b-c)+(b^2-b+1)(c-a)+(c^2-c+1)(a-b)$ and $(a^2-a+1)(b^2-c^2)+(b^2-b+1)(c^2-a^2)+(c^2-c+1)(a^2-b^2)$ are equal.

4 Show that

$$\frac{ab}{(x-a)(x-b)}+\frac{bc}{(x-b)(x-c)}+\frac{ca}{(x-c)(x-a)}=0,$$

$$\text{when } \frac{1}{x}=\frac{1}{3}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

5 Show that if $a+b+c=0$, then

$$a(b-c)^3+b(c-a)^3+c(a-b)^3=0.$$

6. Solve the equations —

(i) $$\frac{x^2-4x+4}{x-1}+\frac{x^2-3x-1}{x-2}=\frac{2(x^2-5x+5)}{x-3};$$

(ii) $$\left.\begin{aligned} y+z &= x+4a \\ z+x &= y+2a \\ x+y &= z \end{aligned}\right\}.$$

7. If $\dfrac{a-1}{x}-\dfrac{a-2}{y}=\dfrac{1}{b}$ and $\dfrac{b-1}{x}-\dfrac{b-2}{y}=\dfrac{1}{a}$,

show that $\dfrac{c-1}{x}-\dfrac{c-2}{y}=\dfrac{c}{ab}$.

8. Two trains, 92 feet long and 84 feet long respectively, are moving with uniform velocities on parallel rails in opposite directions, and are observed to pass each other in one second and a half, but when they are moving in the same direction, their velocities being the same as before, the faster train is observed to pass the other in 6 seconds Find the rate in miles per hour at which each train moves.

XI

1. Transform $(x^2+y^2+z^2+2xy)^2-2(x+y)^2z^2$ into the sum of two perfect squares.

2. Find the H C F of

$$6x^5-9x^4+19x^3-12x^2+19x-15 \text{ and}$$
$$4x^4-2x^3+10x^2+x+15.$$

3. If $a+b+c=0$, show that

$$\frac{a^2+b^2+c^2}{a^3+b^3+c^3}+\frac{2}{3}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=0$$

4. If $xy = ab(a+b)$ and $x^2-xy+y^2 = a^3+b^3$, show that $\left(\frac{x}{a}-\frac{y}{b}\right)\left(\frac{x}{b}-\frac{y}{a}\right) = 0$.

5. Prove that $\frac{(a+b)^3-(b+c)^3+(c+d)^3-(d+a)^3}{(a+b)^2-(b+c)^2+(c+d)^2-(d+a)^2}$

$$= \tfrac{3}{2}(a+b+c+d).$$

6. Resolve into factors :—

(i) $27x^2-48x-512$;

(ii) $(a+b-c-d)^2-(a-b+c-d)^2$;

(iii) $m^4+m^2n^2+n^4$

7. Solve $\left.\begin{aligned} x-\ y+2z &= 11 \\ 2x-\ y+\ z &= \ 9 \\ x-2y+\ z &= \ 0 \end{aligned}\right\}$.

8. A hare is 40 of her own leaps before a greyhound, and takes 5 leaps for the greyhound's 4 , but 3 of the greyhound's leaps are equal to 4 of the hare's How many leaps must the greyhound take to catch the hare ?

XII.

1. Simplify $\frac{x^3y-y^4}{xy^3+x^3y}-\left\{\frac{x^4+x^3y+x^2y^2}{(x^2-y^2)^3}\times\left(1+\frac{y}{x}\right)^2\right\}$.

2 If $x = \frac{a+1}{a-1}$, $y = \frac{b+1}{b-1}$, $z = \frac{c+1}{c-1}$, show that

$$\frac{(x^2+1)(y^2+1)(z^2+1)}{(xy+1)(yz+1)(zx+1)} = \frac{(a^2+1)(b^2+1)(c^2+1)}{(ab+1)(bc+1)(ca+1)}.$$

3 Find the H. C F and L C M of

$2x^4+x^3-9x^2+8x-2$ and $2x^4-7x^3+11x^2-8x+2$.

4 Show that $a^4+b^4+c^4+d^4-2a^2(b^2+c^2+d^2)$
$-2b^2(c^2+d^2)-2c^2d^2+8abcd$

$$= \{(a+b)^2-(c+d)^2\}\{(a-b)^2-(c-d)^2\}.$$

5. If $x = \frac{b^2+c^2-a^2}{2bc}$, $y = \frac{c^2+a^2-b^2}{2ca}$ and $z = \frac{a^2+b^2-c^2}{2ab}$, find in its simplest form the value of $\frac{x+yz}{y+zx}$.

6. If $a+b+c=1$, $ab+bc+ca=\frac{1}{3}$ and $abc=\frac{1}{27}$, prove that $\dfrac{1}{a+bc}+\dfrac{1}{b+ca}+\dfrac{1}{c+ab}=\dfrac{27}{4}$.

7. Given $\dfrac{4}{x}+\dfrac{6}{y}+\dfrac{3}{2z}=4\frac{4}{5}$, $-\dfrac{5}{x}-\dfrac{9}{4y}+\dfrac{1}{z}=3\frac{3}{4}$ and $\dfrac{25}{x}-\dfrac{5}{y}-\dfrac{1}{z}=2\frac{2}{3}$, find the values of x, y, z

8. A and B start simultaneously from two towns and meet after five hours if A had travelled one mile per hour faster and B had started one hour sooner, or if B had travelled one mile per hour slower and A had started one hour later, they would, in either case, have met at the same spot they actually met at What was the distance between the towns ?

CHAPTER XVI.

INDICES

1. Definition The product of m factors each equal to a is represented by a^m. [See Art 11 page 9]

Thus the meaning of a^m is clear when m is a *positive integer*.

2. The Index Law and the truths necessarily following from it.

To prove that $a^m \times a^n = a^{m+n}$, where m and n are any two positive integers

$$
\begin{array}{llll}
\text{Since,} & a^m = a.\,a\,a \quad \ldots & & \text{to } m \text{ factors,} \\
\text{and} & a^n = a.\,a\,a\,a \quad \ldots & & \text{to } n \text{ factoas;} \\
\therefore & a^m \times a^n = (a\,a\,a \quad \ldots & & \text{to } m \text{ factors)} \\
& \qquad\quad \times (a\,a\,a\,a \quad \ldots & & \text{to } n \text{ factors)} \\
& \qquad\quad = a\,a\,a\,a.\,a.\,a\,a. \quad \ldots & & \text{to } (m+n) \text{ factors.} \\
& \qquad\quad = a^{m+n}. & &
\end{array}
$$

This result is called the *Index Law.*

Cor. 1. $a^m \times a^n \times a^p = a^{m+n+p}$, when m, n and p are positive integers.

For $a^m \times a^n = a^{m+n}$, $\therefore$ $a^m \times a^n \times a^p = a^{m+n} \times a^p = a^{(m+n)+p}$ $= a^{m+n+p}$.

Hence, $a^m \times a^n \times a^p \times a^q \ldots = a^{m+n+p+q+} \ldots\ldots\ldots\ldots$

Thus, *the product of any number of powers of a given quantity is that power of the quantity whose index is equal to the sum of the indices of the factors.*

Cor. 2. $(a^m)^n = a^{mn}$, when m and n are any two positive integers.

For $(a^m)^n = a^m \times a^m \times a^m \times \ldots$ to n factors

$= a^{m+m+m+} \ldots$ to n terms [by Cor. 1.]

and $\therefore$ $= a^{mn}$.

Cor. 3. $a^m \div a^n = a^{m-n}$, when m and n are positive integers and m is greater than n.

For $a^{m-n} \times a^n = a^{(m-n)+n}$ [because $m-n$ is a positive integer].

$= a^m$

$\therefore$ $a^m \div a^n = a^{m-n}$.

3. Assuming the formula $a^m \times a^n = a^{m+n}$ to be true for *all* values of m and n, to find meanings for quantities with fractional or negative indices.

(1) To find the meaning of $a^{\frac{p}{q}}$, when p and q are any two positive integers.

Since $a^m \times a^n = a^{m+n}$ for *all* values of m and n, putting $\frac{p}{q}$ for each of them, we have

$$a^{\frac{p}{q}} \times a^{\frac{p}{q}} = a^{\frac{p}{q}+\frac{p}{q}} = a^{\frac{2p}{q}}.$$

Similarly, $a^{\frac{p}{q}} \times a^{\frac{p}{q}} \times a^{\frac{p}{q}} = a^{\frac{2p}{q}} \times a^{\frac{p}{q}} = a^{\frac{3p}{q}}$, and so on.

Hence, $a^{\frac{p}{q}} \times a^{\frac{p}{q}} \times a^{\frac{p}{q}}$ . . . . to q factors

$$= a^{\frac{qp}{q}} = a^p$$

Thus, $a^{\frac{p}{q}}$ is equal to the q^{th} root of a^p, and is therefore, equivalent to $\sqrt[q]{a^p}$

Cor. Hence, $a^{\frac{1}{2}} = \sqrt{a}$, $a^{\frac{1}{3}} = \sqrt[3]{a}$, $a^{\frac{1}{4}} = \sqrt[4]{a}$, and so on.

Generally, $a^{\frac{1}{n}} = \sqrt[n]{a}$

Note 1 From the Index Law it is also easy to see that

$$a^{\frac{1}{q}} \times a^{\frac{1}{q}} \times a^{\frac{1}{q}} \times \text{ to } p \text{ factors} = a^{\frac{p}{q}}$$

Thus $a^{\frac{p}{q}}$ may as well be regarded as the p^{th} power of $a^{\frac{1}{q}}$. *i e*, equivalent to $(\sqrt[q]{a})^p$ Thus $a^{\frac{p}{q}}$ may be interpreted either as the q^{th} root of the p^{th} power of a, or as the p^{th} power of the q^{th} root of a

Note 2 The operation of raising any given expression to any proposed power is called ***involution***, whilst the operation of finding any proposed root of any given expression is called ***evolution***

(ii) To find the meaning of a^0

Since $a^m \times a^n = a^{m+n}$ is true for *all* values of m and n, putting $m = 0$, we have

$$a^0 \times a^n = a^{0+n} = a^n,$$

$$\therefore \quad a^0 = a^n \div a^n = 1$$

Thus *any quantity raised to the power zero is equivalent to 1.*

(iii) To find the meaning of a^{-n}, where n is any positive quantity

Since $a^m \times a^n = a^{m+n}$ is true for *all* values of m and n, putting $m = -n$, we have

$$a^{-n} \times a^n = a^{-n+n} = a^0 = 1\,;$$

$$\therefore \quad a^{-n} = \frac{1}{a^n}, \text{ and } a^n = \frac{1}{a^{-n}}.$$

Cor. Hence, $a^m \div a^n = a^{m-n}$ for *all* values of m and n.

For $a^m \div a^n = \dfrac{a^m}{a^n} = a^m \times a^{-n} = a^{m-n}$.

Example 1. Find the value of $8^{\frac{5}{3}}$.

$$8^{\frac{5}{3}} = \left(\sqrt[3]{8}\right)^5 = 2^5 = 32$$

Example 2. Find the value of $4^{-\frac{5}{2}}$

$$4^{-\frac{5}{2}} = \frac{1}{4^{\frac{5}{2}}} = \frac{1}{(\sqrt{4})^5} = \frac{1}{2^5} = \frac{1}{32}.$$

Example 3. Multiply together $\sqrt{a^5}$, $a^{\frac{3}{4}}$, $\sqrt[4]{a^{-5}}$ and $\frac{1}{a^{-3}}$.

$$\text{The required product} = a^{\frac{5}{2}} \times a^{\frac{3}{4}} \times a^{-\frac{5}{4}} \times a^3$$
$$= a^{\frac{5}{2}+\frac{3}{4}-\frac{5}{4}+3}$$
$$= a^{\frac{5}{2}-\frac{1}{2}+3} = a^{2+3} = a^5.$$

Exercise (78).

Express the following avoiding fractional or negative indices :—

1. $a^{\frac{5}{7}}$ 2. $x^{-\frac{1}{2}}$. 3. $\frac{3}{x^{-\frac{4}{5}}}$.

4. $x^{-\frac{2}{5}} \times 3a^{-\frac{1}{2}}$. 5. $8m^{-2} \times m^{-\frac{2}{3}}$ 6. $x^{-\frac{1}{5}} \div 3a^{-\frac{5}{4}}$.

7. $x^{-\frac{2}{3}} - 2x^{-\frac{1}{5}}$. 8. $\sqrt[5]{x^2} \div \sqrt[5]{x^{-3}}$.

9. $\sqrt[2m]{a^{-5}} \times \sqrt[m]{a^8}$ 10. $\sqrt[4a]{x^4} - \sqrt[2a]{x^{-5}}$

Express the following avoiding radical signs and negative indices :—

11. $\left(\sqrt[8]{x}\right)^7$. 12. $\left(\sqrt[4]{a}\right)^{-6}$. 13 $\frac{1}{\sqrt[3]{x^{-2}}}$.

14. $\frac{1}{(\sqrt[5]{a})^{-2}}$. 15. $\sqrt[3]{x^4} \div \left(\sqrt[5]{x}\right)^{-1}$. 16. $\sqrt[4]{a^{-3}} \div \left(\sqrt[8]{a}\right)^{-12}$.

Find the value of —

17. $4^{-\frac{3}{2}}$. 18. $8^{\frac{2}{3}}$. 19. $9^{\frac{3}{2}}$.

20. $16^{\frac{5}{4}}$ 21. $81^{-\frac{3}{4}}$. 22. $\frac{1}{6^{-2}}$

23. $(125)^{-\frac{2}{3}}$. 24. $\left(\frac{1}{27}\right)^{-\frac{4}{3}}$ 25. $\left(\frac{1}{216}\right)^{-\frac{2}{3}}$.

26. Simplify $\frac{x^{m+2n}x^{3m-8n}}{x^{5m-6n}}$ (C U Entrance Paper, 1874)

4 To prove that $(a^m)^n = a^{mn}$ is true for *all* values of m and n.

(i) Let n be a *positive integer* Then, whatever may be the value of m, we have

$$\begin{aligned}(a^m)^n &= a^m \times a^m \times a^m \times \ldots \ldots \text{ to } n \text{ factors}\\ &= a^{m+m+m+\ldots\ldots \text{ to } n \text{ terms}}\\ &= a^{mn}.\end{aligned}$$

(ii) Let n be a *positive* fraction equal to $\frac{p}{q}$, where p and q are positive integers Then we have

$$\begin{aligned}\left(a^m\right)^n = \left(a^m\right)^{\frac{p}{q}} &= \sqrt[q]{\left(a^m\right)^p} && \text{[Art 3, (i)]}\\ &= \sqrt[q]{a^{mp}} && \text{[by (i)]}\\ &= a^{\frac{mp}{q}} && \text{[Art 3, (i)]}\\ &= a^{mn}.\end{aligned}$$

(iii) Let n be *any negative* quantity, equal to $-p$, where p is *positive* Then we have

$$\begin{aligned}(a^m)^n = (a^m)^{-p} &= \frac{1}{(a^m)^p} && \text{[Art 3, (iii)]}\\ &= \frac{1}{a^{mp}} && \text{[by (i) and (ii)]}\\ &= a^{-mp} && \text{[Art 3, (iii)]}\\ &= a^{m(-p)} = a^{mn}.\end{aligned}$$

Thus the proposition is established.

5. To prove that $a^n b^n = (ab)^n$ for *all* values of n.

(i) Let n be a positive integer. Then we have

$$\begin{aligned} a^n b^n &= (a.\ a.\ a. \ \ldots \text{ to } n \text{ factors}) \\ &\quad \times (b.\ b.\ b. \ \ldots \text{ to } n \text{ factors}) \\ &= ab.\ ab.\ ab. \ \ldots \text{ to } n \text{ factors}. \\ &= (ab)^n. \end{aligned}$$

(ii) Let n be a *positive fraction* equal to $\dfrac{p}{q}$, where p and q are positive integers. Then putting x for $a^n b^n$ we have $x = a^{\frac{p}{q}} b^{\frac{p}{q}}$, $\therefore$ $x^q = \left(a^{\frac{p}{q}} b^{\frac{p}{q}}\right)^q$

$$\begin{aligned} &= \left(a^{\frac{p}{q}}\right)^q \times \left(b^{\frac{p}{q}}\right)^q && \text{[by (i)]} \\ &= a^p \times b^p && \text{[Art. 4]} \\ &= (ab)^p, && \text{[by (i)]} \end{aligned}$$

$$\therefore \quad x = (ab)^{\frac{p}{q}}, \text{ i.e., } a^n b^n = (ab)^n.$$

(iii) Let n be *any negative* quantity, equal to $-p$, where p is positive. Then we have

$$\begin{aligned} a^n b^n &= a^{-p} b^{-p} \\ &= \frac{1}{a^p b^p} \quad \text{[Art. 3, (iii)]} \\ &= \frac{1}{(ab)^p} \quad \text{[by (i) and (ii)]} \\ &= (ab)^{-p} \quad \text{[Art. 3, (iii)]} \\ &= (ab)^n. \end{aligned}$$

Thus the proposition is established.

Cor. 1. $\dfrac{a^n}{b^n} = a^n b^{-n} = a^n. \left(b^{-1}\right)^n = \left(ab^{-1}\right)^n = \left(\dfrac{a}{b}\right)^n.$

Cor. 2. $a^n b^n c^n = (ab)^n c^n = (abc)^n$; generally, $a^n b^n c^n d^n \ldots = (abcd \ldots)^n$.

6. Applications of the results proved in last two articles.

Example 1. Simplify $(a^8b^{\frac{5}{3}})^{-\frac{3}{4}}$.

$$(a^8b^{\frac{5}{3}})^{-\frac{3}{4}} = (a^8)^{-\frac{3}{4}} \times (b^{\frac{5}{3}})^{-\frac{3}{4}}$$
$$= a^{8\cdot(-\frac{3}{4})} \times b^{\frac{5}{3}\cdot(-\frac{3}{4})}$$
$$= a^{-6}b^{-\frac{5}{4}}$$

Example 2. Simplify $\sqrt{a^{-2}b} \times \sqrt[3]{ab^{-3}}$.

$$\sqrt{a^{-2}b} = (a^{-2}b)^{\frac{1}{2}} = (a^{-2})^{\frac{1}{2}} \times b^{\frac{1}{2}} = a^{-1}b^{\frac{1}{2}},$$
$$\sqrt[3]{ab^{-3}} = (ab^{-3})^{\frac{1}{3}} = a^{\frac{1}{3}} \times (b^{-3})^{\frac{1}{3}} = a^{\frac{1}{3}}b^{-1}.$$

Hence, the given expression

$$= a^{-1}b^{\frac{1}{2}} \times a^{\frac{1}{3}}b^{-1}$$
$$= a^{-1+\frac{1}{3}} \times b^{\frac{1}{2}-1} = a^{-\frac{2}{3}}b^{-\frac{1}{2}}.$$

Example 3. Simplify $\sqrt{a^3b^{-\frac{2}{3}}c^{-\frac{7}{6}}} - \sqrt[3]{a^4b^{-1}c^{\frac{5}{4}}}$.

$$\sqrt{a^3b^{-\frac{2}{3}}c^{-\frac{7}{6}}} = \left(a^3b^{-\frac{2}{3}}c^{-\frac{7}{6}}\right)^{\frac{1}{2}}$$
$$= \left(a^3\right)^{\frac{1}{2}}\left(b^{-\frac{2}{3}}\right)^{\frac{1}{2}}\left(c^{-\frac{7}{6}}\right)^{\frac{1}{2}}$$
$$= a^{\frac{3}{2}}b^{-\frac{1}{3}}c^{-\frac{7}{12}},$$

and
$$\sqrt[3]{a^4b^{-1}c^{\frac{5}{4}}} = \left(a^4b^{-1}c^{\frac{5}{4}}\right)^{\frac{1}{3}}$$
$$= \left(a^4\right)^{\frac{1}{3}}\left(b^{-1}\right)^{\frac{1}{3}}\left(c^{\frac{5}{4}}\right)^{\frac{1}{3}}$$
$$= a^{\frac{4}{3}}b^{-\frac{1}{3}}c^{\frac{5}{12}}.$$

Hence, the given expression

$$= a^{\frac{3}{2}}b^{-\frac{1}{3}}c^{-\frac{7}{12}} - a^{\frac{4}{3}}b^{-\frac{1}{3}}c^{\frac{5}{12}}$$

$$= a^{\frac{3}{2}}b^{-\frac{1}{3}}c^{-\frac{7}{12}} + a^{-\frac{3}{4}}b^{\frac{1}{3}}c^{-\frac{5}{12}}$$

$$= a^{\frac{3}{2}-\frac{4}{3}}b^{-\frac{1}{3}+\frac{1}{3}}c^{-\frac{7}{12}-\frac{5}{12}}$$

$$= a^{\frac{1}{6}}b^{0}c^{-1} = a^{\frac{1}{6}}c^{-1}.$$

Exercise (79).

Simplify—

1. $\left(a^{-\frac{3}{4}}\right)^{8}$. 2. $\left(a^{-\frac{2}{3}}b^{\frac{5}{6}}\right)^{\frac{3}{4}}$. 3. $\left(a^{-\frac{1}{2}}b^{-3}\right)^{-2}$.

4. $\left(a^{6}b^{\frac{5}{4}}\right)^{-\frac{4}{3}}$. 5. $\left(\sqrt[3]{a^{4}b^{3}}\right)^{8}$. 6. $\left(\sqrt[6]{x^{9}y^{-8}}\right)^{-3}$.

7. $\sqrt[8]{x^{2}.\sqrt[4]{x^{-3}}}$. 8. $\sqrt{a^{-3}b^{4}} \times \sqrt[4]{a^{2}b^{-8}}$.

9. $\sqrt[4]{x^{-2}\sqrt{y^{5}}} \times \sqrt{x\sqrt[4]{y^{3}}}$.

10. $(8x^{3} \div 27a^{-3})^{\frac{2}{3}}$. 11. $(64x^{3} \div 27a^{-3})^{-\frac{2}{3}}$.

12. $\sqrt[3]{a^{6}b^{-2}c^{-4}} \times \sqrt[4]{a^{-6}b^{4}c^{8}}$ 13. $\sqrt{a^{-\frac{2}{3}}b^{4}c^{-\frac{1}{3}}} \div \sqrt[3]{a^{2}b^{4}c^{-1}}$.

14. $\sqrt{ab^{-4}c^{3}} \div \left(\sqrt[3]{a^{3}b^{4}c^{-3}}\right)^{-1}$. 15. $\left(\dfrac{a^{-1}b^{2}}{a^{2}b^{-4}}\right)^{7} \div \left(\dfrac{a^{3}b^{-5}}{a^{-2}b^{3}}\right)^{-5}$.

7. Miscellaneous Examples.

Example 1. Divide $a+b+c+3a^{\frac{1}{3}}b^{\frac{2}{3}}+3a^{\frac{2}{3}}b^{\frac{1}{3}}$ by

$$a^{\frac{1}{3}}+b^{\frac{1}{3}}+c^{\frac{1}{3}}.$$

Let us proceed by arranging the dividend and the divisor according to descending powers of a —

$$
\begin{array}{l|l|l}
a^{\frac{1}{3}}+(b^{\frac{1}{3}}+c^{\frac{1}{3}}) & a+3a^{\frac{2}{3}}b^{\frac{1}{3}}+3a^{\frac{1}{3}}b^{\frac{2}{3}}+(b+c) & a^{\frac{2}{3}}+a^{\frac{1}{3}}(2b^{\frac{1}{3}}-c^{\frac{1}{3}}) \\
 & a+\ a^{\frac{2}{3}}(b^{\frac{1}{3}}+c^{\frac{1}{3}}) & +(b^{\frac{2}{3}}-b^{\frac{1}{3}}c^{\frac{1}{3}}+c^{\frac{2}{3}}) \\
\hline
 & a^{\frac{2}{3}}(2b^{\frac{1}{3}}-c^{\frac{1}{3}})+3a^{\frac{1}{3}}b^{\frac{2}{3}}+(b+c) & \\
 & a^{\frac{2}{3}}(2b^{\frac{1}{3}}-c^{\frac{1}{3}})+\ a^{\frac{1}{3}}(2b^{\frac{2}{3}}+b^{\frac{1}{3}}c^{\frac{1}{3}}-c^{\frac{2}{3}}) & \\
\hline
 & a^{\frac{1}{3}}(b^{\frac{2}{3}}-b^{\frac{1}{3}}c^{\frac{1}{3}}+c^{\frac{2}{3}})+(b+c) & \\
 & a^{\frac{1}{3}}(b^{\frac{2}{3}}-b^{\frac{1}{3}}c^{\frac{1}{3}}+c^{\frac{2}{3}})+(b+c) & \\
\hline
\end{array}
$$

Thus the required quotient

$$= a^{\frac{2}{3}}+2a^{\frac{1}{3}}b^{\frac{1}{3}}-a^{\frac{1}{3}}c^{\frac{1}{3}}+b^{\frac{2}{3}}-b^{\frac{1}{3}}c^{\frac{1}{3}}+c^{\frac{2}{3}}.$$

Note In multiplication as well as in division the arrangement of the expressions concerned according to ascending or descending powers of some common letter should never be overlooked Such arrangements invariably give neatness to the required operations, if not always indispensable

Example 2. Divide $x+y^{\frac{1}{2}}+z^{\frac{1}{3}}-3x^{\frac{1}{3}}y^{\frac{1}{6}}z^{\frac{1}{9}}$ by $x^{\frac{1}{3}}+y^{\frac{1}{6}}+z^{\frac{1}{9}}$.

Putting a for $x^{\frac{1}{3}}$, b for $y^{\frac{1}{6}}$ and c for $z^{\frac{1}{9}}$, we have

$$
\begin{aligned}
& x+y^{\frac{1}{2}}+z^{\frac{1}{3}}-3x^{\frac{1}{3}}y^{\frac{1}{6}}z^{\frac{1}{9}} \\
&= a^3+b^3+c^3-3abc \\
&= (a+b+c)(a^2+b^2+c^2-ab-bc-ca) \\
&= (x^{\frac{1}{3}}+y^{\frac{1}{6}}+z^{\frac{1}{9}})\left\{(x^{\frac{1}{3}})^2+(y^{\frac{1}{6}})^2+(z^{\frac{1}{9}})^2-x^{\frac{1}{3}}y^{\frac{1}{6}}-y^{\frac{1}{6}}z^{\frac{1}{9}}-z^{\frac{1}{9}}x^{\frac{1}{3}}\right\} \\
&= (x^{\frac{1}{3}}+y^{\frac{1}{6}}+z^{\frac{1}{9}})(x^{\frac{2}{3}}+y^{\frac{1}{3}}+z^{\frac{2}{9}}-x^{\frac{1}{3}}y^{\frac{1}{6}}-y^{\frac{1}{6}}z^{\frac{1}{9}}-z^{\frac{1}{9}}x^{\frac{1}{3}})
\end{aligned}
$$

Hence, the required quotient

$$
\begin{aligned}
&= x^{\frac{2}{3}}+y^{\frac{1}{3}}+z^{\frac{2}{9}}-x^{\frac{1}{3}}y^{\frac{1}{6}}-y^{\frac{1}{6}}z^{\frac{1}{9}}-z^{\frac{1}{9}}x^{\frac{1}{3}} \\
&= x^{\frac{2}{3}}-x^{\frac{1}{3}}(y^{\frac{1}{6}}+z^{\frac{1}{9}})+(y^{\frac{1}{3}}-y^{\frac{1}{6}}z^{\frac{1}{9}}+z^{\frac{2}{9}}).
\end{aligned}
$$

Example 3. Divide $x^{2^n}+a^{2^{n-1}}x^{2^{n-1}}+a^{2^n}$ by

$$x^{2^{n-1}}-a^{2^{n-2}}x^{2^{n-2}}+a^{2^{n-1}}.$$

Let $\quad m = x^{2^{n-2}}$ and $n = a^{2^{n-2}}$.

Then $\quad m^2 = \left(x^{2^{n-2}}\right)^2 = x^{2\times 2^{n-2}} = x^{2^{n-1}}$,

and $\quad m^4 = \left(m^2\right)^2 = \left(x^{2^{n-1}}\right)^2 = x^{2\times 2^{n-1}} = x^{2^n}$.

Similarly, $\quad n^2 = a^{2^{n-1}}$ and $n^4 = a^{2^n}$.

Hence, $\dfrac{x^{2^n}+a^{2^{n-1}}x^{2^{n-1}}+a^{2^n}}{x^{2^{n-1}}-a^{2^{n-2}}x^{2^{n-2}}+a^{2^{n-1}}}$

$$= \frac{m^4+m^2n^2+n^4}{m^2-mn+n^2} = \frac{(m^2+n^2)^2-m^2n^2}{m^2-mn+n^2}$$

$$= \frac{(m^2+n^2+mn)(m^2+n^2-mn)}{m^2-mn+n^2}$$

$$= m^2+mn+n^2$$

$$= x^{2^{n-1}}+x^{2^{n-2}}a^{2^{n-2}}+a^{2^{n-1}}.$$

Example 4. Find the H. C F of

$a^2+2b^2+(a+2b)\sqrt{ab}$ and $a^2-b^2+(a-b)\sqrt{ab}$.

The 1st expression $= a^2+a\sqrt{ab}+2b\sqrt{ab}+2b^2$

$$= a^2+a^{\frac{3}{2}}b^{\frac{1}{2}}+2a^{\frac{1}{2}}b^{\frac{3}{2}}+2b^2$$

$$= a^{\frac{3}{2}}(a^{\frac{1}{2}}+b^{\frac{1}{2}})+2b^{\frac{3}{2}}(a^{\frac{1}{2}}+b^{\frac{1}{2}})$$

$$= (a^{\frac{1}{2}}+b^{\frac{1}{2}})(a^{\frac{3}{2}}+2b^{\frac{3}{2}})$$

The 2nd expression $= a^2+a\sqrt{ab}-b\sqrt{ab}-b^2$

$$= a^2+a^{\frac{3}{2}}b^{\frac{1}{2}}-a^{\frac{1}{2}}b^{\frac{3}{2}}-b^2$$

$$= a^{\frac{3}{2}}(a^{\frac{1}{2}}+b^{\frac{1}{2}})-b^{\frac{3}{2}}(a^{\frac{1}{2}}+b^{\frac{1}{2}})$$

$$= (a^{\frac{1}{2}}+b^{\frac{1}{2}})(a^{\frac{3}{2}}-b^{\frac{3}{2}})$$

Hence, since $a^{\frac{3}{2}}+2b^{\frac{3}{2}}$ and $a^{\frac{3}{2}}-b^{\frac{3}{2}}$ have no common factor, the H C F. required

$$= a^{\frac{1}{2}}+b^{\frac{1}{2}} = \sqrt{a}+\sqrt{b}$$

Example 5. Simplify $\dfrac{x+(xy^2)^{\frac{1}{3}}-(x^2y)^{\frac{1}{3}}}{x+y}$.

The numerator $= x+x^{\frac{1}{3}}y^{\frac{2}{3}}-x^{\frac{2}{3}}y^{\frac{1}{3}}$

$$= x^{\frac{1}{3}}(x^{\frac{2}{3}}+y^{\frac{2}{3}}-x^{\frac{1}{3}}y^{\frac{1}{3}}),$$

and the denominator

$$= (x^{\frac{1}{3}})^3+(y^{\frac{1}{3}})^3$$

$$= (x^{\frac{1}{3}}+y^{\frac{1}{3}})\left\{(x^{\frac{1}{3}})^2-(x^{\frac{1}{3}})(y^{\frac{1}{3}})+(y^{\frac{1}{3}})^2\right\}$$

$$= (x^{\frac{1}{3}}+y^{\frac{1}{3}})(x^{\frac{2}{3}}-x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}})$$

Hence, the given expression $= \dfrac{x^{\frac{1}{3}}}{x^{\frac{1}{3}}+y^{\frac{1}{3}}}$.

Example 6. Shew that

$$\frac{1}{1+x^{m-n}+x^{m-p}}+\frac{1}{1+x^{n-m}+x^{n-p}}+\frac{1}{1+x^{p-m}+x^{p-n}} = 1.$$

The 1st term $= \dfrac{x^{-m}}{x^{-m}(1+x^{m-n}+x^{m-p})}$

$$= \frac{x^{-m}}{x^{-m}+x^{-n}+x^{-p}},$$

The 2nd term $= \dfrac{x^{-n}}{x^{-n}(1+x^{n-m}+x^{n-p})}$

$$= \frac{x^{-n}}{x^{-n}+x^{-m}+x^{-p}},$$

and the 3rd term $= \dfrac{x^{-p}}{x^{-p}(1+x^{p-m}+x^{p-n})}$

$$= \frac{x^{-p}}{x^{-p}+x^{-m}+x^{-n}}.$$

Hence, the given expression

$$= \frac{x^{-m}}{x^{-m}+x^{-n}+x^{-p}}+\frac{x^{-n}}{x^{-n}+x^{-m}+x^{-p}}+\frac{x^{-p}}{x^{-p}+x^{-m}+x^{-n}}$$

$$= \frac{x^{-m}+x^{-n}+x^{-p}}{x^{-m}+x^{-n}+x^{-p}} = 1.$$

Example 7. Solve the equation

$$a^{-x}.(a^x+b^{-x}) = \frac{a^2b^2+1}{a^2b^2}.$$

We have $a^{-x}.a^x+a^{-x}b^{-x} = 1+\dfrac{1}{a^2b^2}$,

or, $1+(ab)^{-x} = 1+a^{-2}b^{-2}$

$= 1+(ab)^{-2}$

Hence $(ab)^{-x} = (ab)^{-2}$, $\therefore\ x = 2$

Example 8. Solve
$$\left.\begin{array}{lr} a^x a^{y+1} = a^7 & \ldots (1) \\ a^{2y}.a^{3x+5} = a^{20} & \ldots (2) \end{array}\right\}$$

(Calcutta University Entrance Paper 1879.)

From the 1st equation, we have

$$a^{x+(y+1)} = a^7,$$

$$\therefore\quad x+y+1 = 7 \quad \ldots (3)$$

From the 2nd equation, we have

$$a^{2y+(3x+5)} = a^{20},$$

$$\therefore\quad 2y+3x+5 = 20 \quad \ldots (4)$$

Now from (3) and (4), we have

$$\left.\begin{array}{r} x+y-6 = 0 \\ \text{and}\quad 3x+2y-15 = 0 \end{array}\right\}$$

Therefore, by cross multiplication

$$\frac{x}{-15+12} = \frac{y}{-18+15} = \frac{1}{2-3},$$

$$\text{or,}\quad \frac{x}{-3} = \frac{y}{-3} = -1.$$

Hence $\qquad x = 3$ and $y = 3$

Example 9. If $a^b = b^a$, shew that $\left(\frac{a}{b}\right)^{\frac{a}{b}} = a^{\frac{a}{b}-1}$, and if $a = 2b$, shew that $b = 2$.

Since $a^b = b^a$,

$\therefore \quad a = b^{\frac{a}{b}}$ [extracting the bth root of both sides].

Hence, $\left(\frac{a}{b}\right)^{\frac{a}{b}} = \dfrac{a^{\frac{a}{b}}}{b^{\frac{a}{b}}} = \dfrac{a^{\frac{a}{b}}}{a} = a^{\frac{a}{b}-1}$

If $a = 2b$, from the given relation we have

$$(2b)^b = (b)^{2b} = (b^2)^b,$$

$$\therefore \quad 2b = b^2, \qquad \therefore \quad b = 2$$

Example 10. If $x = \left(a+\sqrt{a^2+b^3}\right)^{\frac{1}{3}} + \left(a-\sqrt{a^2+b^3}\right)^{\frac{1}{3}}$, shew that $x^3+3bx-2a = 0$.

Putting m for $a+\sqrt{a^2+b^3}$,

and $\qquad n$ for $a-\sqrt{a^2+b^3}$, we have

$$x^3 = \left(m^{\frac{1}{3}}+n^{\frac{1}{3}}\right)^3$$

$$= \left(m^{\frac{1}{3}}\right)^3+\left(n^{\frac{1}{3}}\right)^3+3m^{\frac{1}{3}}n^{\frac{1}{3}}\left(m^{\frac{1}{3}}+n^{\frac{1}{3}}\right)$$

$$= m+n+3(mn)^{\frac{1}{3}}\left(m^{\frac{1}{3}}+n^{\frac{1}{3}}\right)$$

$$= m+n+3(mn)^{\frac{1}{3}}x.$$

But $\qquad m+n = 2a,$

and $\qquad (mn)^{\frac{1}{3}} = \{a^2-(a^2+b^3)\}^{\frac{1}{3}}$

$$= (-b^3)^{\frac{1}{3}} = -b\ ,$$

$\therefore \qquad x^3 = 2a-3bx,$

$\therefore \qquad x^3+3bx-2a = 0.$

Exercise (80.)

Multiply —

1. $x^{\frac{2}{3}}+2x^{\frac{1}{2}}+3x^{\frac{1}{3}}+2x^{\frac{1}{6}}+1$ by $x^{\frac{1}{3}}-2x^{\frac{1}{6}}+1$

2. $a^{\frac{2}{3}}+3a^{\frac{1}{3}}b^{\frac{1}{3}}+9b^{\frac{2}{3}}$ by $a^{\frac{1}{3}}-3b^{\frac{1}{3}}$.

3. $1+ab^{-1}+a^2b^{-2}$ by $1-ab^{-1}+a^2b^{-2}$.

4. $x+2y^{\frac{1}{2}}+3z^{\frac{1}{3}}$ by $x-2y^{\frac{1}{2}}+3z^{\frac{1}{3}}$

5. $x^{-1}+x^{-\frac{1}{2}}y^{-\frac{1}{2}}+y^{-1}$ by $x^{-1}-x^{-\frac{1}{2}}y^{-\frac{1}{2}}+y^{-1}$.

6. $a^{\frac{2}{3}}-a^{\frac{1}{3}}+1-a^{-\frac{1}{3}}+a^{-\frac{2}{3}}$ by $a^{\frac{1}{3}}+1+a^{-\frac{1}{3}}$

7. $a^{\frac{2}{3}}+y^{\frac{2}{3}}+z^{\frac{2}{3}}-y^{\frac{1}{3}}z^{\frac{1}{3}}-z^{\frac{1}{3}}x^{\frac{1}{3}}-x^{\frac{1}{3}}y^{\frac{1}{3}}$ by $x^{\frac{1}{3}}+y^{\frac{1}{3}}+z^{\frac{1}{3}}$

8. $a^m+3b^n-2c^p$ by $a^m-3b^n+2c^p$

9 $a^{\frac{5}{2}}+8ab+4a^{\frac{3}{2}}b^{\frac{2}{3}}+2a^2b^{\frac{1}{3}}+32b^{\frac{5}{3}}+16a^{\frac{1}{2}}b^{\frac{4}{3}}$ by $a^{\frac{1}{2}}-2b^{\frac{1}{3}}$.

10. $a^{\frac{5}{8}}+a^{\frac{1}{4}}x^{-\frac{3}{8}}+x^{-\frac{5}{8}}+a^{\frac{3}{8}}x^{-\frac{1}{4}}+a^{\frac{1}{8}}x^{-\frac{1}{2}}+a^{\frac{1}{2}}x^{-\frac{1}{8}}$ by

$$a^{\frac{3}{8}}+a^{\frac{1}{8}}x^{-\frac{1}{4}}-x^{-\frac{3}{8}}-a^{\frac{1}{4}}x^{-\frac{1}{8}}.$$

Divide —

11. $x^{\frac{5}{2}}-4x^{\frac{3}{2}}-2x^{\frac{1}{2}}+6x-x^2$ by $x^{\frac{3}{2}}+2-4x^{\frac{1}{2}}$

12. $8+12x^{-1}+2x^{-2}+2x^{-4}$ by $x^{-2}-2x^{-1}+4$.

13. $xy^{-1}+2x^{\frac{1}{2}}y^{-\frac{1}{2}}+3+2x^{-\frac{1}{2}}y^{\frac{1}{2}}+x^{-1}y$ by $x^{-1}+x^{-\frac{1}{2}}y^{-\frac{1}{2}}+y^{-1}$.

14. $a^{\frac{5}{2}}-a^{\frac{3}{2}}b+ab^{\frac{3}{2}}-2a^{\frac{1}{2}}b^{2}+b^{\frac{5}{2}}$ by $a^{\frac{3}{2}}-ab^{\frac{1}{2}}+a^{\frac{1}{2}}b-b^{\frac{3}{2}}$

15. $8x^{-n}-8x^{n}+5x^{3n}-3x^{-3n}$ by $5x^{n}-3x^{-n}$

16. $8x^{\frac{3}{2}}+y^{-\frac{3}{2}}-z+6x^{\frac{1}{2}}y^{-\frac{1}{2}}z^{\frac{1}{3}}$ by $2x^{\frac{1}{2}}+y^{-\frac{1}{2}}-z^{\frac{1}{3}}$.

17. Show that $x^{3}+a^{3}+x^{\frac{3}{2}}a^{\frac{3}{2}}$ is divisible by $x^{\frac{3}{4}}+a^{\frac{3}{4}}+x^{\frac{3}{8}}a^{\frac{3}{8}}$.

18. Multiply $x^{2^{n-1}}+a^{2^{n-1}}$ by $x^{2^{n-1}}-a^{2^{n-1}}$.

19. Divide $x^{2^{n}}-y^{2^{n}}$ by $x^{2^{n-1}}+y^{2^{n-1}}$.

(Calcutta University Entrance Paper 1879.)

20. Simplify $\left\{\left(a^{m}\right)^{m-\frac{1}{m}}\right\}^{\frac{1}{m+1}}$.

21. Divide $2x^{-\frac{1}{4}}+3x^{\frac{3}{4}}-7x^{\frac{1}{4}}+x-2x^{\frac{1}{2}}$ by $x^{\frac{1}{4}}-2x^{-\frac{1}{4}}$.

22. Find the square of $x^{\frac{3}{4}}-x^{\frac{1}{2}}y^{-\frac{1}{4}}+y^{\frac{1}{2}}$.

23. Divide $x^{\frac{3n}{2}}-a^{\frac{3n}{2}}$ by $x^{\frac{n}{2}}-a^{\frac{n}{2}}$.

24. Find the square of $x^{\frac{1}{3}}-2x^{\frac{1}{2}}+x^{\frac{2}{3}}$.

25. Divide $ax^{-1}+a^{-1}x+2$ by $a^{\frac{1}{3}}x^{-\frac{1}{3}}+a^{-\frac{1}{3}}x^{\frac{1}{3}}-1$.

26. Simplify $\left\{\dfrac{a-b}{a^{\frac{1}{2}}-b^{\frac{1}{2}}}-\dfrac{a^{\frac{3}{2}}-b^{\frac{3}{2}}}{a-b}\right\}^{-1}$.

27. Simplify $\dfrac{x^{\frac{1}{3}}+3y^{\frac{1}{3}}}{x^{\frac{1}{3}}-3y^{\frac{1}{3}}}+\dfrac{x^{\frac{2}{3}}-3x^{\frac{1}{3}}y^{\frac{1}{3}}+9y^{\frac{2}{3}}}{x^{\frac{2}{3}}+3x^{\frac{1}{3}}y^{\frac{1}{3}}+9y^{\frac{2}{3}}}$

28. Simplify $\dfrac{a^{\frac{3}{2}}-ax^{\frac{1}{2}}+a^{\frac{1}{2}}x-x^{\frac{3}{2}}}{a^{\frac{5}{2}}-a^2x^{\frac{1}{2}}+3a^{\frac{3}{2}}x-3ax^{\frac{3}{2}}+a^{\frac{1}{2}}x^2-x^{\frac{5}{2}}}$.

29. Simplify $\dfrac{a^2+b^2-a^{-2}-b^{-2}}{a^2b^2-a^{-2}b^{-2}}+\dfrac{(a-a^{-1})(b-b^{-1})}{ab+a^{-1}b^{-1}}$.

30. Simplify $\dfrac{x-y}{x^{\frac{3}{4}}+x^{\frac{1}{2}}y^{\frac{1}{4}}}-\dfrac{x^{\frac{1}{2}}+y^{\frac{1}{2}}}{x^{\frac{1}{2}}y^{\frac{1}{4}}+x^{\frac{1}{4}}y^{\frac{1}{2}}}$.

31. Simplify
$(a+b+c)(a^{-1}+b^{-1}+c^{-1})-a^{-1}b^{-1}c^{-1}(b+c)(c+a)(a+b)$.

Solve —

32. $2^{x+7}=4^{x+2}$.

33. $(\sqrt{3})^{x+5}=\left(\sqrt[3]{3}\right)^{2x+5}$.

34. $\left(\sqrt[3]{4}\right)^{4x+7}=\left(\sqrt[11]{64}\right)^{2x+7}$.

35. $\left(\sqrt[3]{25}\right)^{2x+1}=\left(\sqrt[5]{125}\right)^{x+6}$.

36. $\left.\begin{aligned} 2^{3x-1}&=4^{y-1}\\ 3x-y&=1\end{aligned}\right\}$.

37. $\left.\begin{aligned} 9^{2x-3}&=(\sqrt{3})^{2y-x}\\ 2^{3x}&=4^{y}\end{aligned}\right\}$.

38. $\left.\begin{aligned} 4^{3y-1}&=16^{x+y}\\ 3^{x+3y}&=9^{2x+3}\end{aligned}\right\}$

39. $\left.\begin{aligned} 2^{x+y+z}&=8^{x+z-y}\\ 5^{3y+2}&=25^{x+z}\\ 3^{2x+2z+y}&=9^{3x+y}\end{aligned}\right\}$.

40. $\left.\begin{aligned} \left(\sqrt{a}\right)^{x+y}&=\left(\sqrt[3]{a}\right)^{y+z-1}\\ \left(\sqrt[3]{b}\right)^{x+z-2}&=\left(\sqrt[5]{b}\right)^{y+z}\\ \left(\sqrt[4]{c}\right)^{y}&=\left(\sqrt[7]{c}\right)^{x+y+1}\end{aligned}\right\}$.

CHAPTER XVII

SURDS.

1. Definition. Any root of any arithmetical number which cannot be exactly found is called a *surd* or an *irrational quantity* Thus $\sqrt{2}$, $\sqrt{6}$, $\sqrt[3]{4}$, and $\sqrt[4]{5}$ are all surds

Note Quantities which are not surds are called *rational quantities* Hence every root of an arithmetical number is either *rational* or *irrational* Thus $\sqrt[3]{8}$ $\sqrt{25}$ and $\sqrt[4]{16}$ are *rational* quantities, whilst $\sqrt{2}$, $\sqrt[3]{5}$ and $\sqrt[4]{9}$ are all *irrational* quantities

An algebraical expression also, such as $\sqrt{x}$, is called a surd although the value of x may be such that $\sqrt{x}$ is not in reality a surd For instance, if $x = 4$, $\sqrt{x} = \sqrt{4} = 2$, and is therefore not really a surd

2. To express in the form of a surd the product of a rational quantity and a surd.

Example 1. $5\sqrt{3} = \left(5^2\right)^{\frac{1}{2}} \times 3^{\frac{1}{2}}$

$= \left(5^2 \times 3\right)^{\frac{1}{2}}$ [Art 5, Chap XVI.]

$= \sqrt{5^2 \times 3} = \sqrt{75}.$

Example 2. $2\sqrt[3]{9} = \left(2^3\right)^{\frac{1}{3}} \times 9^{\frac{1}{3}}$

$= \left(2^3 \times 9\right)^{\frac{1}{3}}$ [Art 5, Chap XVI.]

$= \sqrt[3]{2^3 \times 9} = \sqrt[3]{72}.$

Exercise (81).

Express as a complete surd —

1. $3\sqrt{5}$ 2. $2\sqrt[3]{3}$ 3. $2\sqrt[4]{6}$. 4. $4\sqrt[4]{5}$
5. $a\sqrt[n]{b}$ 6. $a^3\sqrt[a]{y}$. 7. $a^4\sqrt[5]{b^2}$.

3. A surd may sometimes be expressed as the product of a rational quantity and a surd.

Example 1. $\sqrt{32} = \sqrt{16 \times 2}$

$$= (4^2 \times 2)^{\frac{1}{2}}$$

$$= (4^2)^{\frac{1}{2}} \times 2^{\frac{1}{2}} \qquad \text{[Art 5, Chap XVI]}$$

$$= 4 \times 2^{\frac{1}{2}} = 4\sqrt{2}$$

Example 2. $\sqrt[3]{40} = \sqrt[3]{8 \times 5}$

$$= (2^3 \times 5)^{\frac{1}{3}}$$

$$= (2^3)^{\frac{1}{3}} \times 5^{\frac{1}{3}} \qquad \text{[Art 5, Chap XVI]}$$

$$= 2 \times 5^{\frac{1}{3}} = 2\sqrt[3]{5}$$

Exercise (82).

Simplify —

1. $\sqrt{18}$	2. $\sqrt{80}$	3. $\sqrt[3]{250}$.
4. $\sqrt{128}$.	5. $\sqrt[4]{405}$	6. $\sqrt[3]{1372}$.
7. $\sqrt[4]{1875}$.	8. $\sqrt[3]{a^6 b}$.	9. $\sqrt[n]{x^{4n}a}$
10. $\sqrt[3]{-2560}$	11. $\sqrt[3]{-192a^3b^4}$	12. $\sqrt[3]{500a^7x^4}$.

4. **Similar surds.** Two or more surds are said to be *similar* when they can be so reduced as to have the same irrational factor Thus $\sqrt{45}$ and $\sqrt{80}$ are similar surds for they are respectively equivalent to $3\sqrt{5}$ and $4\sqrt{5}$. The sum of any number of similar surds may be found as follows :—

Example 1. $\sqrt{147} + \sqrt{27}$

$$= \sqrt{49 \times 3} + \sqrt{9 \times 3}$$

$$= 7\sqrt{3} + 3\sqrt{3} = 10\sqrt{3}.$$

Example 2. $\sqrt[3]{625} - \sqrt[3]{135} + \sqrt[3]{40}$

$$= \sqrt[3]{125 \times 5} - \sqrt[3]{27 \times 5} + \sqrt[3]{8 \times 5}$$

$$= \sqrt[3]{5^3 \times 5} - \sqrt[3]{3^3 \times 5} + \sqrt[3]{2^3 \times 5}$$
$$= 5\sqrt[3]{5} - 3\sqrt[3]{5} + 2\sqrt[3]{5}$$
$$= 4\sqrt[3]{5}.$$

Exercise (83).

Simplify —

1. $\sqrt{12}+\sqrt{75}$.
2. $\sqrt{18}+\sqrt{32}$.
3. $\sqrt{20}+\sqrt{180}$
4. $\sqrt{98}-\sqrt{50}$.
5. $\sqrt[3]{128}-\sqrt[3]{54}$.
6. $\sqrt[4]{80}+\sqrt[4]{405}$
7. $\sqrt[4]{768}-\sqrt[4]{243}$
8. $2\sqrt{27}-\sqrt{75}+\sqrt{12}$.
9. $2\sqrt{405}-3\sqrt{125}+\sqrt{45}$
10. $4\sqrt[3]{192}-4\sqrt[3]{375}+2\sqrt[3]{24}$.
11. $3\sqrt[3]{40}+2\sqrt[3]{625}-4\sqrt[3]{320}$.
12. $5\sqrt[3]{-54}-2\sqrt[3]{-16}+4\sqrt[3]{686}$.
13. $\sqrt{45x^3}+\sqrt{80x^3}+\sqrt{5xy^2}$
14. $x\sqrt[3]{x^3a}+y\sqrt[3]{-8y^3a}-z\sqrt[3]{-27z^3a}$
15. $2\sqrt[4]{32a^4x}+3\sqrt[4]{512a^4x}-4a\sqrt[4]{162x}$

5. Surds of the same order. Surds are said to be *of the same order* or *equiradical* when they have all got the same root symbol. Thus $\sqrt{5}$, $\sqrt{a^3}$ and $(a+x)^{\frac{5}{2}}$ are all surds of the same (the *second*) order

A surd of the second order is often called a *quadratic* surd, whilst one of the third order, as $\sqrt[3]{4}$ or $\sqrt[3]{a^2}$, is called a *cubic* surd.

Surds of different orders may be reduced to equivalent surds of the same order

Example 1. Reduce $\sqrt{5}$ and $\sqrt[3]{4}$ to surds of the same order

The given surds are respectively of the 2nd and 3rd orders, and the L C. M of 2 and 3 is 6. Hence we can at once reduce them to surds of the 6th order, thus —

$$\sqrt{5}=5^{\frac{1}{2}}=5^{\frac{3}{6}}=\sqrt[6]{5^3}=\sqrt[6]{125},$$
$$\sqrt[3]{4}=4^{\frac{1}{3}}=4^{\frac{2}{6}}=\sqrt[6]{4^2}=\sqrt[6]{16}.$$

Thus the required surds are $\sqrt[6]{125}$ and $\sqrt[6]{16}$.

Example 2. Reduce $\sqrt[6]{3}$ and $\sqrt[8]{2}$ to surds of the same order

The L C M of 6 and 8 is 24

Thus we have $\sqrt[6]{3} = 3^{\frac{1}{6}} = 3^{\frac{4}{24}} = \sqrt[24]{3^4} = \sqrt[24]{81}$,

and $\sqrt[8]{2} = 2^{\frac{1}{8}} = 2^{\frac{3}{24}} = \sqrt[24]{2^3} = \sqrt[24]{8}$.

Thus the required surds are $\sqrt[24]{81}$ and $\sqrt[24]{8}$

Example 3. Which is the greater $\sqrt[3]{9}$ or $\sqrt[4]{20}$?

We have $\sqrt[3]{9} = 9^{\frac{1}{3}} = 9^{\frac{4}{12}} = \sqrt[12]{9^4} = \sqrt[12]{6561}$,

and $\sqrt[4]{20} = 20^{\frac{1}{4}} = 20^{\frac{3}{12}} = \sqrt[12]{20^3} = \sqrt[12]{8000}$.

Thus the given surds are respectively equivalent to $\sqrt[12]{6561}$ and $\sqrt[12]{8000}$, and as the latter is greater than the former, therefore $\sqrt[4]{20} > \sqrt[3]{9}$.

Exercise (34)

Reduce to surds of the same order —

1. $\sqrt{3}$ and $\sqrt[3]{2}$ **2.** $\sqrt[3]{4}$ and $\sqrt[4]{5}$ **3.** $\sqrt[5]{2}$ and $\sqrt[3]{3}$.

4. $\sqrt[4]{3}$ and $\sqrt[6]{5}$. **5.** $\sqrt[6]{4}$ and $\sqrt[8]{6}$.

Which is the greater —

6. $\sqrt{2}$ or $\sqrt[3]{3}$? **7.** $\sqrt[3]{3}$ or $\sqrt[4]{4}$? **8.** $\sqrt[3]{6}$ or $\sqrt[4]{10}$?

Arrange according to descending order of magnitude —

9. $\sqrt[4]{6}$, $\sqrt{2}$ and $\sqrt[3]{4}$ **10.** $\sqrt[4]{3}$, $\sqrt[8]{10}$ and $\sqrt[12]{25}$.

6. Multiplication and division of surds.

Example 1. $\sqrt[3]{6} \times \sqrt[3]{10} = 6^{\frac{1}{3}} \times 10^{\frac{1}{3}}$

$$= (6 \times 10)^{\frac{1}{3}}$$
$$= \sqrt[3]{60}$$

Note In this example the given surds are of the *same* order.

Example 2. $\sqrt[4]{5}\times\sqrt[6]{8} = 5^{\frac{1}{4}}\times 8^{\frac{1}{6}}$

$$= 5^{\frac{3}{12}}\times 8^{\frac{2}{12}}$$

$$= \left(5^3\right)^{\frac{1}{12}}\times\left(8^2\right)^{\frac{1}{12}} \text{ [Art 4 Chap XVI]}$$

$$= \left(5^3\times 8^2\right)^{\frac{1}{12}} \text{ [Art 6 Chap XVI]}$$

$$= \sqrt[12]{125\times 64} = \sqrt[12]{8000}.$$

Note In this example the given surds are of *different* orders

Example 3. $\sqrt[3]{2}\times\sqrt[5]{2} = 2^{\frac{1}{3}}\times 2^{\frac{1}{5}}$

$$= 2^{\frac{1}{3}+\frac{1}{5}} = 2^{\frac{8}{15}}$$

$$= \sqrt[15]{2^8} = \sqrt[15]{256}$$

Note In this example the given surds have got the same quantity under the radical sign They may as well be regarded as surds of different orders and treated like those in the last example

Example 4. $4\sqrt{18}\times\sqrt{75}$

$$= 4.3\sqrt{2}\times 5\sqrt{3}$$

$$= 63\sqrt{2}.\sqrt{3} = 60\sqrt{6}$$

Note In this example the given surds have been reduced to simpler forms before multiplication

Example 5. $\sqrt[6]{4}-\sqrt[4]{6} = 4^{\frac{1}{6}}-6^{\frac{1}{4}}$

$$= 4^{\frac{2}{12}}-6^{\frac{3}{12}}$$

$$= \frac{\left(4^2\right)^{\frac{1}{12}}}{\left(6^3\right)^{\frac{1}{12}}} \text{ [Art. 4, Chap XVI]}$$

$$= \left(\frac{4^2}{6^3}\right)^{\frac{1}{12}} \text{ [Cor 1 Art 5, Chap XVI]}$$

$$= \sqrt[12]{\frac{2}{27}}.$$

Example 6 Express $\sqrt{5} \div 3\sqrt{3}$ as a fraction with a rational denominator

We have
$$\sqrt{5} \div 3\sqrt{3} = \frac{\sqrt{5}}{3\sqrt{3}} = \frac{\sqrt{5}\times\sqrt{3}}{3\sqrt{3}\times\sqrt{3}}$$
$$= \frac{\sqrt{15}}{3\times 3} = \frac{\sqrt{15}}{9}.$$

Note For arithmetical calculations it is always most convenient to reduce the quotient of one surd by another to the form of a fraction with a rational denominator Hence even when the numerical value of a surd fraction is not required it is usual to express it in the above form

Exercise (85).

Simplify :—

1. $\sqrt{5}\times\sqrt{10}$
2. $\sqrt{8}\times\sqrt{6}$.
3. $\sqrt{27}\times\sqrt{3}$.
4. $\sqrt{15}\times\sqrt{6}$.
5. $\sqrt{20}\times\sqrt{45}$.
6. $\sqrt[3]{5}\times\sqrt[3]{25}$.
7. $\sqrt[3]{6ax}\times\sqrt[3]{27a^2x^4}$
8. $\sqrt[3]{2}\times\sqrt[2]{6}$
9. $\sqrt[2]{2}\times\sqrt[3]{6}$.
10. $\sqrt[3]{4}\times\sqrt[2]{8}$.
11. $\sqrt[3]{9}\times\sqrt[2]{27}$
12. $\sqrt[6]{2}\times\sqrt[9]{3}$.
13. $\sqrt[4]{8}\times\sqrt[6]{3}$
14. $\sqrt[6]{2}\times\sqrt[9]{2}$.
15. $\sqrt[4]{4}\times\sqrt[6]{4}$.
16. $5\sqrt{8}\times 2\sqrt{6}$
17. $8\sqrt{12}\times 3\sqrt{24}$
18. $4\sqrt[3]{72}\times 5\sqrt[3]{576}$.
19. $7\sqrt[3]{8a^3x^2}\times 5\sqrt[3]{27b^3x^3}$.
20. $8\sqrt{10}-4\sqrt{15}$
21. $3\sqrt{12}-6\sqrt{27}$.
22. $\sqrt[3]{36}\div\sqrt[3]{18}$
23. $\sqrt[9]{8}-\sqrt[6]{6}$

Given $\sqrt{2} = 1{\cdot}414$, $\sqrt{3} = 1{\cdot}732$, $\sqrt{5} = 2{\cdot}236$, find to 3 places of decimals the numerical value of .—

24. $\sqrt{2}-\sqrt{6}$.
25. $\sqrt{72}-\sqrt{10}$.
26. $\sqrt{275}\div\sqrt{22}$.
27. $10\sqrt{108}-\sqrt{15}$

7. Compound surds. An expression consisting of two or more simple surds connected by the sign + or − is called a *compound* surd. Thus $5\sqrt{2}$ and $4\sqrt{3}$ are simple surds, but $5\sqrt{2}+1\sqrt{3}$ and $5\sqrt{2}-4\sqrt{3}$ are compound surds.

Two or more compound surds are multiplied together in the same way as two or more compound algebraical expressions.

Example 1. Multiply $3\sqrt{x}+2\sqrt{3}$ by $\sqrt{x}-\sqrt{3}$.

$$\begin{aligned}(3\sqrt{x}+2\sqrt{3})(\sqrt{x}-\sqrt{3}) &= 3\sqrt{x}\ \sqrt{x}+2\sqrt{3}\ \sqrt{x} \\ &\quad -3\sqrt{x}\ \sqrt{3}-2\sqrt{3}\ \sqrt{3} \\ &= 3x+2\sqrt{3x}-3\sqrt{3x}-6 \\ &= 3x-\sqrt{3x}-6.\end{aligned}$$

Example 2. Multiply $7\sqrt{2}+\sqrt{3}$ by $7\sqrt{2}-\sqrt{3}$

$$\begin{aligned}(7\sqrt{2}+\sqrt{3})(7\sqrt{3}-\sqrt{3}) &= (7\sqrt{2})^2-(\sqrt{3})^2 \\ &= 49\ 2-3 \\ &= 98-3 = 95\end{aligned}$$

Example 3. Find the square of $\sqrt{3a+x}+\sqrt{3a-x}$

$$\begin{aligned}\left(\sqrt{3a+x}+\sqrt{3a-x}\right)^2 &= (\sqrt{3a+x})^2+(\sqrt{3a-x})^2 \\ &\quad +2\sqrt{3a+x}.\sqrt{3a-x} \\ &= (3a+x)+(3x-x)-2\sqrt{9a^2-x^2} \\ &= 6a+2\sqrt{9a^2-x^2}.\end{aligned}$$

Exercise (86).

Multiply —

1. $\sqrt{a}+\sqrt{b}$ by $\sqrt{ab}$ **2.** $\sqrt{a}+\sqrt{b}$ by $\sqrt{a}-\sqrt{b}$.

3. $3\sqrt{a}-5$ by $2\sqrt{a}$ **4.** $4\sqrt{x}+3\sqrt{y}$ by $4\sqrt{x}-3\sqrt{y}$

5. $2\sqrt{x-5}+4$ by $3\sqrt{x-5}-6$

6. $3\sqrt{5}-4\sqrt{2}$ by $2\sqrt{5}+3\sqrt{2}$.

7. $\sqrt{2}+2\sqrt{3}+\sqrt{7}$ by $\sqrt{2}+2\sqrt{3}-\sqrt{7}$.

8. $3-\sqrt{5}+\sqrt{8}$ by $3-\sqrt{5}-\sqrt{8}$.

9. $\sqrt{11}+\sqrt{6}-\sqrt{3}$ by $\sqrt{11}-\sqrt{6}+\sqrt{3}$.

10. $\sqrt[3]{4}+\sqrt[3]{9}+\sqrt[3]{48}$ by $\sqrt[3]{2}+\sqrt[3]{3}$.

Find the square of —

11. $\sqrt{x+a}-\sqrt{x-a}$ **12.** $2\sqrt{8}+5\sqrt{6}$.

13. $2\sqrt{5}+3\sqrt{7}$ **14.** $\sqrt{a^2+2b^2}-\sqrt{a^2-2b^2}$.

15. $2\sqrt{x^2+y^2}+5\sqrt{x^2-y^2}$.

8. **Rationalisation** If two surds be such that their product is rational, each of them is said to be rationalised when multiplied by the other. Thus $2\sqrt{5}$ and $\sqrt{3}+\sqrt{2}$ are rationalised when respectively multiplied by $\sqrt{5}$ and $\sqrt{3}-\sqrt{2}$,

$$\text{for} \quad 2\sqrt{5}\times\sqrt{5}=10,$$
$$\text{and} \quad (\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2})=3-2=1.$$

Two binomial quadratic surds which differ only in the sign which connects their terms are said to be **conjugate** or **complementary** to each other Thus $\sqrt{3}+\sqrt{2}$ and $2\sqrt{5}-\sqrt{7}$ are respectively *conjugate* (or *complementary*) to $\sqrt{3}-\sqrt{2}$ and $2\sqrt{5}+\sqrt{7}$.

Evidently therefore every binomial quadratic surd is rationalised when multiplied by the complementary surd.

Hence a fraction with a binomial quadratic surd for its denominator can be easily reduced to an equivalent fraction with a rational denominator

Example 1. Given $\sqrt{2}=1\cdot414$, find to three places of decimals the value of $\dfrac{1+\sqrt{2}}{3-2\sqrt{2}}$.

$$\frac{1+\sqrt{2}}{3-2\sqrt{2}}=\frac{(1+\sqrt{2})(3+2\sqrt{2})}{(3-2\sqrt{2})(3+2\sqrt{2})}$$
$$=\frac{3+3\sqrt{2}+2\sqrt{2}+4}{9-8}$$
$$=7+5\sqrt{2}$$
$$=7+5\times1\cdot414$$
$$=7+7\cdot070=14\cdot070.$$

Example 2. Rationalise the denominator of

$$\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}$$

The given expression

$$=\frac{(\sqrt{1+x^2}-\sqrt{1-x^2})^2}{(\sqrt{1+x^2}+\sqrt{1-x^2})(\sqrt{1+x^2}-\sqrt{1-x^2})}$$

$$= \frac{(1+x^2)+(1-x^2)-2\sqrt{1-x^4}}{(1+x^2)-(1-x^2)}$$

$$= \frac{2-2\sqrt{1-x^4}}{2x^2} = \frac{1-\sqrt{1-x^4}}{x^2}.$$

Example 3 Simplify $\dfrac{3+\sqrt{6}}{5\sqrt{3}-2\sqrt{12}-\sqrt{32}+\sqrt{50}}$.

The denominator $= 5\sqrt{3}-2\times 2\sqrt{3}-4\sqrt{2}+5\sqrt{2}$

$= \sqrt{3}+\sqrt{2}$

Hence, the given fraction $= \dfrac{3+\sqrt{6}}{\sqrt{3}+\sqrt{2}}$

$$= \frac{(3+\sqrt{6})(\sqrt{3}-\sqrt{2})}{(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2})}$$

$$= \frac{3\sqrt{3}+3\sqrt{2}-3\sqrt{2}-2\sqrt{3}}{3-2}$$

$$= \sqrt{3}$$

Example 4. Simplify $\dfrac{3\sqrt{2}}{\sqrt{3}+\sqrt{6}}-\dfrac{4\sqrt{3}}{\sqrt{6}+\sqrt{2}}+\dfrac{\sqrt{6}}{\sqrt{2}+\sqrt{3}}$

The 1st term

$$= \frac{3\sqrt{2}}{\sqrt{3}(1+\sqrt{2})} = \frac{\sqrt{6}}{\sqrt{2}+1} = \frac{\sqrt{6}(\sqrt{2}-1)}{(\sqrt{2}+1)(\sqrt{2}-1)} = 2\sqrt{3}-\sqrt{6}.$$

The 2nd term

$$= \frac{4\sqrt{3}}{\sqrt{2}(\sqrt{3}+1)} = \frac{2\sqrt{6}}{\sqrt{3}+1} = \frac{2\sqrt{6}(\sqrt{3}-1)}{(\sqrt{3}+1)(\sqrt{3}-1)}$$

$$= \frac{2(3\sqrt{2}-\sqrt{6})}{2} = 3\sqrt{2}-\sqrt{6}.$$

The 3rd term

$$= \frac{\sqrt{6}}{\sqrt{2}+\sqrt{3}} = \frac{\sqrt{6}(\sqrt{3}-\sqrt{2})}{(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2})} = 3\sqrt{2}-2\sqrt{3}$$

Hence the given expression

$$= (2\sqrt{3}-\sqrt{6})-(3\sqrt{2}-\sqrt{6})+(3\sqrt{2}-2\sqrt{3})$$
$$= 0.$$

Exercise (87).

Reduce to an equivalent fraction with a rational denominator —

1 $\dfrac{5\sqrt{3}+\sqrt{7}}{4\sqrt{3}+2\sqrt{7}}$. 2. $\dfrac{3\sqrt{2}+2\sqrt{3}}{3\sqrt{2}-2\sqrt{3}}$. 3. $\dfrac{4+3\sqrt{2}}{3-2\sqrt{2}}$.

4. $\dfrac{3\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}}$. 5 $\dfrac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}$

6. $\dfrac{\sqrt{x^2+1}-\sqrt{x^2-1}}{\sqrt{x^2+1}+\sqrt{x^2-1}}$. 7. $\dfrac{1}{1+\sqrt{2}+\sqrt{3}}$

Given $\sqrt{2}=1.414$, $\sqrt{3}=1.732$, $\sqrt{5}=2.236$, find to three places of decimals the value of —

8. $\dfrac{\sqrt{2}+1}{\sqrt{2}-1}$. 9 $\dfrac{\sqrt{3}}{2-\sqrt{3}}$. 10. $\dfrac{8-5\sqrt{2}}{3-2\sqrt{2}}$.

11. $\dfrac{3}{\sqrt{5}-\sqrt{2}}$. 12. $\dfrac{3+\sqrt{5}}{3-\sqrt{5}}$. 13. $\dfrac{\sqrt{5}+\sqrt{\ }}{4+\sqrt{1}}$

Simplify —

14. $\dfrac{1}{x+\sqrt{x^2-1}}+\dfrac{1}{x-\sqrt{x^2-1}}$

15 $\dfrac{15}{\sqrt{10}+\sqrt{20}+\sqrt{40}-\sqrt{5}-\sqrt{80}}$.

16. $\dfrac{\sqrt{2}(\sqrt{3}+1)(2-\sqrt{3})}{(\sqrt{2}-1)(3\sqrt{3}-5)(2+\sqrt{2})}$

17. $\dfrac{4}{\sqrt{3}+\sqrt{5}-\sqrt{2}}$. 18. $(3+2\sqrt{2})^{-3}+(3-2\sqrt{2})^{-3}$

19. $\dfrac{x+\sqrt{x^2-1}}{x-\sqrt{x^2-1}}-\dfrac{x-\sqrt{x^2-1}}{x+\sqrt{x^2-1}}$.

20. $\dfrac{\sqrt{x^2+1}+\sqrt{x^2-1}}{\sqrt{x^2+1}-\sqrt{x^2-1}}+\dfrac{\sqrt{x^2+1}-\sqrt{x^2-1}}{\sqrt{x^2+1}+\sqrt{x^2-1}}$

Rationalise the denominator of :—

21 $\dfrac{1}{\sqrt[3]{3}+\sqrt[3]{2}}$. 22. $\dfrac{1}{\sqrt[3]{4}-\sqrt[3]{3}}$.

9. The square root of a rational quantity cannot be partly rational and partly a quadratic surd

If possible let $\sqrt{n} = a + \sqrt{m}$,

then, squaring both sides we must have

$$n = a^2 + m + 2a\sqrt{m},$$

whence $$\sqrt{m} = \frac{n - a^2 - m}{2a}.$$

Thus a surd is equal to a rational quantity, which is impossible

10 If $a + \sqrt{b} = x + \sqrt{y}$, where a and x are rational, and $\sqrt{b}$ and $\sqrt{y}$ are irrational, then will $a = x$, and $b = y$.

For if a be not equal to x, let $a = x + m$,

then we have $$x + m + \sqrt{b} = x + \sqrt{y},$$

$$\therefore\ m + \sqrt{b} = \sqrt{y}.$$

Thus $\sqrt{y}$ is partly rational and partly a quadratic surd, which is impossible by the last article

Therefore $a = x$, and consequently $\sqrt{b} = \sqrt{y}$, or $b = y$.

Note It should be distinctly borne in mind that the results proved above are true *only when* $\sqrt{b}$ and $\sqrt{y}$ are *really* irrational. For instance, from the relation $5 + \sqrt{9} = 3 + \sqrt{25}$, we cannot conclude that $5 = 3$ and $9 = 25$

11. To find the square root of $a + \sqrt{b}$, where $\sqrt{b}$ is a surd.

Let $\sqrt{a + \sqrt{b}} = \sqrt{x} + \sqrt{y}$.

Then, squaring both sides we have

$$a + \sqrt{b} = x + y + 2\sqrt{xy}$$

Hence, by the last article,

$$\left.\begin{aligned} a &= x + y \\ \text{and}\quad \sqrt{b} &= 2\sqrt{xy} \end{aligned}\right\} \quad \ldots\ldots (1)$$

Hence, $$a^2 - b = (x+y)^2 - 4xy$$
$$= (x-y)^2,$$

$$\therefore \sqrt{a^2 - b} = x - y.$$

Thus we have $$\left.\begin{aligned} x + y &= a \\ \text{and}\quad x - y &= \sqrt{a^2 - b} \end{aligned}\right\}.$$

Hence, by addition and subtraction,

$$2x = a + \sqrt{a^2 - b}, \qquad \text{and } 2y = a - \sqrt{a^2 - b},$$

$$\therefore \quad x = \tfrac{1}{2}(a + \sqrt{a^2 - b}), \quad \text{and} \quad y = \tfrac{1}{2}(a - \sqrt{a^2 - b}).$$

$$\text{Thus } \sqrt{a + \sqrt{b}} = \sqrt{\tfrac{1}{2}(a + \sqrt{a^2 - b})} + \sqrt{\tfrac{1}{2}(a - \sqrt{a^2 - b})}.$$

Note From the values of x and y found above it is clear that unless $\sqrt{a^2 - b}$ is rational the square root obtained is by far more complicated than the original expression Thus the process given above is of no great practical value except when $a^2 - b$ is a perfect square

Cor. From (1), we have

$$a - \sqrt{b} = x + y - 2\sqrt{xy} = (\sqrt{x} - \sqrt{y})^2,$$

$$\therefore \quad \sqrt{a - \sqrt{b}} = \sqrt{x} - \sqrt{y}.$$

Thus if $\sqrt{a + \sqrt{b}} = \sqrt{x} + \sqrt{y}$, then will

$$\sqrt{a - \sqrt{b}} = \sqrt{x} - \sqrt{y}$$

Example 1. Find the square root of $7 + 2\sqrt{10}$.

Let $\sqrt{7 + 2\sqrt{10}} = \sqrt{x} + \sqrt{y}$

Then, squaring both sides,

$$7 + 2\sqrt{10} = x + y + 2\sqrt{xy}$$

Hence, $\left.\begin{aligned} x + y &= 7 \\ \text{and } xy &= 10 \end{aligned}\right\}$

These relations are evidently satisfied by the numbers 5 and 2

Hence, the required root $= \sqrt{5} + \sqrt{2}$

Example 2. Find the square root of $19 - 8\sqrt{3}$

Let $\sqrt{19 - 8\sqrt{3}} = \sqrt{x} - \sqrt{y}$

Then, $19 - 8\sqrt{3} = x + y - 2\sqrt{xy}$

Hence, $\left.\begin{aligned} x + y &= 19, \quad \ldots \quad \ldots & (1) \\ \text{and } 2\sqrt{xy} &= 8\sqrt{3}, \text{ or } xy = 48 & (2) \end{aligned}\right\}$

Now, (1) and (2) are obviously satisfied by the numbers 16 and 3.

Hence, the required root $= \sqrt{16} - \sqrt{3} = 4 - \sqrt{3}$.

Example 3 Find the square root of $16-5\sqrt{7}$.

Let $\sqrt{16-5\sqrt{7}} = \sqrt{x}-\sqrt{y}$

Then, $16-5\sqrt{7} = x+y-2\sqrt{xy}$

Therefore, $x+y = 16$
and $2\sqrt{xy} = 5\sqrt{7}$

Hence,
$$\begin{aligned}(x-y)^2 &= (x+y)^2-4xy \\ &= (16)^2-(5\sqrt{7})^2 \\ &= 256-175 = 81,\end{aligned}$$

$$x-y = 9.$$

Thus we have $x+y = 16$
and $x-y = 9$

Hence, $x = \frac{25}{2}$ and $y = \frac{7}{2}$

Thus the required root $= \sqrt{\frac{25}{2}}-\sqrt{\frac{7}{2}}$.

Example 4. Find the square root of $\sqrt{27}+\sqrt{15}$

$$\begin{aligned}\sqrt{27}+\sqrt{15} &= 3\sqrt{3}+\sqrt{3}\sqrt{5} \\ &= \sqrt{3}(3+\sqrt{5})\end{aligned}$$

Hence $\sqrt{\sqrt{27}+\sqrt{15}} = \sqrt[4]{3}\sqrt{3+\sqrt{5}}$

Now proceeding as in the last example, we find that

$$\sqrt{3+\sqrt{5}} = \sqrt{\tfrac{1}{2}}+\sqrt{\tfrac{5}{2}}$$

Therefore $\sqrt{\sqrt{27}+\sqrt{15}} = \sqrt[4]{3}.(\sqrt{\frac{1}{2}}+\sqrt{\frac{5}{2}})$

Example 5. Find the value of

$$\frac{1+x}{1+\sqrt{1+x}}+\frac{1-x}{1-\sqrt{1-x}}, \quad \text{when } x = \frac{\sqrt{3}}{2}.$$

We have

$$1+x = 1+\frac{\sqrt{3}}{2} = \frac{2+\sqrt{3}}{2} = \frac{4+2\sqrt{3}}{4} = \left(\frac{\sqrt{3}+1}{2}\right)^2,$$

and $1-x = 1-\dfrac{\sqrt{3}}{2} = \dfrac{2-\sqrt{3}}{2} = \dfrac{4-2\sqrt{3}}{4} = \left(\dfrac{\sqrt{3}-1}{2}\right)^2.$

Hence the given expression

$$=\frac{\frac{1}{2}(2+\sqrt{3})}{1+\frac{1}{2}(\sqrt{3}+1)}+\frac{\frac{1}{2}(2-\sqrt{3})}{1-\frac{1}{2}(\sqrt{3}-1)}$$

$$=\frac{2+\sqrt{3}}{3+\sqrt{3}}+\frac{2-\sqrt{3}}{3-\sqrt{3}}$$

$$=\frac{(2+\sqrt{3})(3-\sqrt{3})+(2-\sqrt{3})(3+\sqrt{3})}{(3+\sqrt{3})(3-\sqrt{3})}$$

$$=\frac{(6+\sqrt{3}-3)+(6-\sqrt{3}-3)}{9-3}=\frac{6}{6}=1.$$

Example 6. Find the value of $\dfrac{2a\sqrt{1+x^2}}{x+\sqrt{1+x^2}}$, when $x=\dfrac{1}{2}\left(\sqrt{\dfrac{a}{b}}-\sqrt{\dfrac{b}{a}}\right)$.

$$\frac{2a\sqrt{1+x^2}}{x+\sqrt{1+x^2}}=\frac{2a\sqrt{1+x^2}\{x-\sqrt{1+x^2}\}}{x^2-(1+x^2)}$$

$$=-2ax\sqrt{1+x^2}+2a(1+x^2)$$

Now since $x=\dfrac{1}{2}\left(\sqrt{\dfrac{a}{b}}-\sqrt{\dfrac{b}{a}}\right)$,

$$\therefore\ x^2=\frac{1}{4}\left(\frac{a}{b}+\frac{b}{a}-2\right),$$

$$\sqrt{1+x^2}=\sqrt{1+\frac{1}{4}\left(\frac{a}{b}+\frac{b}{a}-2\right)}$$

$$=\sqrt{\frac{1}{4}\left(\frac{a}{b}+\frac{b}{a}+2\right)}$$

$$=\frac{1}{2}\left(\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{a}}\right)$$

Hence, the required value

$$=-2a.\frac{1}{4}\left(\frac{a}{b}-\frac{b}{a}\right)+2a.\frac{1}{4}\left(\frac{a}{b}+\frac{b}{a}+2\right)$$

$$=2a\left(\frac{1}{2}+\frac{1}{2}.\frac{b}{a}\right)=a+b$$

Exercise (88).

Find the square root of —

1. $4-2\sqrt{3}$ 2 $7+4\sqrt{3}$. 3 $11-6\sqrt{2}$
4 $8+2\sqrt{15}$ 5 $14-6\sqrt{5}$. 6 $28+10\sqrt{3}$.
7. $21-8\sqrt{5}$. 8 $17+12\sqrt{2}$. 9 $41+12\sqrt{5}$.
10 $37-20\sqrt{3}$. 11 $31+4\sqrt{21}$. 12. $73-12\sqrt{35}$.
13 $47+4\sqrt{33}$ 14. $4-\sqrt{7}$. 15 $6-\sqrt{35}$
16 $\sqrt{28}-\sqrt{16}$. 17 $\sqrt{32}-\sqrt{24}$
18 $\sqrt{27}+\sqrt{24}$ 19 $5\sqrt{5}+\sqrt{120}$

20 Simplify $\dfrac{2+\sqrt{3}}{\sqrt{2}+\sqrt{2+\sqrt{3}}}+\dfrac{2-\sqrt{3}}{\sqrt{2}-\sqrt{2-\sqrt{3}}}$.

21. Find the value of

$$\frac{1+x}{1+\sqrt{1+x}}+\frac{1-x}{1+\sqrt{1-x}},\text{ when } x=\frac{\sqrt{3}}{2}.$$

22. Find the value of

$$\frac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}},\text{ when } x=\frac{2ab}{b^2+1}.$$

Find the square root of —

23. $a^2+2x\sqrt{a^2-x^2}$ 24 $2a+2\sqrt{a^2-b^2}$.
25 $a+x+\sqrt{2ax+x^2}$ 26 $2x-1+2\sqrt{x^2-x-6}$.
27. $x+y+z+2\sqrt{xz+yz}$

12 Equations involving surds.

Example 1. Solve $\sqrt{x+12}=\sqrt{x}+2$.

Squaring both sides, we have

$$x+12=x+4+4\sqrt{x}.$$

Hence, $4\sqrt{x}=8$,

or, $\sqrt{x}=2$, $\therefore\ x=4$.

Example 2. Solve $2(x+2)=1+\sqrt{4x^2+9x+14}$

(Calcutta University Entrance Paper, 1877)

By transposition, we have

$$2x+3 = = \sqrt{4x^2+9x+14}$$

Squaring both sides,

$$4x^2+12x+9 = 4x^2+9x+14,$$

$$\text{or, } 3x = 5, \quad \therefore \quad x = \tfrac{5}{3}$$

Example 3. Solve $\sqrt{x+6}+\sqrt{x-5} = 11.$

By transposition,

$$\sqrt{x+6} = 11-\sqrt{x-5}$$

Squaring both sides,

$$x+6 = 121-22\sqrt{x-5}+(x-5),$$

$$\therefore \quad 22\sqrt{x-5} = 110, \qquad \text{(by transposition.)}$$

$$\text{or,} \quad \sqrt{x-5} = 5,$$

$$\therefore \quad x-5 = 25, \quad \therefore \quad x = 30$$

Example 4. Solve $\sqrt{x^2+11x+20}-\sqrt{x^2+5x-1} = 3.$

(Calcutta University Entrance Paper, 1881.)

By transposition,

$$\sqrt{x^2+11x+20} = 3+\sqrt{x^2+5x-1}.$$

Squaring both sides,

$$x^2+11x+20 = 9+(x^2+5x-1)+6\sqrt{x^2+5x-1},$$

$$\text{or,} \quad 6x+12 = 6\sqrt{x^2+5x-1},$$

$$\text{or,} \quad x+2 = \sqrt{x^2+5x-1},$$

$$\therefore \quad x^2+4x+4 = x^2+5x-1, \quad \text{whence} \quad x = 5.$$

Example 5. Solve $\dfrac{3x-1}{\sqrt{3x}+1} = 1+\dfrac{\sqrt{3x}-1}{2}.$

Since $\quad 3x-1 = (\sqrt{3x}+1)(\sqrt{3x}-1),$

$$\therefore \quad \frac{3x-1}{\sqrt{3x}+1} = = \sqrt{3x}-1.$$

Hence, from the given equation, we have

$$\sqrt{3x}-1 = 1+\frac{\sqrt{3x}-1}{2}$$

or, $$(\sqrt{3x}-1)(1-\tfrac{1}{2}) = 1, \qquad \text{(by transposition)}$$

or, $$\frac{\sqrt{3x}-1}{2} = 1,$$

or, $$\sqrt{3x}-1 = 2,$$

or, $$\sqrt{3x} = 3,$$

$$3x = 9, \qquad x = 3$$

Example 6. Solve $$\sqrt[3]{a+x}+\sqrt[3]{a-x} = b$$

Since $$\left(\sqrt[3]{a+x}+\sqrt[3]{a-x}\right)^3$$

$$= \left(a+x\right)+\left(a-x\right)+3\sqrt[3]{a^2-x^2}\left\{\sqrt[3]{a+x}+\sqrt[3]{a-x}\right\}$$

$$= 2a+3\sqrt[3]{a^2-x^2}\,b,$$

therefore, cubing both sides of the equation, we have

$$2x+3\sqrt[3]{a^2-x^2}.b = b^3$$

or, $$3b\sqrt[3]{a^2-x^2} = b^3-2a,$$

$$\therefore \quad a^2-x^2 = \left(\frac{b^3-2a}{3b}\right)^3,$$

$$\therefore \quad x^2 = a^2-\left(\frac{b^3-2a}{3a}\right)^3,$$

$$\therefore \quad x = \sqrt{a^2-\left(\frac{b^3-2a}{3b}\right)^3}.$$

Example 7. Solve $$\frac{x-8}{\sqrt{x+1}-3}+\frac{x-26}{\sqrt{x-1}+5}$$

$$= \frac{4x-5}{\sqrt{4x-1}+2}$$

$$\frac{x-8}{\sqrt{x+1}-3} = \frac{(x-8)(\sqrt{x+1}+3)}{(x+1)-9} = \sqrt{x+1}+3;$$

$$\frac{x-26}{\sqrt{x-1}+5} = \frac{(x-26)(\sqrt{x-1}-5)}{(x-1)-4} = \sqrt{x-1}-5,$$

$$\frac{4x-5}{\sqrt{4x-1}+2} = \frac{(4x-5)(\sqrt{4x-1}-2)}{(4x-1)-4} = \sqrt{4x-1}-2$$

Hence, from the given equation, we have

$$(\sqrt{x+1}+3)+(\sqrt{x-1}-5) = \sqrt{4x-1}-2,$$

or, $$\sqrt{x+1}+\sqrt{x-1} = \sqrt{4x-1},$$

$$\therefore\ (x+1)+(x-1)+2\sqrt{x^2-1} = 4x-1,$$

or, $$2\sqrt{x^2-1} = 2x-1,$$

or, $$4(x^2-1) = 4x^2-4x+1,$$

or, $$4x = 5,\quad \therefore\ x = \tfrac{5}{4}$$

Example 8 Solve $\sqrt{2x^2+9}+\sqrt{2x^2-9} = 9+3\sqrt{7}$.

We have, for *all* values of x,

$$(2x^2+9)-(2x^2-9) = 18,$$

and hence this relation is also true for the particular value which x has in the given equation

Therefore the required value of x will also satisfy the equation

$$\frac{(2x^2+9)-(2x^2-9)}{\sqrt{2x^2+9}+\sqrt{2x^2-9}} = \frac{18}{9+3\sqrt{7}}$$

or, $$\sqrt{2x^2+9}-\sqrt{2x^2-9} = \frac{18(9-3\sqrt{7})}{81-63} = 9-3\sqrt{7}$$

Adding together the given equation and this, we have

$$2\sqrt{2x^2+9} = 18,$$

or, $$\sqrt{2x^2+9} = 9,$$

$$\therefore\quad 2x^2+9 = 81,$$

$$\therefore\quad x^2 = 36,\quad \therefore\ x = 6.$$

Exercise (89).

Solve the following equations —

1. $\sqrt{x+7} = 1+\sqrt{x}$ 2. $\sqrt{3x+16} = \sqrt{3x}+2$.

3. $\sqrt{x+9} = 1+\sqrt{x}$. 4. $\sqrt{3x}-4 = \sqrt{3x+4}$.

5. $\sqrt{5x+10} = \sqrt{5x}+2$ 6. $\sqrt{x-16}+\sqrt{x} = 8$.

7. $\sqrt{2x+9}+\sqrt{2x} = 9$ 8. $\sqrt{x+11}-\sqrt{x} = 1$.

9. $\sqrt{8x+33}-3 = 2\sqrt{2x}$ 10. $x+\sqrt{2ax+x^2} = a$

11. $x+a+\sqrt{2ax+x^2} = b$ 12. $\sqrt{x-4}+3 = \sqrt{x+11}$

13. $\sqrt{x-5} = 6-\sqrt{x+7}$ 14. $\sqrt{x+9}-\sqrt{x+2} = 1$.

15. $\sqrt{3x+1}-\sqrt{3x-11} = 2$.

16. $\sqrt{5x+6}+\sqrt{5x-14} = 10$.

17. $\sqrt{7x+4}+\sqrt{7x-12} = 8$.

18. $\sqrt{x^2-3x+5}-\sqrt{x^2-x+1} = 1$

19. $\dfrac{x-1}{\sqrt{x}+1} = 4+\dfrac{\sqrt{x}-1}{2}$ 20. $\dfrac{ax-1}{\sqrt{ax}+1} = 4+\dfrac{\sqrt{ax}-1}{2}$.

(C U Entrance Paper, 1885)

21. $\dfrac{ax-b^2}{\sqrt{ax}+b} = c+\dfrac{\sqrt{ax}-b}{c}$

22. $\dfrac{200+120\sqrt{5x}}{9x-5} = \left(3\sqrt{x}-\sqrt{5}\right)^2$.

23. $\sqrt{4a+x}-\sqrt{a+x} = 2\sqrt{x-2a}$.

24. $\sqrt{x}+\sqrt{a+x} = \dfrac{3a}{\sqrt{a+x}}$.

25. $\sqrt{x}+\sqrt{x+13} = \dfrac{91}{\sqrt{x+13}}$.

26. $\sqrt{x+a}+\sqrt{x-a} = \dfrac{b}{\sqrt{x+a}}$

27. $\dfrac{3\sqrt{x}-4}{\sqrt{x}+2}=\dfrac{15+3\sqrt{x}}{\sqrt{x}+40}$.

28. $\sqrt{x}+\sqrt{x-\sqrt{1-x}}=1$.

29. $\sqrt{x}+\sqrt{8-\sqrt{x^2+8x}}=2\sqrt{2}$.

30. $\sqrt{1-x}+\sqrt{1-x+\sqrt{1+x}}=\sqrt{1+x}$.

31. $\dfrac{1}{a}\sqrt{a+x}+\dfrac{1}{x}\sqrt{a+x}=\dfrac{1}{b}\sqrt{x}$.

32. $\sqrt[5]{x+8}=\sqrt[10]{x^2+64x+36}$.

33. $\left(1+x\right)^{\frac{1}{3}}+\left(1-x\right)^{\frac{1}{3}}=2^{\frac{1}{3}}$. (C U Entr Paper, 1885)

34. $\left(a+x\right)^{\frac{2}{3}}+\left(a-x\right)^{\frac{2}{3}}=3\left(a^2-x^2\right)^{\frac{1}{3}}$.

35. $\left(\dfrac{x}{a}+\dfrac{a}{b}\right)^{\frac{1}{2}}+9\left(\dfrac{x}{a}-\dfrac{a}{b}\right)^{\frac{1}{2}}=6\left(\dfrac{x^2}{a^2}-\dfrac{a^2}{b^2}\right)^{\frac{1}{4}}$.

36. $\dfrac{x-47}{\sqrt{x+2}-7}+\dfrac{x-19}{\sqrt{x-3}-4}=\dfrac{4x-124}{\sqrt{4x-3}-11}$.

37. $\dfrac{2x-49}{\sqrt{2x+15}-8}+\dfrac{18x+22}{\sqrt{18x+31}+3}=\dfrac{8x+191}{2\sqrt{2x+54}-5}$.

38. $x=\sqrt{a^2+x\sqrt{b^2+x^2}}-a$.

39. $\sqrt{x^2+9}+\sqrt{x^2-9}=4+\sqrt{34}$.

40. $\sqrt{3x^2+16}-\sqrt{3x^2-16}=8-4\sqrt{2}$.

CHAPTER XVIII

SQUARE AND CUBE ROOTS.

1. The ordinary method of finding the square root of a compound algebraical expression. From our previous knowledge of formulæ the following results are obvious —

$$(a+b)^2 = a^2+(2a+b)b\ ,$$
$$(a+b+c)^2 = a^2+(2a+b)b+(2a+2b+c)c\ ;$$
$$(a+b+c+d)^2 = a^2+(2a+b)a+(2a+2b+c)c$$
$$+(2a+2b+2c+d)d\ ;$$

and so on.

Clearly therefore we must have

$$(ax^2+bx+c)^2 = a^2x^4+(2ax^2+bx)bx+(2ax^2+2bx+c)c,$$

and this latter, when arranged according to the descending powers of $x = a^2x^4+2abx^3+(b^2+2ac)x^2+2bcx+c^2$.

Now if it is proposed to find the square root of the above expression, let us see what means we have of discovering successively the several terms of the root —

The first term of the root, *viz*, ax^2, is evidently the square root of the first term of the given expression which is a^2x^4 ;

if we subtract a^2x^4 from the given expression the remainder is $\{(2ax^2+bx)bx+(2ax^2+2bx+c)c\}$, in which the term containing the highest power of $x = 2ax^2\times bx$, *i e*, = twice the first term of the root into the second term , this enables us to get the second term after having obtained the first ,

if now from the above remainder we subtract $(2ax^2+bx)bx$ the second remainder is $(2ax^2+2bx+c)c$, in which the term containing the highest power of $x = 2ax^2\times c$, *i e*, = twice the first term of the root *into* the third , this shows how to get the 3rd term after having obtained the 1st and 2nd.

Thus we are furnished with a clue for successively discovering the terms of the expression ax^2+bx+c when its square is given.

The operation may be performed as follows —

$$
\begin{array}{r|l}
 & a^2x^4+2abx^3+(b^2+2ac)x^2+2bcx+c^2 \;\big(\, ax^2+bx+c \\
 & a^2x^4 \\
\hline
2ax^2+bx & 2abx^3+(b^2+2ac)x^2+2bcx+c^2 \\
 & 2abx^3+b^2x^2 \\
\hline
2ax^2+2bx+c & 2acx^2+2bcx+c^2 \\
 & 2acx^2+2bcx+c^2 \\
\hline
\end{array}
$$

(1) Find the square root of a^2x^4, the first term of the proposed expression and set it down as the first term of the the required root ;

(2) Subtract a^2x^4 from the given expression and bring down the remainder $2abx^3+(b^2+2ac)x^2+2bcx+c^2$;

(3) Set down $2ax^2$, *i.e.*, twice the 1st term of the root, on the left of the above remainder as the first term of a divisor ;

(4) Divide the first term of the remainder by $2ax^2$ and set down the quotient, bx, as the second term of the root and also as the second term of the divisor ;

(5) Multiply the divisor thus obtained by the second term of the root and subtract the product from the first remainder ;

(6) Bring down the second remainder $2acx^2+2bcx+c^2$ and put $2ax^2+2bx$ (*i.e.*, twice the sum of the two terms of the root already obtained) on the left of this remainder for the first two terms of a divisor ;

(7) Divide the first term of the new remainder by the first term of the new divisor and set down the quotient, c, as the third term of the root and also as the third term of the divisor ;

(8) Multiply the complete divisor thus obtained by the third term of the root and subtract the product from the second remainder.

After this nothing remains, and we obtain ax^2+bx+c for the required root.

Note. The expression considered above stands arranged according to descending powers of x. Similarly, every expression of which the square root is sought must be arranged according to descending or ascending order of the powers of some letter.

Example 1. Extract the square root of

$$x^6+8x^4-2x^3+16x^2-8x+1.$$

$$\begin{array}{r|l}
 & x^6+8x^4-2x^3+16x^2-8x+1\,(x^3+4x-1 \\
 & x^6 \\
\hline
2x^3+4x\,) & 8x^4-2x^3+16x^2-8x+1 \\
 & 8x^4 \qquad\quad +16x^2 \\
\hline
2x^3+8x-1\,) & \quad -2x^3 \qquad\quad -8x+1 \\
 & \quad -2x^3 \qquad\quad -8x+1 \\
\hline
\end{array}$$

Thus the required root $= x^3+4x-1$

Example 2. Extract the square root of

$$x^4+2(y+z)x^3+(3y^2+2yz+3z^2)x^2+2(y^3+y^2z+yz^2+z^3)x$$
$$+y^4+2y^2z^2+z^4.$$

(Calcutta University Entrance Paper, 1888)

The given expression

$$= x^4+2(y+z)x^3+(3y^2+2yz+3z^2)x^2$$
$$+2(y+z)(y^2+z^2)x+(y^2+z^2)^2,$$

which stands arranged according to descending powers of x, so we can at once proceed thus —

$$\begin{array}{r|l}
 & \qquad\qquad +2(y+z)(y^2+z^2)x+(y^2+z^2)^2\,(x^2+(y+z)x \\
 & x^4+2(y+z)x^3+(3y^2+2yz+3z^2)x^2 \qquad\qquad +(y^2+z^2) \\
 & x^4 \\
\hline
2x^2+(y+z)x\,) & 2(y+z)x^3+(3y^2+2yz+3z^2)x^2 \\
 & 2(y+z)x^3+(y^2+2yz+z^2)x^2 \\
\hline
2x^2+2(y+z)x+(y^2+z^2)\,) & 2(y^2+z^2)x^2+2(y+z)(y^2+z^2)x+(y^2+z^2)^2 \\
 & 2(y^2+z^2)x^2+2(y+z)(y^2+z^2)x+(y^2+z^2)^2 \\
\hline
\end{array}$$

Thus the required root $= x^2+xy+xz+y^2+z^2$

Example 3 Find the square root of

$$\frac{x^4}{4}+4x^2+\frac{ax^2}{3}+\frac{a^2}{9}-2x^3-\frac{4ax}{3}.$$

(C U Entr Paper, 1889)

Arrange the expression according to descending powers of x and then proceed thus :—

$$\frac{x^4}{4}-2x^3+4x^2+\frac{ax^2}{3}-\frac{4ax}{3}+\frac{a}{9}\left(\frac{x^2}{2}-2x+\frac{a}{3}\right.$$

$$\frac{x^4}{4}$$

$$x^2-2x\,\Big)\;\begin{array}{l}-2x^3+4x^2\\-2x^3+4x^2\end{array}$$

$$x^3-4x+\frac{a}{3}\,\Bigg)\;\begin{array}{l}\dfrac{ax^2}{3}-\dfrac{4ax}{3}+\dfrac{a^2}{9}\\[2ex]\dfrac{ax^2}{3}-\dfrac{4ax}{3}+\dfrac{a^2}{9}\end{array}$$

Thus the required root $=\dfrac{x^4}{2}-2x+\dfrac{a}{3}$

Example 4 Extract the square root of

$$\frac{x^4}{4y^4}+\frac{4y^4}{x^4}+\frac{x^2}{y^2}+\frac{4y^2}{x^2}+3$$

The expression when arranged according to descending powers of x stands thus —

$$\frac{x^4}{4y^4}+\frac{x^2}{y^2}+3+\frac{4y^2}{x^2}+\frac{4y^4}{x^4},$$

for *now* the indices of the powers of x in the successive terms are respectively 4, 2, 0, -2 and -4, which numbers evidently are in descending order of magnitude. Hence we proceed as follows —

$$\frac{x^4}{4y^4}+\frac{x^2}{y^2}+3+\frac{4y^2}{x^2}+\frac{4y^4}{x^4}\left(\frac{x^2}{2y^2}+1+\frac{2y^2}{x^2}\right.$$

$$\frac{x^4}{4y^4}$$

$$\frac{x^2}{y^2}+1\,\Bigg)\;\begin{array}{l}\dfrac{x^2}{y^2}+3\\[2ex]\dfrac{x^2}{y^2}+1\end{array}$$

$$\frac{x^2}{y^2}+2+\frac{2y^2}{x^2}\,\Bigg)\;\begin{array}{l}2+\dfrac{4y^2}{x^2}+\dfrac{4y^4}{x^4}\\[2ex]2+\dfrac{4y^2}{x^2}+\dfrac{4y^4}{x^4}\end{array}$$

Thus the required root $= \dfrac{x^2}{2y^2} + 1 + \dfrac{2y^2}{x^2}$.

Example 5 Extract the square root of

$$x^{\frac{8}{5}} - 2a^{-\frac{3}{5}}x^{\frac{11}{5}} + 2a^{\frac{4}{5}}x^{\frac{4}{5}} + a^{-\frac{6}{5}}x^{\frac{14}{5}} - 2a^{\frac{1}{5}}x^{\frac{7}{5}} + a^{\frac{8}{5}}$$

(Calcutta University Entrance Paper, 1880)

Let us proceed by arranging the expression according to descending powers of x, thus —

$$\begin{array}{l}
\qquad a^{-\frac{6}{5}}x^{\frac{14}{5}} - 2a^{-\frac{3}{5}}x^{\frac{11}{5}} + x^{\frac{8}{5}} - 2a^{\frac{1}{5}}x^{\frac{7}{5}} + 2a^{\frac{4}{5}}x^{\frac{4}{5}} + a^{\frac{8}{5}}\left(a^{-\frac{3}{5}}x^{\frac{7}{5}} - x^{\frac{4}{5}} - a^{\frac{4}{5}}\right. \\
\qquad a^{-\frac{6}{5}}x^{\frac{14}{5}} \\
\hline
2a^{-\frac{3}{5}}x^{\frac{7}{5}} - x^{\frac{4}{5}}\Big) -2a^{-\frac{3}{5}}x^{\frac{11}{5}} + x^{\frac{8}{5}} \\
\qquad\qquad\qquad -2a^{-\frac{3}{5}}x^{\frac{11}{5}} + x^{\frac{8}{5}} \\
\hline
\qquad 2a^{-\frac{3}{5}}x^{\frac{7}{5}} - 2x^{\frac{4}{5}} - a^{\frac{4}{5}}\Big) -2a^{\frac{1}{5}}x^{\frac{7}{5}} + 2a^{\frac{4}{5}}x^{\frac{4}{5}} - a^{\frac{8}{5}} \\
\qquad\qquad\qquad\qquad\qquad\qquad -2a^{\frac{1}{5}}x^{\frac{7}{5}} + 2a^{\frac{4}{5}}x^{\frac{4}{5}} + x^{\frac{8}{5}} \\
\hline
\end{array}$$

Thus the required root $= a^{-\frac{3}{5}}x^{\frac{7}{5}} - x^{\frac{4}{5}} - a^{\frac{4}{5}}$.

Exercise (90).

Find the square root of —

1. $4x^2z^2 + 12xyz + 9y^2$
2. $x^4 - 4x^3 + 10x^2 - 12x + 9$.
3. $x^6 - 2x^4 + 2x^3 + x^2 - 2x + 1$
4. $4x^4 - 12x^3 + 25x^2 - 24x + 16$
5. $4x^4 + 8ax^3 + 4a^2x^2 + 16b^2x^2 + 16ab^2x + 16b^4$

(Calcutta University Entrance Paper, 1870)

6. $9x^4 - 2x^3y + \frac{163}{9}x^2y^2 - 2xy^3 + 9y^4$

(C U Entr Paper, 1874)

7. $x^4 - 2x^3 + \dfrac{3x^2}{2} - \dfrac{x}{4} + \dfrac{1}{16}$.

8. $\frac{1051x^2}{25}-\frac{6x}{5}-\frac{14x^3}{5}+49x^4+9$

9. $x^4+\frac{4}{x^2}-2+4x-x^3+\frac{x^2}{4}$.

10. $\frac{a^2}{x^2}+\frac{x^2}{a^2}+\frac{a^4}{4}+\frac{a^3}{x}-2-ax$.

11. $\frac{a^2}{4b^2}-\frac{a}{b}+\frac{4b^2}{a^2}-1+\frac{4b}{a}$.

12. $\frac{9a^2}{x^2}-\frac{6a}{5x}+\frac{101}{25}-\frac{4x}{15a}+\frac{4x^2}{9a^2}$.

13. $4x^4-8x^3y^2+4xy^6+y^8$.

14. $\frac{49x^2}{y^2}+\frac{y^2}{49x^2}-\frac{42x}{y}+\frac{6y}{7x}+7$

15. $\frac{x^2}{y^2}+\frac{y^2}{x^2}-\frac{x}{y}+\frac{y}{x}-1\frac{3}{4}$.

16. $25\frac{3}{4}-\frac{20x}{7y}+\frac{9y^2}{16x^2}-\frac{15y}{2x}+\frac{4x^2}{49y^2}$

17. $x^2-2x^{\frac{3}{2}}+3x-2x^{\frac{1}{2}}+1$.

18. $x^{\frac{5}{3}}-4x^{\frac{4}{3}}+2x+4x^{\frac{2}{3}}+x^{\frac{1}{3}}$.

19. $a^2x^{-2}+2ax^{-1}+a^{-2}x^2+3+2a^{-1}x$.

20. $x^{\frac{3}{2}}+xy^{-\frac{1}{2}}-2x^{\frac{5}{4}}y^{-\frac{1}{4}}-2x^{\frac{1}{2}}y^{\frac{1}{4}}+2x^{\frac{3}{4}}y^{\frac{1}{2}}+y$.

21. $\frac{9x^3}{4}-5x^{\frac{5}{2}}y^{\frac{1}{2}}+\frac{179x^2y}{45}-\frac{4x^{\frac{3}{2}}y^{\frac{3}{2}}}{3}+\frac{4xy^2}{25}$.

22. $a^{2m}-4a^{m+n}+4a^{2n}$.

23. $9a^{2m}+6a^{3m+1}+25c^{2n-4}-30a^mc^{n-2}$
$+a^{4m+2}-10a^{2m+1}c^{n-2}$.

2. Extraction of square roots by the application of the formula $a^2 \pm 2ab + b^2 = (a \pm b)^2$

Example 1 Find the square root of

$$4-4c+2b+c^2-bc+\frac{b^2}{4}.$$

(Calcutta University Entrance Paper, 1876.)

The given expression, arranged according to powers of b,

$$=\frac{b^2}{4}-b(c-2)+(c^2-4c+4)$$

$$=\left(\frac{b}{2}\right)^2-2\left\{\frac{b}{2}(c-2)\right\}+(c-2)^2$$

$$=\left\{\frac{b}{2}-(c-2)\right\}^2=\left(\frac{b}{2}-c+2\right)^2.$$

Therefore the required root $=\frac{b}{2}-c+2$.

Example 2. Extract the square root of

$$\left(x^2+\frac{1}{x^2}\right)^2-4\left(x+\frac{1}{x}\right)^2+12$$

The given expression

$$=\left(x^2+\frac{1}{x^2}\right)^2-4\left(x^2+\frac{1}{x^2}+2\right)+12$$

$$=\left(x^2+\frac{1}{x^2}\right)^2-4\left(x^2+\frac{1}{x^2}\right)+4$$

$$=\left(x^2-2+\frac{1}{x^2}\right)^2.$$

Therefore the required root $=x^2-2+\frac{1}{x^2}$

Example 3 Extract the square root of

$$\frac{(a^2+b^2)^2}{a^4+b^4-2a^2b^2}+4\frac{a}{a+b}\times\frac{b}{a-b}.$$

(Calcutta University Entrance Paper, 1886.)

The given expression

$$=\frac{(a^2+b^2)^2}{(a^2-b^2)^2}+\frac{4ab}{a^2-b^2}=\frac{(a^2+b^2)^2+4ab(a^2-b^2)}{(a^2-b^2)^2}$$

of which the numerator $=\{(a^2-b^2)^2+4a^2b^2\}+4ab(a^2-b^2)$

$$=(a^2-b^2)^2+4ab(a^2-b^2)+4a^2b^2$$

$$=\{(a^2-b^2)+2ab\}^2,$$

$\therefore$ the given expression $=\dfrac{(a^2+2ab-b^2)^2}{(a^2-b^2)^2}$.

Therefore the required root $=\dfrac{a^2+2ab-b^2}{a^2-b^2}$.

Example 4. Extract the square root of

$$(ab+ac+bc)^2-4ac(a+c).$$

(Calcutta University Entrance Paper, 1888)

The given expression

$$=\{b(a+c)+ac\}^2-4abc(a+c)$$

$$=b^2(a+c)^2+a^2c^2-2abc(a+c)$$

$$=\{b(a+c)-ac\}^2=(ab-ac+bc)^2.$$

Therefore the required root $=ab-ac+bc$

Example 5. Extract the square root of

$$a^4+b^4+c^4+d^4-2(a^2+c^2)(b^2+d^2)+2a^2c^2+2b^2d^2.$$

Arranging the given expression according to descending powers of a, we have

$$a^4-2a^2(b^2+d^2-c^2)+\{b^4+c^4+d^4-2c^2(b^2+d^2)+2b^2d^2\};$$

and the expression within the braces arranged according to descending powers of b,

$$=b^4-2b^2(c^2-d^2)+(c^4+d^4-2c^2d^2)$$

$$=b^4-2b^2(c^2-d^2)+(c^2-d^2)^2$$

$$=\{b^2-(c^2-d^2)\}^2$$

Hence, the given expression

$$=a^4-2a^2(b^2-c^2+d^2)+(b^2-c^2+d^2)^2$$

$$=\{a^2-(b^2-c^2+d^2)\}^2$$

$$=(a^2-b^2+c^2-d^2)^2.$$

Therefore the required root $=a^2-b^2+c^2-d^2$.

Example 6. Find the square root of

$$4\{(a^2-b^2)cd+ab(c^2-d^2)\}^2+\{(a^2-b^2)(c^2-d^2)-4abcd\}^2.$$

The given expression

$$=4\{(a^2-b^2)^2c^2d^2+2abcd(a^2-b^2)(c^2-d^2)+a^2b^2(c^2-d^2)^2\}$$
$$+\{(a^2-b^2)^2(c^2-d^2)^2-8abcd(a^2-b^2)(c^2-d^2)+16a^2b^2c^2d^2\}$$
$$=4(a^2-b^2)^2c^2d^2+4a^2b^2(c^2-d^2)^2\}+\{(a^2-b^2)^2(c^2-d^2)^2+16a^2b^2c^2d^2\}$$
$$=(a^2-b^2)^2\{(c^2-d^2)^2+4c^2d^2\}+4a^2b^2\{(c^2-d^2)^2+4c^2d^2\}$$
$$=\{(a^2-b^2)^2+4a^2b^2\}\{c^2-d^2)^2+4c^2d^2\}$$
$$=(a^4+2a^2b^2+b^4)(c^4+2c^2d^2+d^4)$$
$$=(a^2+b^2)^2(c^2+d^2)^2.$$

Therefore the required root $=(a^2+b^2)(c^2+d^2)$.

Exercise (91).

Find the square root of —

1. $25x^2y^2-40xy+16$.
2. $49a^2x^4-42ab^2x^2+9b^4$.
3. $49a^6b^8+126a^7b^7+81a^8b^6$
4. $\frac{1}{4}x^8y^4-\frac{1}{5}x^7y^7+\frac{1}{25}x^6y^{10}$.
5. $\frac{25a^2b^2}{4}+\frac{c^4}{9}-\frac{5abc^2}{3}$
6. $a^2+b^2+c^2+2ab+2ac+2bc$
7. $a^2+b^2+c^2-2ab+2ac-2bc$
8. $4a^2+b^2+9c^2+6bc-12ac-4ab$
9. $a^4+4b^4+9c^4+4a^2b^2-6a^2c^2-12b^2c^2$
10. $4a^4+9b^4+25c^4-12a^2b^2+20a^2c^2-30b^2c^2$
11. $x^2+\frac{a^2}{9}-bx+\frac{b^2}{4}-\frac{ab}{3}+\frac{2ax}{3}$
12. $\left(x+\frac{1}{x}\right)^2-4\left(x-\frac{1}{x}\right)$.

13. $x^4+\frac{1}{x^4}+2\left(x^2+\frac{1}{x^2}\right)+3.$

14. $\frac{a^2}{b^2}+\frac{b^2}{a^2}+\frac{2a}{b}+\frac{2b}{a}+3$

15. $\frac{x^2}{y^2}+\frac{y^2}{x^2}-\left(\frac{x}{y}+\frac{y}{x}\right)\sqrt{2}+2\frac{1}{2}$

16. $\frac{9x^2}{a^2}+\frac{a^2}{9x^2}-6\frac{x}{a}-\frac{2a}{3x}+3$

17. $x^2+\frac{1}{x^2}+4\left(x+\frac{1}{x}\right)+6.$

18. $-2+a^{2\sqrt{2}}+a^{-2\sqrt{2}}.$

19. $a^2+b^2+c^2+d^2-2a(b-c+d)-2b(c-d)-2cd$

20. $(a-b)^4-2(a^2+b^2)(a-b)^2+2(a^4+b^4)$

21. $a^4+b^4+c^4+d^4-2a^2(b^2+d^2)-2b^2(c^2-d^2)$
$+2c^2(a^2-d^2).$

22. $a^4+2a^3-a+\frac{1}{4}$

23. $2a^2(b+c)^2+2b^2(c+a)^2+2c^2(a+b)^2+4abc(a+b+c)$

3. The ordinary method of finding the cube root of a compound algebraical expression.

Evidently we have $(ax^2+bx+c)^3$

$$= (ax^2+bx)^3+3(ax^2+bx)^2c+3(ax^2+bx)c^2+c^3$$
$$= a^3x^6+3(a^2x^4)(bx)+3(ax^2)(bx)^2+(bx)^3$$
$$+3(ax^2+bx)^2c+3(ax^2+bx)c^2+c^3.$$

Hence if we are asked to find the cube root of the above expression we see that we have the following means of discovering successively the several terms of the root —

The first term of the root, *viz*, ax^2, is evidently the cube root of the first term of the given expression, which is a^3x^6

If we subtract a^3x^6 from the given expression the term containing the highest power of x in the remainder is $3(a^2x^4)(bx)$, *i e*, equal to three times the square of the first term of the root *into* the second term, the second term is therfore discovered

If from the above remainder we now subtract $\{3(a^2x^4)+3(ax^2)(bx)+(bx)^2\}(bx)$, the second remainder is $3(ax^2+bx)^2c+3(ax^2+bx)c^2+c^3$, the term containing the highest power of x in this remainder is $3a^2x^4c$, *i e*, equal to three times the square of the first term of the root *into* the third.

Hence, the third term is discovered

If from the second remainder we now subtract $\{3(ax^2+bx)^2+3(ax^2+bx)c+c^2\}c$, nothing is left and we obtain the required root $= ax^2+bx+c$

Let us illustrate the process by an example.

Example. Find the cube root of

$$x^6-6x^5y+24x^4y^2-56x^3y^3+96x^2y^4-96xy^5+64x^6.$$

The given expression stands arranged according to descending powers of x, we need not therefore change the order of the terms

The second term of the root, *viz*, $-2xy$, as shewn on the next page, is obtained by dividing $-6x^5y$ by $3x^4$(*i e*, three times the square of the first term)

Then the divisor, $3x^4-6x^3y+4x^2y^2$, is formed as shewn on the next page

The product of this divisor by $(-2xy)$, *viz*, $-6x^5y+12x^4y^2-8x^3y^3$, is now subtracted from the expression which stands above it and the remainder is put down below the line.

Now take three times the square of the part of the root already obtained and put down the result, $3x^4-12x^3y+12x^2y^2$, as part of a divisor

The third term of the root, *viz*, $4y^2$ is obtained by dividing $12x^4y^2$, the first term of the remainder, by $3x^4$, the first term of the divisor

The complete divisor is then formed as shewn on the next page, and the product of this divisor by the third term of the root is subtracted from the expression which stands above it

As no remainder is now left we find the required root $= x^2-2xy+4y^2$.

$$
\begin{array}{r|l}
 & x^6 - 6x^5y + 24x^4y^2 - 56x^3y^3 + 96x^2y^4 - 96xy^5 + 64y^6 \,\big(\, x^2 - 2xy + 4y^2 \\
 & x^6 \\
\hline
3 \times (x^2)^2 = 3x^4 \qquad\qquad & -6x^5y + 24x^4y^2 - 56x^3y^3 + 96x^2y^4 - 96xy^5 + 64y^6 \\
3 \times x^2 \times (-2xy) = \qquad -6x^3y \qquad & \\
(-2xy)^2 = \qquad\qquad + 4x^2y^2 & \\
\hline
3x^4 - 6x^3y + 4x^2y^2 & -6x^5y + 12x^4y^2 - 8x^3y^3 \\
\hline
3 \times (x^2 - 2xy)^2 = 3x^4 - 12x^3y + 12x^2y^2 \qquad\qquad & 12x^4y^2 - 48x^3y^3 + 96x^2y^4 - 96xy^5 + 64y^6 \\
3 \times (x^2 - 2xy) \times (4y^2) = \qquad\qquad + 12x^2y^2 - 24xy^3 \qquad & \\
(4y^2)^2 = \qquad\qquad\qquad\qquad + 16y^4 & \\
\hline
3x^4 - 12x^3y + 24x^2y^2 - 24xy^3 + 16y^4 & 12x^4y^2 - 48x^3y^3 + 96x^2y^4 - 96xy^5 + 64y^6 \\
\hline
\end{array}
$$

Exercise (92).

Find the cube root of —

1. $x^3+27x^2+243x+729$
2. $27x^3-216x^2+576x-512$
3. $64a^3-144a^2b+108ab^2-27b^3$
4. $33x^4-36x+x^6-63x^3+8-9x^5+66x^2$
5. $8x^6+12x^5-30x^4-34x^3+45x^2+27x-27$
6. $1-9x^2+33x-63x^6+66x^8-36x^{10}+8x^{12}$
7. $c^6-63c^3x^3+8x^6-9c^5x+66c^2x^4-36cx^5+33c^4x^2$.

CHAPTER XIX

RATIO AND PROPORTION

1. Definitions. The ratio of one quantity to another of the same kind is defined to be the *abstract number* (integral or fractional) which expresses what multiple, part or parts, the former is of the latter. Thus,

since 2 hours is a portion of time which is three times as large as 40 minutes, the ratio of 2 hours to 40 minutes $= 3$;

since a length of 9 inches is a fourth part of 3 feet, the ratio of 9 inches to 3 feet $= \frac{1}{4}$;

since the sum of £1 4*s* is obtained by dividing 18*s.* into 3 equal parts and taking 4 of those parts, the ratio of £1 4*s* to 18*s* $= \frac{4}{3}$,

and so on.

Hence it is clear that the ratio of one *concrete* quantity to another (of the same kind) is a fraction, of which the numerator and denominator are respectivily the *measures* of those quantities (*referred to one and the same unit*); and the ratio of one *abstract* quantity to another is a fraction, of which the numerator and denominator are respectively the quantities themselves.

The ratio of any number a to any other number b is usually expressed by the notation $a:b$, thus $a:b$ is the same as $\frac{a}{b}$. The quantities a and b are respectively called the **Antecedent** and the **Consequent** (or the *first term* and the *second term*) of the ratio $a:b$.

A ratio is called a ratio of *greater inequality*, of *less inequality* or of *equality* according as it is *greater* than, *less* than or *equal* to 1.

Note. Since a ratio is only a fraction, there is no difficulty in seeing that the value of a ratio remains unaltered if its terms be multiplied or divided by the same number. Thus the ratios $3:4$, $6:8$, $15:20$, and $3n:4n$ are equal to one another. Hence also two or more ratios can be easily compared with one another, for instance the ratios $2:3$, $4:5$ and $7:10$ being respectively equivalent to $20:30$, $24:30$, and $21:30$, we see at once that the second of them is the greatest and the first the least.

2. A ratio of less inequality is increased, and a ratio of greater inequality is diminished, by adding the same number to both its terms.

Let $\frac{a}{b}$ be any given ratio, and let $\frac{a+x}{b+x}$ be the new ratio formed by adding x to both its terms.

Then, $\frac{a+x}{b+x}-\frac{a}{b}=\frac{x(b-a)}{b(b+x)}$,

and therefore it is positive or negative according as a is less or greater than b.

Hence, if $a<b$, $\frac{a+x}{b+x}>\frac{a}{b}$,

and if $a>b$, $\frac{a+x}{b+x}<\frac{a}{b}$,

which proves the proposition.

Note. Similarly it can be proved that *a ratio of less inequality is diminished, and a ratio of greater inequality is increased by subtracting from both its terms any number which is less than each of those terms.* This is left as an exercise for the student.

3. Composition of Ratios. The ratio of the product of the antecedents of any number of ratios to the product of their consequents is called the ratio *compounded* of the given ratios.

Thus, the ratio compounded of the ratios

$$3 : 4, \qquad 8 : 9, \qquad 2x : 3y$$

is

$$3 \times 8 \times 2x : 4 \times 9 \times 3y$$

$$\text{or, } 4x : 9y$$

When the ratio $a : b$ is compounded with itself the resulting ratio $a^2 : b^2$ is called the *duplicate* ratio of $a : b$. Similarly, $a^3 : b^3$ is called the *triplicate* ratio of $a : b$; $a^{\frac{1}{2}} : b^{\frac{1}{2}}$ is called the *subduplicate* ratio of $a : b$, and $a^{\frac{1}{3}} : b^{\frac{1}{3}}$ is called the *subtriplicate* ratio of $a : b$.

4. Approximate values of Ratios. If x is *very small* comrared with a, to shew that the ratio $(a+x)^2 : a^2$ is approximately the same as $a+2x : a$

$$\text{We have } \frac{(a+x)^2}{a^2} = \frac{a^2+2ax+x^2}{a^2} = 1+\frac{2x}{a}+\frac{x^2}{a^2}$$

$$\text{and } \therefore \textit{ approximately } = 1+\frac{2x}{a},$$

since $\frac{x^2}{a^2}\left(\text{which} = \frac{x}{a}\times\frac{x}{a}\right)$ is very small compared with $\frac{2x}{a}$ and smaller still than 1.

Thus approximately we have

$$\frac{(a+x)^2}{a^2} = 1+\frac{2x}{a} = \frac{a+2x}{a} \quad \ldots \quad \ldots \quad (1)$$

Cor From (1) we have $\sqrt{\frac{a+2x}{a}} = \frac{a+x}{a}$. Hence if x is very small compared with a, we have

$$\sqrt{a+x} : \sqrt{a} = a+\tfrac{1}{2}x : a.$$

Note By a similar mode of reasoning it can be shewn that when x is very small compared with a, $(a+x)^3 : a^3 = a+3x : a$,

$(a+x)^4 : a^4 = a+4x : a$, $(a+x)^{\frac{1}{3}} : a^{\frac{1}{3}} = a+\frac{1}{3}x : a$, and so on.

5. Incommensurable quantities. If two quantities be such that their ratio cannot be exactly expressed by the ratio of two integers, they are said to be *incommensurable quantities* Thus $\sqrt{3}$ and 2 are incommensurable quantities, since no two integers can be found whose ratio is *exactly* equal to $\sqrt{3} : 2$.

Although the ratio of two incommensurable quantities can not be *exactly* expressed by the ratio of two integers, we can always find two integers however, whose ratio differs from such a ratio by as small a quantity as we please.

For instance, $\frac{\sqrt{3}}{2} = \frac{1.73205\ldots}{2} = \cdot 86602\ldots$,

and therefore $\frac{\sqrt{3}}{2} > \frac{86602}{100000}$ and $< \frac{86603}{100000}$;

thus, $\sqrt{3} \div 2$ differs from either $86602 \div 100000$ or $86603 \div 100000$ by even less than a hundred-thousandth part of unity. A further approximation might evidently be arrived at by calculating the value of $\sqrt{3}$ to more places of decimals.

Note.—Any number which cannot be *exactly* expressed as the ratio of two whole numbers is also sometimes called incommensurable. From this point of view every surd is an incommensurable quantity.

Examples.

Example 1. Two numbers are in the ratio of 2 to 3, and if 9 be added to each they are in the ratio of 3 to 4. Find the numbers.

Since the numbers are in the ratio of 2 to 3, evidently we can represent them by $2x$ and $3x$ respectively.

Hence, by the second condition, we have

$$\frac{2x+9}{3x+9} = \frac{3}{4}.$$

Hence, $8x+36 = 9x+27$, whence $x = 9$.

Therefore the numbers are 18 and 27.

Example 2. What is the ratio of x to y, if

$$10x+3y : 5x+2y = 9 : 5 ?$$

We have $\frac{9}{5} = \frac{10x+3y}{5x+2y} = \frac{10\frac{x}{y}+3}{5\frac{x}{y}+2}.$

Hence, $45\frac{x}{y}+18 = 50\frac{x}{y}+15$,

$$\therefore \quad 5\frac{x}{y} = 3, \qquad \therefore \quad \frac{x}{y} = \frac{3}{5}$$

Example 3. Which is the greater (x and y being positive $x^3+y^3 : x^2+y^2$, or $x^2+y^2 : x+y$?

We have $\frac{x^3+y^3}{x^2+y^2}-\frac{x^2+y^2}{x+y} = \frac{xy^3+x^3y-2x^2y^2}{(x^2+y^2)(x+y)}$

$$= \frac{xy(x-y)^2}{(x^2+y^2)(x+y)},$$

which evidently is a positive quantity, since $(x-y)^2$ is positive whether x is greater or less than y.

Hence, $x^3+y^3 : x^2+y^2 > x^2+y^2 : x+y$

Example 4. Two armies number 11000 and 7000 men respectively, before they fight each is reinforced by 1000 men: in favour of which army is the increase?

(Calcutta University Entrance Paper, 1879

The new strength of the 1st army : its original strength

$$= 12000 : 11000 = 12 : 11,$$

whilst, the new strength of the 2nd army : its original strength

$$= 8000 : 7000 = 8 : 7.$$

Now since $12 : 11 = 84 : 77$,

and $8 : 7 = 88 : 77$,

it is clear that $8 : 7 > 12 : 11$.

Thus, *compared* with the original strength, the new strength of the second army is greater than that of the first.

Hence the increase is in favour of the second army

Exercise (93).

Which is the greater :

1. $4 : 5$ or $7 : 8$?
2. $7 : 10$ or $11 : 14$?
3. $9 : 5$ or $13 : 8$?
4. $22 : 27$ or $32 : 45$?
5. $28 : 39$ or $49 : 65$?

Find the ratio compounded of :—

6. $a : b$, $b : c$ and $c : d$.

7. $3 : 5$, $7 : 9$ and $15 : 28$.

8. $a+x : a-x$, $a^2+x^2 : (a+x)^2$ and $(a^2-x^2)^2 : a^4-x^4$.

9. $16 : 5$, the triplicate ratio of $5 : 4$ and the subduplicate ratio of $9 : 4$.

10. $25 : 18$, the subduplicate ratio of $81 : 49$, the triplicate ratio of $2 : 3$ and the duplicate ratio of $7 : 5$.

11. If $2x+5y : 3x+5y = 9 : 10$, find $x : y$.

12. If $x : y = 3 : 4$, find the value of $5x+9y : 16x+5y$.

13. Two numbers are in the ratio of $7 : 8$, and their sum is 135. Find the numbers.

14. Find two numbers which are in the ratio of $5 : 3$ and whose difference is 34.

15. Two numbers are in the ratio of $4 : 5$, and if 7 be added to each, the sums are in the ratio of $5 : 6$. Find the numbers.

16. Two numbers are in the ratio of $7 : 9$, and if 10 be subtracted from each, the remainders are in the ratio of $8 : 11$. Find the numbers.

17. For what value of x will the ratio $23+x : 19+x$ be equal to 2?

18. What number must be added to each term of the ratio $25 : 37$ that it may become equal to $5 : 6$?

19. What number must be added to each term of the ratio $29 : 38$ that it may become equal to $4 : 7$?

20. What quantity must be added to each of the terms of the ratio $a : b$, that it may become equal to $c : d$?

21. Show that if $a>x$ the ratio $a^2-x^2 : a^2+x^2$ is greater than the ratio $a-x : a+x$.

22. Show that the ratio $a^2+b^2 : a+b$ is less than the ratio $a^2-b^2 : a-b$.

Find approximately the value of :—

23. $(226)^3 : (225)^3$. 24. $\sqrt{(3546)} : \sqrt{(3542)}$.

25 A, B, C are three school boys getting monthly allowances of Rs 15, Rs 20 and Rs 25 respectively, out of these amounts they respectively spend Rs $8\frac{3}{4}$, Rs $11\frac{1}{4}$ and Rs $15\frac{5}{8}$ per month Which of them is the most frugal ?

PROPORTION.

2. **Definitions.** Four quantities are said to be *proportionals* when the ratio of the first to the second is equal to the ratio of the third to the fourth Thus, a, b, c, d are proportionals if $a : b = c : d$ This is often expressed as $a : b :: c : d$ and read "a is to b as c is to d"

The terms a and d are called the *extremes* and the terms b and c, the *means* The term d is also called the *fourth proportional* to a, b, c.

Three or more quantities are said to be in continued proportion when the first is to the second as the second is to the third, as the third is to the fourth, and so on. Thus a, b, c, d are in continued proportion when $a : b = b : c = c : d$.

If three quantities a, b, c are in continued proportion $(a : b :: b : c)$, then b is called the *mean proportional* between a and c, and c is called the *third proportional* to a and b

7 If $a : b :: c : d$, then will $ad = bc$

Since $$\frac{a}{b} = \frac{c}{d},$$

multiplying both sides by bd, we have $ad = bc$.

Thus, *if four quantities are proportional, the product of the extremes is equal to the product of the means.*

[Conversely, if $ad = bc$, then $a : b :: c : d$ This is obvious by dividing both sides of the equation by bd]

Cor. If $a : b :: b : c$, then $ac = b^2$, i e, *if three quantities are in continued proportion, the product of the extreme is equal to the square of the mean.*

Note From the result above established we can at once find a third proportional to, or a mean proportional between, two given quantities, as well as a fourth proportional to three given quantities

Exercise (94).

Find a third proportional to —

1. 9, 6. 2. 8, 12. 3. 6, 15. 4. 16, 24.

Find a fourth proportional to —

5. 6, 8, 15. 6. 14, 24, 35. 7. 0014, 1 4, 02

Find a mean proportional between —

8. 4, 9. 9. 7, 28. 10. 6, 54.

8. If $a : b :: b : c$, then $a : c :: a^2 : b^2$.

For $$\frac{a}{b} = \frac{b}{c},$$

$$\therefore \quad \frac{a}{b} \times \frac{b}{c} = \frac{a}{b} \times \frac{a}{b}$$

or, $$\frac{a}{c} = \frac{a^2}{b^2}$$

Thus, *if three quantities are in continued proportion, the first is to the third in the duplicate ratio of the first to the second*

Note Similarly, if $a : b = b : c = c : d$ it can be easily proved that $a : d = a^3 : b^3$ which is left as an exercise for the student

9. If $a : b :: c : d$, then $b : a :: d : c$.

For $$\frac{a}{b} = \frac{c}{d},$$

$$\therefore \quad 1 - \frac{a}{b} = 1 - \frac{c}{d}, \text{ whence } \frac{b}{a} = \frac{d}{c}.$$

Thus, *if four quantities be proportionals, they are also proportionals when taken inversely*

This operation is called **Invertendo.**

10. $a : b :: c : d$, then $a : c :: b : d$

For $$\frac{a}{b} = \frac{c}{d},$$

$$\therefore \quad \frac{a}{b} \times \frac{b}{c} = \frac{c}{d} \times \frac{b}{c}, \text{ or, } \frac{a}{c} = \frac{b}{d}.$$

Thus, *if four quantities be proportionals, they are proportionals when taken alternately.*

This operation is called **Alternando.**

11. If $a : b :: c : d$, then $a+b : b :: c+d : d$.

For $$\frac{a}{b} = \frac{c}{d},$$

$$\therefore \quad \frac{a}{b}+1 = \frac{c}{d}+1, \quad \text{or, } \frac{a+b}{b} = \frac{c+d}{d}.$$

Thus, *when four quantities are proportionals, the first together with the second is to the second as the third together with the fourth is to the fourth.*

This operation is called **Componendo.**

12. If $a : b :: c : d$, then $a-b : b :: c-d : d$.

For $$\frac{a}{b} = \frac{c}{d},$$

$$\therefore \quad \frac{a}{b}-1 = \frac{c}{d}-1, \quad \text{or, } \frac{a-b}{b} = \frac{c-d}{d}.$$

Thus, *when four quantities are proportionals, the excess of the first over the second is to the second as the excess of the third over the fourth is to the fourth.*

This operation is called **Dividendo.**

***Cor.* If $a : b :: c : d$, then $a : a-b :: c : c-d$.**

For $\frac{a-b}{b} = \frac{c-d}{d}$, $\therefore$ inversely, $\frac{b}{a-b} = \frac{d}{c-d}$.

Hence, $$\frac{b}{a-b} \times \frac{a}{b} = \frac{d}{c-d} \times \frac{c}{d},$$

or, $$\frac{a}{a-b} = \frac{c}{c-d}.$$

Thus, *when four quantities are proportionals, the first is to the excess of the first above the second as the third is to the excess of the third above the fourth.*

This operation is called **Convertendo.**

13. If $a : b :: c : d$, then
$$a+b : a-b :: c+d : c-d$$

From Art 11, $\dfrac{a+b}{b} = \dfrac{c+d}{d}$ ... .. (1)

From Art 12, $\dfrac{a-b}{b} = \dfrac{c-d}{d}$ .. ... (2)

Hence, dividing (1) by (2),
$$\frac{a+b}{a-b} = \frac{c+d}{c-d}.$$

Thus, *when four quantities are proportionals, the sum of the first and second is to their difference as the sum of the third and fourth is to their difference.*

This result is often spoken of as **Componendo and Dividendo.**

Note The result proved in this article is of great use in solving a certain class of equations This will be illustrated in some of the following examples

Example 1. Solve $\dfrac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}} = b.$

By componendo and dividendo, we have
$$\frac{2\sqrt{a+x}}{2\sqrt{a-x}} = \frac{b+1}{b-1}.$$

Hence $\dfrac{a+x}{a-x} = \left(\dfrac{b+1}{b-1}\right)^2 = \dfrac{b^2+2b+1}{b^2-2b+1}.$

Again applying componendo and dividendo,
$$\frac{2a}{2x} = \frac{2(b^2+1)}{4b}, \text{ or, } \frac{a}{x} = \frac{b^2+1}{2b};$$
$$\therefore\quad x(b^2+1) = 2ab, \quad \therefore\quad x = \frac{2ab}{b^2+1}.$$

Example 2. Solve $\dfrac{1-ax}{1+ax}\sqrt{\dfrac{1+bx}{1-bx}} = 1.$

We have $\sqrt{\dfrac{1+bx}{1-bx}} = \dfrac{1+ax}{1-ax},$
$$\therefore\quad \frac{1+bx}{1-bx} = \frac{1+2ax+a^2x^2}{1-2ax+a^2x^2}.$$

Hence, by componendo and dividendo,

$$\frac{1}{bx} = \frac{1+a^2x^2}{2ax},$$

$$\therefore\ b(1+a^2x^2) = 2a, \text{ or, } a^2x^2 = \frac{2a}{b}-1.$$

$$x = \frac{1}{a}\sqrt{\frac{2a}{b}-1}.$$

Example 3. Find the value of $\dfrac{x+2a}{x-2a}+\dfrac{x+2b}{x-2b}$, when

$x = \dfrac{4ab}{a+b}$. (Allahabad University Entrance Paper, 1892.)

From the given relation, we have

$$\frac{x}{2a} = \frac{2b}{a+b}, \text{ and } \frac{x}{2b} = \frac{2\cancel{b}a}{a+b}.$$

Hence, by componendo and dividendo,

$$\frac{x+2a}{x-2a} = \frac{a+3b}{b-a}, \text{ and } \frac{x+2b}{x-2b} = \frac{3a+b}{a-b}.$$

Hence, the given expression

$$= \frac{-(a+3b)}{a-b}+\frac{3a+b}{a-b}$$

$$= \frac{2(a-b)}{a-b} = 2.$$

Note. For a different solution of this example see page 183

Example 4. If $(a+b+c+d)(a-b-c+d)$
$= (a-b+c-d)(a+b-c-d)$, show that $a : b :: c : d$.

From the given relation, we have

$$\frac{a+b+c+d}{a+b-c-d} = \frac{a-b+c-d}{a-b-c+d}.$$

Hence, by componendo and dividendo,

$$\frac{a+b}{c+d} = \frac{a-b}{c-d},$$

$$\therefore\ \frac{a+b}{a-b} = \frac{c+d}{c-d};$$

whence by a second application of componendo and dividendo,

$$\frac{a}{b} = \frac{c}{d}.$$

Example 5. If $x = \dfrac{\sqrt[3]{m+1}+\sqrt[3]{m-1}}{\sqrt[3]{m+1}-\sqrt[3]{m-1}}$, shew that

$$x^3-3mx^2+3x-m = 0.$$

From the given relation, by componendo and dividendo we have

$$\frac{x+1}{x-1} = \frac{\sqrt[3]{m+1}}{\sqrt[3]{m-1}},$$

$$\therefore \quad \frac{m+1}{m-1} = \frac{(x+1)^3}{(x-1)^3} = \frac{x^3+3x^2+3x+1}{x^3-3x^2+3x-1}.$$

Hence, by a second application of componendo and dividendo, we have

$$\frac{m}{1} = \frac{x^3+3x}{3x^2+1},$$

$$\therefore \quad m(3x^2+1) = x^3+3x,$$

$$\text{whence,} \quad x^3-3mx^2+3x-m = 0$$

Exercise (95).

Solve the following equations —

1. $\left.\begin{aligned} \frac{x+y}{x-y} &= 5 \\ 2x+3y &= 36 \end{aligned}\right\}.$

2. $\left.\begin{aligned} \frac{3x-5y}{3x+5y} &= \tfrac{1}{4} \\ 4x-9y &= 19 \end{aligned}\right\}.$

3. $\left.\begin{aligned} \frac{5x-7y}{5x+7y} &= \tfrac{1}{7} \\ 3x-5y &= 16 \end{aligned}\right\}$

4. $16\left(\dfrac{a-x}{a+x}\right)^3 = \dfrac{a+x}{a-x}$

(C U Entrance Paper, 1886)

5. $\dfrac{2x+\sqrt{4x^2-1}}{2x-\sqrt{4x^2-1}} = 4.$

6. $\dfrac{1-\sqrt{1-x}}{1+\sqrt{1-x}} = \tfrac{1}{3}$

7. $\dfrac{\sqrt{36x+1}+\sqrt{36x}}{\sqrt{36x+1}-\sqrt{36x}} = 9$

8. $\dfrac{1+x+x^2}{1-x+x^2} = \dfrac{62}{63}\,\dfrac{1+x}{1-x}.$

9. $\dfrac{\sqrt{5}+\sqrt{5-x}}{\sqrt{5}-\sqrt{5-x}} = 5.$ 10. $\dfrac{a+x+\sqrt{a^2-x^2}}{a+x-\sqrt{a^2-x^2}} = \dfrac{b}{x}.$

11. $\dfrac{a^{\frac{1}{2}}-\{a-(a^2-ax)^{\frac{1}{2}}\}^{\frac{1}{2}}}{a^{\frac{1}{2}}+\{a-(a^2-ax)^{\frac{1}{2}}\}^{\frac{1}{2}}} = b$

Prove that $a : b :: c : d$ —

12. If $(a+3b+2c+6d)(a-3b-2c+6d)$

$$= (a-3b+2c-6d)(a+3b-2c-6d).$$

13. If $(2a+b+4c+2d)(2a-b-4c+2d)$

$$= (2a-b+4c-2d)(2a+b-4c-2d).$$

14. If $x = \dfrac{\sqrt{2a+3b}+\sqrt{2a-3b}}{\sqrt{2a+3b}-\sqrt{2a-3b}}$, shew that

$$3bx^2-4ax+3b = 0.$$

15. If $x = \dfrac{2\sqrt{24}}{\sqrt{2}+\sqrt{3}}$, find the value of $\dfrac{x+\sqrt{8}}{x-\sqrt{8}}+\dfrac{x+\sqrt{12}}{x-\sqrt{12}}$.

14. An important Theorem. If $\dfrac{a}{b} = \dfrac{c}{d} = \dfrac{e}{f}$,

then each of the ratios $= \left(\dfrac{pa^n+qc^n+re^n}{pb^n+qd^n+rf^n}\right)^{\frac{1}{n}}$, where p, q, r, n are any quantities whatever.

Supposing each of the given ratios $= k$, we have $a = bk$, $c = dk$, $e = fk$.

Hence, $\left.\begin{aligned} pa^n &= p(bk)^n = pb^n k^n \\ qc^n &= q(dk)^n = qd^n k^n \\ re^n &= r(fk)^n = rf^n k^n \end{aligned}\right\}$ $\begin{aligned} &\therefore pa^n+qc^n+re^n \\ &= (pb^n+qd^n+rf^n)k^n, \end{aligned}$

whence, $k^n = \dfrac{pa^n+qc^n+re^n}{pb^n+qd^n+rf^n}$,

and $\therefore$ $k = \left(\dfrac{pa^n+qc^n+re^n}{pb^n+qd^n+rf^n}\right)^{\frac{1}{n}}$, which proves the proposition.

Cor. As a particular case if p, q, r, n, be each equal to 1, we have each of the given ratios $= \dfrac{a+c+e}{b+d+f}$.

Similarly, giving different sets of values to p, q, r, n, several particular cases may be at once deduced

Note What is proved above for three equal ratios is obviously true for *any number* of equal ratios the same reasoning being applicable to all cases It is always a very good exercise for the student however to work out independently every fresh example of this class applying the mode of demonstration illustrated above Hence an exercise is added below with a recommendation to the student that he should find the result in *each* case *without using the formula* established in this article

Exercise (96).

If $\dfrac{a}{b}=\dfrac{c}{d}=\dfrac{e}{f}$, prove that each of these ratios is equal to —

1. $\dfrac{a-c+e}{b-d+f}$.
2. $\dfrac{a+3c-5e}{b+3d-5f}$.
3. $\dfrac{5a-7c-13e}{5b-7d-13f}$.
4. $\dfrac{ka+lc+me}{kb+ld+mf}$.

(C U. Entr Paper, 1875)

5. $\left(\dfrac{a^2+c^2+e^2}{b^2+d^2+f^2}\right)^{\frac{1}{2}}$.
6. $\left(\dfrac{a^3-2c^3+3e^3}{b^3-2d^3+3f^3}\right)^{\frac{1}{3}}$.
7. $\dfrac{\sqrt[3]{a^3+c^3+e^3}}{\sqrt[3]{b^3+d^3+f^3}}$. (C U Entr Paper, 1882)

If $\dfrac{a}{b}=\dfrac{c}{d}=\dfrac{e}{f}=\dfrac{g}{h}$, prove that each of these ratios is equal to :—

8. $\left(\dfrac{a^{-1}+c^{-1}+e^{-1}+g^{-1}}{b^{-1}+d^{-1}+f^{-1}+h^{-1}}\right)^{-1}$.
9. $\sqrt[4]{\dfrac{a^4-2c^4+3e^4-4g^4}{b^4-2d^4+3f^4-4h^4}}$.
10. $\sqrt{\left(\dfrac{3a^{-2}-7c^{-2}-8e^{-2}+15g^{-2}}{3b^{-2}-7d^{-2}-8f^{-2}+15h^{-2}}\right)^{-1}}$.

15. Miscellaneous Examples.

Example 1 If $x : y :: m^2 : n^2$, and $m : n :: \sqrt{p^2+x^2} : \sqrt{p^2-y^2}$ then $p^2 : xy :: x+y : x-y$.

We have $\dfrac{x}{y} = \dfrac{m^2}{n^2} = \dfrac{p^2+x^2}{p^2-y^2}$;

$$\therefore \quad x(p^2-y^2) = y(p^2+x) \qquad \text{[Art 7]}$$

$$\text{or,} \quad p^2(x-y) = xy(x+y),$$

$$\therefore \quad \frac{p^2}{xy} = \frac{x+y}{x-y} \qquad \text{[Art 7, Converse]}$$

$$\text{i e,} \quad p^2 : xy :: x+y : x-y$$

Example 2 If $a : b :: c : d$, show that

$$ma+nc : mb+nd :: (a^2+c^2)^{\frac{1}{2}} : (b^2+d^2)^{\frac{1}{2}}.$$

(Calcutta University Entrance Paper, 1880)

Since $\dfrac{a}{b} = \dfrac{c}{d}$, $\quad \therefore \dfrac{ma}{mb} = \dfrac{nc}{nd}$,

and therefore each of them $= \dfrac{ma+nc}{mb+nd}$. [Art 14]

Again, since $\dfrac{a}{b} = \dfrac{c}{d}$, $\quad \therefore \dfrac{a^2}{b^2} = \dfrac{c^2}{d^2}$,

and therefore each of them $= \dfrac{a^2+c^2}{b^2+d^2}$. [Art 14]

Thus we have $\dfrac{ma+nc}{mb+nd} = \dfrac{ma}{mb} = \dfrac{a}{b}$ ... ... (1)

and $\dfrac{a^2+c^2}{b^2+d^2} = \dfrac{a^2}{b^2}$ ... ... (2)

Hence from (1) and (2),

$$\frac{ma+nc}{mb+nd} = \frac{(a^2+c^2)^{\frac{1}{2}}}{(b^2+d^2)^{\frac{1}{2}}},$$ which was to be proved.

Example 3 If $\dfrac{x}{(b-c)(b+c-2a)} = \dfrac{y}{(c-a)(c+a-2b)}$ $= \dfrac{z}{(a-b)(a+b-2c)}$, find the value of $x+y+z$.

(Calcutta University Entrance Paper 1889)

Let each of the given ratios $= k$

Then, $x = k(b-c)(b+c-2a) = k\{(b^2-c^2)-2a(b-c)\}$,

$y = k(c-a)(c+a-2b) = k\{(c^2-a^2)-2b(c-a)\}$,

$z = k(a-b)(a+b-2c) = k\{(a^2-b^2)-2c(a-b)\}$.

Hence, $x+y+z$

$$= k[\{(b^2-c^2)+(c^2-a^2)+(a^2-b^2)\}]$$
$$-2\{a(b-c)+b(c-a)+c(a-b)\}]$$
$$= 0$$

Example 4. If $\dfrac{ay-bx}{c} = \dfrac{cx-az}{b} = \dfrac{bz-cy}{a}$,

shew that $\dfrac{x}{a} = \dfrac{y}{b} = \dfrac{z}{c}$.

Let each of the given ratios $= k$.

Then, we have $(ay-bx)c = kc^2$,
$(cx-az)b = kb^2$,
$(bz-cy)a = ka^2$.

Hence, by addition, $k(a^2+b^2+c^2) = 0$;
$\therefore\ k = 0$

Hence, $ay-bx = 0$, $\therefore\ ay = bx$, $\therefore\ \dfrac{x}{a} = \dfrac{y}{b}$ .. (1)

also, $cx-az = 0$, $\therefore\ cx = az$, $\therefore\ \dfrac{x}{a} = \dfrac{z}{c}$ .. (2)

Hence, from (1) and (2), $\dfrac{x}{a} = \dfrac{y}{b} = \dfrac{z}{c}$.

Example 5. If $\dfrac{a}{b} = \dfrac{b}{c} = \dfrac{c}{d}$, then will

$$(a-d)^2 = (b-c)^2+(c-a)^2+(d-b)^2.$$

From the given relations, we have

(i) $b^2 = ac$, (ii) $c^2 = bd$, (iii) $bc = ad$. [Art 7]

Now, $(b-c)^2+(c-a)^2+(d-b)^2$

$$= (b^2+c^2-2bc)+(c^2+a^2-2ac)+(d^2+b^2-2bd)$$
$$= 2(b^2-ac)+2(c^2-bd)+a^2+d^2-2bc$$

$$= a^2+d^2-2bc \qquad \text{[from (i) and (ii)]}$$
$$= a^2+d^2-2ad \qquad \text{[from (iii)]}$$
$$= (a-d)^2$$

Example 6. If $a : b :: c : d$, show that

$$4(a+b)(c+d) = bd\left\{\frac{a+b}{b}+\frac{c+d}{d}\right\}^2.$$

(Calcutta University Entrance Paper, 1874

Since $\frac{a}{b} = \frac{c}{d}$, $\quad\therefore\quad \frac{a+b}{b} = \frac{c+d}{d}$; [componendo

clearly therefore, $\frac{a+b}{b}+\frac{c+d}{d} = \frac{2(a+b)}{b} = \frac{2(c+d)}{d}$

Hence, $$\left\{\frac{a+b}{b}+\frac{c+d}{d}\right\}^2 = \frac{2(a+b)}{b}\times\frac{2(c+d)}{d}$$
$$= \frac{4(a+b)(c+d)}{bd}.$$

$$\therefore\quad bd\left\{\frac{a+b}{b}+\frac{c+d}{d}\right\}^2 = 4(a+b)(c+d).$$

Example 7. If $a : b :: p : q$, show that

$$a^2+b^2 : \frac{a^3}{a+b} :: p^2+q^2 : \frac{p^3}{p+q}.$$

From the given relation, we have

$$\frac{b}{a} = \frac{q}{p}, \quad\text{and}\quad \therefore\quad \frac{b^2}{a^2} = \frac{q^2}{p^2}.$$

Hence, (i) $\frac{a+b}{a} = \frac{p+q}{p}$, and (ii) $\frac{a^2+b^2}{a^2} = \frac{p^2+q^2}{p^2}$.

Multiplying together (i) and (ii), we have

$$\frac{(a^2+b^2)(a+b)}{a^3} = \frac{(p^2+q^2)(p+q)}{p^3}$$

or, $$\frac{a^2+b^2}{\left(\frac{a^3}{a+b}\right)} = \frac{p^2+q^2}{\left(\frac{p^3}{p+q}\right)},$$

i e, $a^2+b^2 : \frac{a^3}{a+b} :: p^2+q^2 : \frac{p^3}{p+q}$

Example 8. If $m : n :: p : q$, prove that

$$\frac{(m-n)(m-p)}{m} = (m+q)-(n+p).$$

We have $\frac{m}{n} = \frac{p}{q}$, $\therefore$ $\frac{m-n}{m} = \frac{p-q}{p}$;

alternately, $\frac{m}{p} = \frac{n}{q}$, $\therefore$ $\frac{m-p}{p} = \frac{n-q}{q}$.

Hence, $\frac{(m-n)(m-p)}{mp} = \frac{(p-q)(n-q)}{pq}$

or, $$\frac{(m-n)(m-p)}{m} = \frac{(p-q)(n-q)}{q}$$

$$= \frac{pn-q(n+p)+q^2}{q}$$

$$= \frac{mq+q^2-q(n+p)}{q} \quad [\because pn = mq]$$

$$= (m+q)-(n+p).$$

Example 9. If $\frac{a}{b} = \frac{b}{c} = \frac{c}{d}$, show that

$$(a^2+b^2+c^2)(b^2+c^2+d^2) = (ab+bc+cd)^2.$$

[Calcutta University Entrance Paper 1887.]

Let each of the given ratios $= k$

Then $\left.\begin{aligned} k^2b^2 &= a^2 \\ k^2c^2 &= b^2 \\ k^2d^2 &= c^2 \end{aligned}\right\}$ $\therefore$ $k^2(b^2+c^2+d^2) = a^2+b^2+c^2$

$\therefore$ $k^2 = \frac{a^2+b^2+c^2}{b^2+c^2+d^2}$ ... ...(1)

also $\left.\begin{aligned} kb = a, \ \therefore kb^2 &= ab \\ kc = b, \ \therefore kc^2 &= bc \\ kd = c, \ \therefore kd^2 &= cd \end{aligned}\right\}$ $\therefore$ $k(b^2+c^2+d^2) = ab+bc+cd$

$\therefore$ $k = \frac{ab+bc+cd}{b^2+c^2+d^2}$ ... ...(2)

Hence equating the value of k^2 from (1) and (2), we have

$$\frac{a^2+b^2+c^2}{b^2+c^2+d^2} = \frac{(ab+bc+cd)^2}{(b^2+c^2+d^2)^2};$$

$$\therefore \quad (a^2+b^2+c^2)(b^2+c^2+d^2) = (ab+bc+cd)^2.$$

Example 10 If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$, show that

$$\sqrt{(a+c+e)(b+d+f)} = (ab)^{\frac{1}{2}}+(cd)^{\frac{1}{2}}+(ef)^{\frac{1}{2}}.$$

Let each of the given ratios $=k$

$$\text{Then}\quad \left.\begin{aligned} a &= bk \\ c &= dk \\ e &= fk \end{aligned}\right\} \begin{aligned} &\therefore\ a+c+e = k(b+d+f), \\ & \\ &\therefore\ (a+c+e)(b+d+f)=k(b+d+f)^2\ ; \end{aligned}$$

$$\therefore\quad \sqrt{(a+c+e)(b+d+f)} = (b+d+f)\sqrt{k}\quad . \quad . \qquad (1)$$

$$\text{Also, we have}\quad \begin{aligned} ab &= b^2k, \\ cd &= d^2k,\ \therefore \\ ef &= f^2k,\ \therefore \end{aligned} \left.\begin{aligned} (ab)^{\frac{1}{2}} &= b\sqrt{k} \\ (cd)^{\frac{1}{2}} &= d\sqrt{k} \\ (ef)^{\frac{1}{2}} &= f\sqrt{k} \end{aligned}\right\}$$

$$\therefore\quad (ab)^{\frac{1}{2}}+(cd)^{\frac{1}{2}}+(ef)^{\frac{1}{2}} = (b+d+f)\sqrt{k} \quad \ldots\ldots \quad (2)$$

Hence, from (1) and (2),

$$\sqrt{(a+c+e)(b+d+f)} = (ab)^{\frac{1}{2}}+(cd)^{\frac{1}{2}}+(ef)^{\frac{1}{2}}.$$

Exercise (97).

If a be the greatest of the four quantities a, b, c, d and if $a : b :: c : d$, show that —

1. b and c are each $>d$.

2. $a-b>c-d$ 3. $a+d>b+c$.

If $a : b :: c : d$, show that —

4. $ma+nb : b :: mc+nd : d$.

5. $ma+nb : mc+nd :: pa-qb : pc-qd$

6. $a : b :: a+c : b+d$. 7. $a^2 : b^2 :: a^2+c^2 : b^2+d^2$.

8. $a^2+c^2 : b^2+d^2 :: ac : bd$. (C U Entr Paper 1877)

9. $(a-c)^2 : (b-d)^2 = a^2 : b^2$.

10. $(a+c)^3 : (b+d)^3 = a(a-c)^2 : b(b-d)^2$.

(Calcutta University Entrance Paper, 1888

11. $a^2+b^2 : a^2-b^2 = ac+bd : ac-bd.$

12. $a(a+c) : c^2 :: b(b+d) : d^2$

13. $c : d = \sqrt{a^2+c^2} : \sqrt{b^2+d^2}$

14. $a+b : c+d = \sqrt{a^2+b^2} : \sqrt{c^2+d^2}$

15. $a+b : c+d :: \sqrt{3a^2+5b^2} : \sqrt{3c^2+5d^2}.$

16. $a^2+ab+b^2 : a^2-ab+b^2 :: c^2+cd+d^2 : c^2-cd+d^2.$

17. $a^2+ac+c^2 : a^2-ac+c^2 :: b^2+bd+d^2 : b^2-bd+d^2$

If $a : b = c : d = e : f$, show that

18. $\dfrac{ma+nb}{mc+nd} = \dfrac{b^2c}{d^2a}$ (Calcutta University Entrance Paper, 1876.)

19. $ac : bd :: 2a^2+3c^2+5e^2 : 2b^2+3d^2+5f^2.$

20. $a^2+c^2+e^2 : b^2+d^2+f^2 :: ce : df.$

(Calcutta University Entrance Paper, 1876.)

21. $pa+qc+re : pb+qd+rf :: \sqrt[3]{ace} : \sqrt[3]{bdf}$

22. $a^2 : b^2 :: ac+ce+ae : bd+df+bf.$

23. $a^3+c^3+e^3 : b^3+d^3+f^3 :: ace : bdf.$

24. $\sqrt{a^3c^3+c^3e^3+a^3e^3} : \sqrt{b^3d^3+d^3f^3+b^3f^3} :: ace : bdf.$

25. If a, b, c, d, e be in continued proportion,

show that $a : e :: a^4 : b^4.$

26. If $\dfrac{x}{b+c-a} = \dfrac{y}{c+a-b} = \dfrac{z}{a+b-c}$, find the value of

$(b-c)x+(c-a)y+(a-b)z$

(Calcutta University Entrance Paper, 1878.)

27. If $a : b :: c : d$, prove that:

$a^2+c^2 : b^2+d^2 :: \sqrt{a^4+c^4} : \sqrt{b^4+d^4}.$

28. If $a : b = c : d = e : f$, show that

$$27(a+b)(c+d)(e+f) = bdf\left(\frac{a+b}{b}+\frac{c+d}{d}+\frac{e+f}{f}\right)^3.$$

1—23

29. If $a : b :: c : d$, show that
$$ad+bc : 2bd :: a^2+c^2 : ab+cd.$$

30. If $a : b :: c : d$, show that
$$a^2+b^2 : ab+ad-bc :: c^2+d^2 : cd-ad+bc.$$

If $a : b :: b : c$, show that —

31. $a^2+ab+b^2 : b^2+bc+c^2 = a : c$.

32. $a-2b+c = \dfrac{(a-b)^2}{a} = \dfrac{(b-c)^2}{c}$.

33. $a^2b^2c^2\left(\dfrac{1}{a^3}+\dfrac{1}{b^3}+\dfrac{1}{c^3}\right) = a^3+b^3+c^3$.

If $a : b = b : c = c : d$, show that —

34. $(b+c)(b+d) = (c+a)(c+d)$.

35. $(a+d)(b+c)-(a+c)(b+d) = (b-c)^2$.

36. $\left(\dfrac{a-b}{c}+\dfrac{a-c}{b}\right)^2-\left(\dfrac{d-b}{c}+\dfrac{d-c}{b}\right)^2 = (a-d)^2\left(\dfrac{1}{c^2}-\dfrac{1}{b^2}\right)$.

37. $a : b = \dfrac{1}{b}+\dfrac{1}{c}+\dfrac{1}{d} : \dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}$.

38. $a : d :: a^3+b^3+c^3 : b^3+c^3+d^3$.

39. If $a : b :: c : d$ show that
$$a^2+ab : c^2+cd :: b^2-2ab : d^2-2cd.$$

40. If $a : b = c : d = e : f$, show that
$$(a^2+b^2)(ce+df)^2 = (c^2+d^2)(ae+bf)^2 = (e^2+f^2)(ac+bd)^2.$$

CHAPTER XX

ELIMINATION AND MISCELLANEOUS ARTIFICES

1. Elimination. If there be *two* equations involving *one* unknown quantity they will generally not be satisfied by the same value of it. For instance, the same value of x will *not* satisfy the equations $x+3=7$ and $x+4=9$. But this cannot be strictly said of the two equations $x+a=7$ and $x+b=9$, where a and b have no fixed numerical values; the appropriate remark in this case would be "the two equations *will be* satisfied by the same value of x *if* $7-a=9-b$, or $b-a=2$." Thus if *one* unknown quantity occurs in *two* equations which *also* involve other *algebraical symbols*, there always exists a particular relation between these other symbols for which and for which alone *both* the given equations are satisfied by the *same* value of the unknown quantity. The process of finding this relation is called the *Elimination* of the unknown quantity from the given equations, and the relation obtained is called the *Eliminant* of those equations.

Similarly there may be a question of eliminating two unknown quantities from three given equations. For instance, the three equations $x+y=a$, $x+2y=b$, $x+3y=c$, *cannot* be *all* satisfied by the *same* values of x and y *unless* the quantities a, b, c, are connected with one another in a certain way, and this connection it may be necessary to investigate.

A few simple cases of elimination will now be presented to the student, calculated to give him a tolerably clear idea of the subject, as also to familiarise him with some of the various ways of dealing with such questions.

Example 1. Eliminate x from the equations

$$a_1x+b_1=0,\quad a_2x+b_2=0.$$

From the first equation we have $x=-\dfrac{b_1}{a_1}$,

and from the second equation $x=-\dfrac{b_2}{a_2}$.

Evidently therefore both the equations will be satisfied by the same value of x if $\frac{b_1}{a_1} = \frac{b_2}{a_2}$, or, $a_1b_2 = a_2b_1$.

Thus $a_1b_2 = a_2b_1$ is the required Eliminant

Example 2. Eliminate x from the equations

$$a_1x^2+b_1x+c_1 = 0,\ a_2x^2+b_2x+c_2 = 0$$

Let α be the value of x which satisfies both the equations. Then we must have

$$\left.\begin{aligned} a_1\alpha^2+b_1\alpha+c_1 &= 0 \\ a_2\alpha^2+b_2\alpha+c_2 &= 0 \end{aligned}\right\}$$

Hence, by cross multiplication,

$$\frac{\alpha^2}{b_1c_2-b_2c_1} = \frac{\alpha}{c_1a_2-c_2a_1} = \frac{1}{a_1b_2-a_2b_1};$$

$$\therefore \frac{\alpha^2}{b_1c_2-b_2c_1} \times \frac{1}{a_1b_2-a_2b_1} = \left(\frac{\alpha}{c_1a_2-c_2a_1}\right)^2,$$

whence, $(b_1c_2-b_2c_1)(a_1b_2-a_2b_1) = (c_1a_2-c_2a_1)^2$, which is the required Eliminant

Example 3 Eliminate x and y from the equations

$$\left.\begin{aligned} a_1x+b_1y+c_1 &= 0 \\ a_2x+b_2y+c_2 &= 0 \\ a_3x+b_3y+c_3 &= 0 \end{aligned}\right\}$$

From the first two equations, by cross multiplication, we have

$$\frac{x}{b_1c_2-b_2c_1} = \frac{y}{c_1a_2-c_2a_1} = \frac{1}{a_1b_2-a_2b_1},$$

$$\therefore \quad x = \frac{b_1c_2-b_2\,{}_1}{a_1b_2-a_2b_1}, \quad y = \frac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1}.$$

If the third equation also be satisfied by these values of x and y, we must evidently have

$$a_3.\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1}+b_3\frac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1}+c_3 = 0,$$

or, $a_3(b_1c_2-b_2c_1)+b_3(c_1a_2-c_2a_1)+c_3(a_1b_2-a_2b_1) = 0$, which is the required Eliminant.

Example 4. Eliminate x, y, z from the equations

$$\frac{ax}{by+cz} = \frac{by}{cz+ax} = \frac{z}{x+y} = \frac{1}{2}.$$

We have $$\frac{ax}{by+cz} = \frac{1}{2},$$

$$\therefore\ 2ax = by+cz, \text{ or, } 2ax-by-cz=0 \quad ..(1)$$

Also $$\frac{by}{cz+ax} = \frac{1}{2},$$

$$\therefore\ 2by = cz+ax, \text{ or, } ax-2by+cz=0 \quad . \quad (2)$$

Hence, from (1) and (2), by cross multiplication, we have

$$\frac{x}{-bc-2bc} = \frac{y}{-ca-2ca} = \frac{z}{-4ab+ab},$$

or, $$\frac{x}{-3bc} = \frac{y}{-3ca} = \frac{z}{-3ab},$$

or, $$\frac{x}{bc} = \frac{y}{ca} = \frac{z}{ab}.$$

Supposing each of these ratios $= k$, we have

$$x = k\,bc, \quad y = k\,ca, \quad z = k\,ab.$$

Substituting these values of x, y, z in the third equation which is $2z = x+y$, we have

$$2k\,ab = k(bc+ca),$$

or, $$2ab = bc+ca,$$

$$\therefore\quad \frac{2}{c} = \frac{1}{a}+\frac{1}{b},$$

which is the required Eliminant

Note It may be noticed in this example that the three given equations $2ax-by-cx=0$, $ax-2by+cz=0$ and $2z=x+y$ virtually involves *two* unknown quantities instead of three, for they are respectively equivalent to $2a\left(\frac{x}{z}\right)-b\left(\frac{y}{z}\right)-c=0$, $a\left(\frac{x}{z}\right)-2b\left(\frac{y}{z}\right)+c=0$ and $2=\left(\frac{x}{z}\right)+\left(\frac{y}{z}\right)$, in which the only unknown quantities are $\frac{x}{z}$ and $\frac{y}{z}$.

It is owing to this disguised character (so to speak) of the three given equations that we have been able to eliminate from them the *three* unknown quantities, x, y, z, *otherwise* the required elimination would not be possible

Example 5. Eliminate x from the equations

$$x^3+\frac{3}{x} = 4(a^3+b^3),\quad 3x+\frac{1}{x^3} = 4(a^3-b^3)$$

Adding together the equations, we have

$$x^3+3x+\frac{3}{x}+\frac{1}{x^3} = 8a^3$$

$$\text{or,}\quad \left(x+\frac{1}{x}\right)^3 = (2a)^3,$$

$$\therefore\quad x+\frac{1}{x} = 2a \quad \ldots \quad . \quad (1)$$

Subtracting the second equation from the first, we have

$$x^3-3x+\frac{3}{x}-\frac{1}{x^3} = 8b^3,$$

$$\text{or,}\quad \left(x-\frac{1}{x}\right)^3 = (2b)^3,$$

$$\therefore\quad x-\frac{1}{x} = 2b \quad . \quad (2)$$

From (1) and (2), by addition,

$$2x = 2(a+b),\quad \text{or,}\quad x = a+b\ ;$$

and by subtraction,

$$\frac{2}{x} = 2(a-b),\quad \text{or,}\quad \frac{1}{x} = a-b$$

Hence, $$(a+b)(a-b) = x\times\frac{1}{x} = 1.$$

Thus $a^2-b^2 = 1$ is the required Eliminant

Example 6. Eliminate x, y, z from the equations

$$\left.\begin{aligned} x+y+z &= a \quad . & (1)\\ 2(yz+zx+xy) &= b^2 & (2)\\ x^3+y^3+z^3 &= c^3\ . & (3)\\ 3xyz &= d^3\ .\ . & (4)\end{aligned}\right\}$$

Since $$x^2+y^2+z^2 = (x+y+z)^2-2(yz+zx+xy),$$

$\therefore$ from (1) and (2), it $= a^2-b^2$ ... . (5)

Now, since $x^3+y^3+z^3-3xyz$

$$= (x+y+z)(x^2+y^2+z^2-yz-zx-xy)$$
$$= (x+y+z)\{(x^2+y^2+z^2)-(yz+zx+xy)\},$$

$\therefore$ from (3), (4), (1), (5) and (2), we must have

$$c^3-d^3 = a\{(a^2-b^2)-\tfrac{1}{2}b^2\}$$
$$= a^3-\tfrac{3}{2}ab^2,$$

or, $2a^3-3ab^2-2c^3+2d^3 = 0,$

which is the required Eliminant.

Example 7. Eliminate x, y, z from the equations :—

(i) $x^2(y+z) = a^2,$ (ii) $y^2(x+z) = b^2,$

(iii) $z^2(x+y) = c^2,$ (iv) $xyz = abc$

Multiplying the first three equations together we have

$$x^2y^2z^2(y+z)(z+x)(x+y) = a^2b^2c^2.$$

Hence, from (iv), $(y+z)(z+x)(x+y) = 1$ . .. (α)

But $(y+z)(z+x)(x+y)$

$$= (y+z)\{x^2+x(y+z)+yz\}$$
$$= x^2(y+z)+x(y^2+z^2+2yz)+yz(y+z)$$
$$= x^2(y+z)+y^2(x+z)+z^2(x+y)+2xyz,$$

and $\therefore$ from the given equations it $= a^2+b^2+c^2+2abc.$

Hence, from (α), we have $a^2+b^2+c^2+2abc = 1$, as the required Eliminant

Exercise (98).

Eliminate x from the equations :—

1. $\left.\begin{aligned} a^2x^2-b^2 &= 0 \\ cx-d &= 0 \end{aligned}\right\}$ **2.** $\left.\begin{aligned} ax^2-b &= 0 \\ cx^3-d &= 0 \end{aligned}\right\}$

3. $\left.\begin{aligned} mx^3-n &= 0 \\ px^4-q &= 0 \end{aligned}\right\}$ **4** $\left.\begin{aligned} ax^2+bx+c &= 0 \\ x+d &= 0 \end{aligned}\right\}$

5. $\left.\begin{aligned} lx^2+mx+n &= 0 \\ ax+b &= 0 \end{aligned}\right\}$ **6** $\left.\begin{aligned} ax^2+bx+c &= 0 \\ lx^2+mx+n &= 0 \end{aligned}\right\}$

7. $\left.\begin{aligned} x+\frac{1}{x} &= a+b \\ x-\frac{1}{x} &= a-b \end{aligned}\right\}$

8. $\left.\begin{aligned} 2x+\frac{3}{x} &= 5p+7q \\ 2x-\frac{3}{x} &= 5p-7q \end{aligned}\right\}$

9. $\left.\begin{aligned} a_1x^3+b_1x+c_1 &= 0 \\ a_2x^3+b_2x+c_2 &= 0 \end{aligned}\right\}$

10. $\left.\begin{aligned} a_1x^3+b_1x^2+c_1 &= 0 \\ a_2x^3+b_2x^2+c_2 &= 0 \end{aligned}\right\}$

11. $\left.\begin{aligned} a_1x^4+b_1x^3+c_1 &= 0 \\ a_2x^4+b_2x^3+c_2 &= 0 \end{aligned}\right\}$

12. $\left.\begin{aligned} ax^3+bx+c &= 0 \quad (1) \\ x^2+mx+n &= 0 \quad (2) \end{aligned}\right\}$

13. $\left.\begin{aligned} ax^2+bx+c &= 0 \\ x^3+2x^2+3 &= 0 \end{aligned}\right\}$

[Multiply (2) by ax and subtract (1) from the resulting equation we thus get $amx^2+(an-b)x-c=0$ Now eliminate x from this equation and (2)]

Eliminate x and y from the equations —

14. $\left.\begin{aligned} ax+by &= m \\ bx-ay &= n \\ x^2+y^2 &= 1 \end{aligned}\right\}$

15. $\left.\begin{aligned} ax+b &= cy \\ a_1y+b_1 &= c_1x \\ x^2+y^2 &= 1 \end{aligned}\right\}$

16. $\left.\begin{aligned} ax+by &= 0 \\ lx^2+mxy+ny^2 &= 0 \end{aligned}\right\}$

Eliminate x, y, z from the equations —

17. $\dfrac{x}{y+z} = a, \quad \dfrac{y}{z+x} = b, \quad \dfrac{z}{x+y} = c$

18. $\dfrac{y-z}{y+z} = a, \quad \dfrac{z-x}{z+x} = b, \quad \dfrac{x-y}{x+y} = c$

19. $\dfrac{y}{z}+\dfrac{z}{y} = a, \quad \dfrac{z}{x}+\dfrac{x}{z} = b, \quad \dfrac{x}{y}+\dfrac{y}{x} = c.$

[Example 6, page 184, may be consulted with profit]

20. $x^2(y-z) = a, \ y^2(z-x) = b, \ z^2(x-y) = c, \ xyz = a.$

21. Eliminate a, b, c from the equations —

$bz+cy = a, \ az+cx = b, \ ay+bx = c$

2. Miscellaneous Artifices. We shall now work out some examples which require for their solution either the application of some principle with which the student is not already acquainted or some special artifice

Example 1. Express $(x+3a)(x+5a)(x+7a)(x+9a)$ as the difference of two square quantities. (C U Entr Paper, 1887)

The given expression

$$= \{(x+3a)(x+9a)\}\{(x+5a)(x+7a)\}$$
$$= \{x^2+12ax+27a^2\}\{x^2+12ax+35a^2\}$$
$$= \{(x^2+12ax+31a^2)-4a^2\}\{(x^2+12ax+31a^2)+4a^2\}$$
$$= (x^2+12ax+31a^2)^2-16a^4.$$

Example 2. A man receives $\frac{x}{y}$ths of Rs. 10 and afterwards $\frac{y}{x}$ths of Rs 10 He then gives away Rs 20. Show that he cannot lose by the transaction

(C U Entrance Paper 1881)

The man receives altogether $\left(\frac{x}{y}+\frac{y}{x}\right)$. 10 rupees and gives away 20 rupees

Clearly therefore he loses

$$\text{if } \left(\frac{x}{y}+\frac{y}{x}\right).10 < 20$$

$$\text{i e, if } \frac{x}{y}+\frac{y}{x} < 2$$

$$\text{i e, if } x^2+y^2 < 2xy$$

$$\text{i e, if } x^2+y^2-2xy < 0$$

i. e, if $(x-y)^2$ be a negative quantity.

But, whichever of x and y may be the greater, $(x-y)^2$ can *never* be negative

Hence, the man *cannot* lose.

Note It may be observed that there is always a gain in this transaction *except when* $x=y$

Example 3. If $\frac{a}{b+c}+\frac{c}{a+b}=\frac{2b}{c+a}$, prove that

$$a^2+c^2=2b^2, \quad \text{or} \quad a+b+c=0.$$

From the given relation, we have

$$\frac{a}{b+c}-\frac{b}{c+a}=\frac{b}{c+a}-\frac{c}{a+b},$$

$$\text{or,} \quad \frac{c(a-b)+a^2-b^2}{(b+c)(c+a)}=\frac{a(b-c)+b^2-c^2}{(c+a)(a+b)},$$

$$\text{or,}\quad \frac{(a-b)(c+\overline{a-b})}{b+c} = \frac{(b-c)(a+\overline{b+c})}{a+b},$$

$$\text{or,}\quad (a^2-b^2)(a+b+c) = (b^2-c^2)(a+b+c),$$

$$\text{or,}\quad (a+b+c)\{(a^2-b^2)-(b^2-c^2)\} = 0,$$

$$\text{or,}\quad (a+b+c)(a^2+c^2-2b^2) = 0$$

Therefore *either* $\qquad a+b+c = 0$,

or, $\qquad a^2+c^2-2b^2 = 0$,

and $\qquad \therefore \quad a^2+c^2 = 2b^2$

Note It may be observed in this connection that whenever any relation of equality is reduced to the form $xp = xp_1$ [or $x(p-p_1)=0$], it is obviously satisfied *either* (i) when $x=0$, *or* (ii) when $p=p_1$, and that of these two alternative results we cannot accept one as the *only* conclusion to which we are led *unless* it is known that the other is impossible

In the present example we have got $(a^2-b^2)(a+b+c) = (b^2-c^2)(a+b+c)$ as one of the steps in the solution, and it is not difficult to see from this that it would be a mistake to remove the common factor $a+b+c$ from both sides and set down $a^2-b^2 = b^2-c^2$ as the next step, for the above relation may be true *not* on account of a^2-b^2 being equal to b^2-c^2 *but* on account of $a+b+c$ being equal to *zero* We might remove $a+b+c$ from both sides of the equation however *if* we know that owing to certain restrictions on the values of the letters a, b, c, the expression $a+b+c$ could not possibly vanish

Hence the only legitimate conclusion from the relation $xp = xp_1$, [or $x(p-p_1)=0$], is "*either* $x=0$ *or* $p=p_1$" but *not* simply "$p=p_1$", *except when* x is known to be *not equal to zero*

Example 4. Show that if $\dfrac{a-b}{c}+\dfrac{b-c}{a}+\dfrac{c+a}{b} = 1$, and $a-b+c$ is *not* $=0$, then $\dfrac{1}{a} = \dfrac{1}{b}+\dfrac{1}{c}$.

(Calcutta University Entrance Paper, 1875)

From the given relation, we have

$$1-\frac{b-c}{a} = \frac{a-b}{c}+\frac{c+a}{b}$$

$$\text{or,}\quad \frac{a-b+c}{a} = \frac{b(a-b)+c(c+a)}{bc}$$

$$= \frac{b(a-b+c)+c(c+a-b)}{bc}$$

$$= \frac{(a-b+c)(b+c)}{bc}$$

Hence, either $a-b+c = 0$,

or, $\dfrac{1}{a} = \dfrac{b+c}{bc}$ [See note, last example.]

But by hypothesis $a-b+c$ is *not* zero;

therefore, we must have $\dfrac{1}{a} = \dfrac{b+c}{bc} = \dfrac{1}{b}+\dfrac{1}{c}$.

Example 5. Show that if $(x^2+y^2+z^2)(a^2+b^2+c^2) = (ax+by+cz)^2$, then $\dfrac{x}{a} = \dfrac{y}{b} = \dfrac{z}{c}$.

From the given relation, we have

$$a^2(y^2+z^2)+b^2(x^2+z^2)+c^2(x^2+y^2) = 2abxy+2acxz+2bcyz.$$

Hence, by transposition,

$$(a^2y^2+b^2x^2-2abxy)+(a^2z^2+c^2x^2-2acxz)+(b^2z^2+c^2y^2-2bcyz) = 0,$$

or, $(ay-bx)^2+(az-cx)^2+(bz-cy)^2 = 0$

Now, as *none* of the terms of the left-hand expression is *negative*, this equation cannot hold unless *each* of those terms is zero.

Hence,

$$\left.\begin{aligned} ay-bx &= 0, \quad \therefore \ \frac{x}{a} = \frac{y}{b} \\ az-cx &= 0, \quad \therefore \ \frac{x}{a} = \frac{z}{c} \\ bz-cy &= 0, \quad \therefore \ \frac{y}{b} = \frac{z}{c} \end{aligned}\right\}$$

Thus we have $\dfrac{x}{a} = \dfrac{y}{b} = \dfrac{z}{c}$

Example 6. If $a+b+c = 0$, show that

$$2(a^4+b^4+c^4) = (a^2+b^2+c^2)^2.$$

From the given relation, we have

$$a+b = -c, \quad \therefore \ a^2+2ab+b^2 = c^2,$$

$$\therefore \ a^2+b^2-c^2 = -2ab,$$

$$\therefore \ (a^2+b^2-c^2)^2 = 4a^2b^2$$

or, $a^4+b^4+c^4+2a^2b^2-2a^2c^2-2b^2c^2 = 4a^2b^2$;

$$\therefore\ a^4+b^4+c^4 = 2(a^2b^2+b^2c^2+c^2a^2).$$

Hence, $2(a^4+b^4+c^4) = a^4+b^4+c^4+2(a^2b^2+b^2c^2+c^2a^2)$

$$= (a^2+b^2+c^2)^2$$

Example 7. If $a+b+c = 0$, shew that

$$\frac{1}{b^2+c^2-a^2}+\frac{1}{c^2+a^2-b^2}+\frac{1}{a^2+b^2-c^2} = 0$$

From the given relation, we have

$$a+b = -c, \quad \therefore \quad a^2+2ab+b^2 = c^2,$$

$$\therefore \quad a^2+b^2-c^2 = -2ab.$$

Similarly, $b^2+c^2-a^2 = -2bc$, and $c^2+a^2-b^2 = -2ca$

Hence, the proposed expression

$$= \frac{1}{-2bc}+\frac{1}{-2ca}+\frac{1}{-2ab}$$

$$= \frac{a+b+c}{-2abc} = \frac{0}{-2abc} = 0$$

Example 8. If $a+b+c = 0$, shew that

$$\frac{a^2}{2a^2+bc}+\frac{b^2}{2b^2+ca}+\frac{c^2}{2c^2+ab} = 1.$$

We have $2a^2+bc = a^2+a\cdot a+bc$

$$= a^2-a(b+c)+bc \quad [\because\ a=-(b+c)]$$

$$= (a-b)(a-c)$$

Similarly, $2b^2+ca = b^2-b(a+c)+ca$

$$= (b-c)(b-a),$$

and $2c^2+ab = c^2-c(a+b)+ab$

$$= (c-a)(c-b).$$

Hence, the proposed expression

$$= \frac{a^2}{(a-b)(a-c)}+\frac{b^2}{(b-c)(b-a)}+\frac{c^2}{(c-a)(c-b)}$$

$$= \frac{a^2}{(a-b)(a-c)}+\frac{-b^2}{(b-c)(a-b)}+\frac{c^2}{(a-c)(b-c)}$$

$$= \frac{a^2(b-c)-b^2(a-c)+c^2(a-b)}{(a-b)(a-c)(b-c)}$$

$$= \frac{a^2(b-c)+b^2(c-a)+c^2(a-b)}{(a-b)(a-c)(b-c)}$$

$$= \frac{(a-b)(a-c)(b-c)}{(a-b)(a-c)(b-c)} = 1. \qquad \text{[Formula 19, Page 96.]}$$

Example 9. Prove that $\left(x+\frac{1}{x}\right)^2+\left(y+\frac{1}{y}\right)^2+\left(z+\frac{1}{z}\right)^2$

$$= 4+\left(x+\frac{1}{x}\right)\left(y+\frac{1}{y}\right)\left(z+\frac{1}{z}\right), \text{ if } xyz = 1.$$

$$\left(x+\frac{1}{x}\right)^2+\left(y+\frac{1}{y}\right)^2 = \left(x^2+2+\frac{1}{x^2}\right)+\left(y^2+2+\frac{1}{y^2}\right)$$

$$= 4+\left(x^2+y^2\right)+\left(\frac{1}{x^2}+\frac{1}{y^2}\right)$$

$$= 4+xy\left(\frac{x}{y}+\frac{y}{x}\right)+\frac{1}{xy}\left(\frac{y}{x}+\frac{x}{y}\right)$$

$$= 4+\left(\frac{x}{y}+\frac{y}{x}\right)\left(xy+\frac{1}{xy}\right)$$

$$= 4+\left(\frac{x}{y}+\frac{y}{x}\right)\left(\frac{1}{z}+z\right) \qquad [\because xyz=1]$$

Hence, the given expression

$$= 4+\left(\frac{x}{y}+\frac{y}{x}\right)\left(z+\frac{1}{z}\right)+\left(z+\frac{1}{z}\right)^2$$

$$= 4+\left(z+\frac{1}{z}\right)\left(\frac{x}{y}+\frac{y}{x}+z+\frac{1}{z}\right)$$

$$= 4+\left(z+\frac{1}{z}\right)\left\{\left(\frac{x}{y}+\frac{1}{z}\right)+\left(\frac{y}{x}+z\right)\right\}$$

$$= 4+\left(z+\frac{1}{z}\right)\left\{\left(\frac{x}{y}+xy\right)+\left(\frac{y}{x}+\frac{1}{xy}\right)\right\} \qquad [\because xyz=1]$$

$$= 4+\left(z+\frac{1}{z}\right)\left\{x\left(\frac{1}{y}+y\right)+\frac{1}{x}\left(y+\frac{1}{y}\right)\right\}$$

$$= 4+\left(z+\frac{1}{z}\right)\left(y+\frac{1}{y}\right)\left(x+\frac{1}{x}\right).$$

Example 10. If $xy+yz+zx=1$, show that

$$\frac{x}{1-x^2}+\frac{y}{1-y^2}+\frac{z}{1-z^2}=\frac{4xyz}{(1-x^2)(1-y^2)(1-z^2)}.$$

Since $xy+yz+zx=1$, we must have

$$\left.\begin{array}{ll} xy+yz=1-zx, \text{ or, } y(x+z)=1-zx & \text{(i)} \\ yz+zx=1-xy, \text{ or, } z(y+x)=1-xy & \text{(ii)} \\ zx+xy=1-yz, \text{ or, } x(z+y)=1-yz & \text{(iii)} \end{array}\right\},$$

Now, the given expression

$$=\frac{x(1-y^2)(1-z^2)+y(1-z^2)(1-x^2)+z(1-x^2)(1-y^2)}{(1-x^2)(1-y^2)(1-z^2)}$$

of which the numerator

$$=x\{1-(y^2+z^2)+y^2z^2\}+y\{1-(z^2+x^2)+z^2x^2\}$$
$$+z\{1-(x^2+y^2)+x^2y^2\}$$
$$=(x+y+z)-y^2(z+x)-z^2(x+y)-x^2(y+z)$$
$$+xyz(yz+zx+xy)$$
$$=(x+y+z)-y\{y(z+x)\}-z\{z(x+y)\}-x\{x(y+z)\}+xyz\,1$$
$$=(x+y+z)-y(1-zx)-z(1-xy)-x(1-yz)+xyz$$

[by (i), (ii), and (iii)],

$$=(x+y+z)-(y+z+x)+3xyz+xyz$$
$$=4xyz$$

Hence, the given expression $=\dfrac{4xyz}{(1-x^2)(1-y^2)(1-z^2)}$.

Example 11. If $x+a$ be the H. C. F. of x^2+px+q, and $x^2+p'x+q'$, show that $a=\dfrac{q-q'}{p-p'}$.

Dividing x^2+px+q by $x+a$,

$$\begin{array}{l} x+a\,)\,x^2+px+q\,(\,x+p-a \\ \qquad\quad x^2+ax \\ \hline \qquad\quad (p-a)x+q \\ \qquad\quad (p-a)x+(p-a)a \\ \hline \qquad\qquad\qquad q-(p-a)a \end{array}$$

the remainder is found to be $q-(p-a)a$.

Similarly, dividing $x^2+p'x+q'$ by $x+a$, the remainder becomes $q'-(p'-a)a$.

Now, since each of the given expressions is divisible by $x+a$ *without remainder*, the remainders thus found must be zero

$$\left.\begin{array}{ll}\text{Hence,} & q-(p-a)a = 0 \\ \text{and} & q'-(p'-a)a = 0\end{array}\right\}$$

$$\therefore \quad q-q'-(p-p')a = 0,$$

$$\text{or,} \quad q-q' = (p-p')a,$$

$$\therefore \quad a = \frac{q-q'}{p-p'}.$$

Example 12. By performing the operation for extracting the square root, find the value of x which will make $x^4+6x^3+11x^2+3x+31$ a perfect square

$$\begin{array}{rl}
 & x^4+6x^3+11x^2+3x+31\,(x^2+3x+1 \\
 & x^4 \\ \hline
2x^2+3x) & 6x^3+11x^2 \\
 & 6x^3+\ \ 9x^2 \\ \hline
2x^2+6x+1) & 2x^2+3x+31 \\
 & 2x^2+6x+\ \ 1 \\ \hline
 & -3x+30
\end{array}$$

Now, in order that the given expressions may be a perfect square, the remainder $(-3x+30)$ must be $= 0$, and therefore $3x = 30$, or $x = 10$

Hence, when $x = 10$, the given expression is a perfect square.

Example 13. If $x(b-c)+y(c-a)+z(a-b) = 0$,

then will $\dfrac{bz-cy}{b-c} = \dfrac{cx-az}{c-a} = \dfrac{ay-bx}{a-b}$.

$$\left.\begin{array}{ll}\text{We have} & x(b-c)+y(c-a)+z(a-b) = 0 \\ \text{and identically also,} & a(b-c)+b(c-a)+c(a-b) = 0\end{array}\right\}$$

Hence, by cross multiplication,

$$\frac{b-c}{cy-bz} = \frac{c-a}{az-cx} = \frac{a-b}{bx-ay},$$

$$\text{whence,} \quad \frac{bz-cy}{b-c} = \frac{cx-az}{c-a} = \frac{ay-bx}{a-b}.$$

Example 14. Solve

$$\left.\begin{aligned} x+y+z &= a+b+c \quad . \quad (1)\\ \frac{x}{a}+\frac{y}{b}+\frac{z}{c} &= 3 \quad . \quad . \quad (2)\\ \frac{x}{a^2}+\frac{y}{b^2}+\frac{z}{c^2} &= \frac{1}{a}+\frac{1}{b}+\frac{1}{c} \quad (3)\end{aligned}\right\}$$

From (1), $(x-a)+(y-b)+(z-c) = 0$

From (2), $\frac{1}{a}(x-a)+\frac{1}{b}(y-b)+\frac{1}{c}(z-c) = 0.$

Hence, by cross multiplication,

$$\frac{x-a}{\frac{1}{c}-\frac{1}{b}} = \frac{y-b}{\frac{1}{a}-\frac{1}{c}} = \frac{z-c}{\frac{1}{b}-\frac{1}{a}},$$

and supposing each of these fractions $= k$, we have

$$x-a = k\frac{b-c}{bc},\ y-b = k\frac{c-a}{ac};\ z-c = k.\frac{a-b}{ab} \ \ldots\ldots(\alpha)$$

Now from (3), $\frac{1}{a^2}(x-a)+\frac{1}{b^2}(y-b)+\frac{1}{c^2}(z-c) = 0.$

Substituting in this equation the values of $x-a$, $y-b$, $z-c$, found above, we have

$$k\left\{\frac{b-c}{bc}\cdot\frac{1}{a^2}+\frac{c-a}{ca}\cdot\frac{1}{b^2}+\frac{a-b}{ab}\cdot\frac{1}{c^2}\right\} = 0$$

or, $$k.\frac{bc(b-c)+ca(c-a)+ab(a-b)}{a^2b^2c^2} = 0$$

or, $$k.\frac{(b-c)(a-b)(a-c)}{a^2b^2c^2} = 0 \text{ [Formula 20, page 96]}$$

$$\therefore\ k = 0,$$

since a, b, c being impliedly unequal, none of the factors $b-c$, $a-b$, $a-c$ is zero.

Hence from (α),

$$\left.\begin{aligned} x-a &= 0, \text{ or, } x = a\\ y-b &= 0, \text{ or, } y = b\\ z-c &= 0, \text{ or, } z = c\end{aligned}\right\}.$$

Example 15. If $x = cy + bz$, $y = az + cx$, and $z = bx + ay$, show that $\dfrac{x^2}{1-a^2} = \dfrac{y^2}{1-b^2} = \dfrac{z^2}{1-c^2}$.

From the given relations, we have

$$\left.\begin{aligned} x - cy - bz &= 0 \quad \ldots\ldots \quad (1)\\ cx - y + az &= 0 \quad \ldots\ldots \quad (2)\\ bx + ay - z &= 0 \quad \ldots\ldots \quad (3)\end{aligned}\right\}$$

From (1) and (2), by cross multiplication,

$$\frac{x}{-ac-b} = \frac{y}{-bc-a} = \frac{z}{-1+c^2}$$

$$\text{or,}\quad \frac{x}{ac+b} = \frac{y}{bc+a} = \frac{z}{1-c^2} \quad \ldots\ldots \quad (4)$$

Similarly from (2) and (3),

$$\frac{x}{1-a^2} = \frac{y}{ab+c} = \frac{z}{ac+b} \quad \ldots\ldots \quad (5)$$

and from (1) and (3),

$$\frac{x}{ab+c} = \frac{y}{1-b^2} = \frac{z}{bc+a} \quad \ldots\ldots \quad (6)$$

Now, from (4) and (5),

$$\left.\begin{aligned} &\frac{x}{ac+b} = \frac{z}{1-c^2}\\ \text{and}\quad &\frac{x}{1-a^2} = \frac{z}{ac+b}\end{aligned}\right\} \text{whence,}\quad \frac{x^2}{1-a^2} = \frac{z^2}{1-c^2}.$$

Again, from (5) and (6),

$$\left.\begin{aligned} &\frac{x}{1-a^2} = \frac{y}{ab+c}\\ \text{and}\quad &\frac{x}{ab+c} = \frac{y}{1-b^2}\end{aligned}\right\} \text{whence,}\quad \frac{x^2}{1-a^2} = \frac{y^2}{1-b^2}.$$

Hence, $\dfrac{x^2}{1-a^2} = \dfrac{y^2}{1-b^2} = \dfrac{z^2}{1-c^2}$.

Example 16. Show that if $ax + by + cz = 0$, an

$$\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 0, \text{ then will}$$

$$ax^2 + by^2 + cz^2 + (a+b+c)(xy + yz + zx) = 0.$$

From the given relations, we have

$$\left.\begin{aligned} ax+by+cz &= 0 \\ \text{and} \qquad ayz+bzx+cxy &= 0 \end{aligned}\right\}$$

Hence, by cross multiplication,

$$\frac{a}{x(y^2-z^2)} = \frac{b}{y(z^2-x^2)} = \frac{c}{z(x^2-y^2)}$$

and $\therefore$ each of these ratios

$$= \frac{ax^2+by^2+cz^2}{x^3(y^2-z^2)+y^3(z^2-x^2)+z^3(x^2-y^2)},$$

and also $= \dfrac{a+b+c}{x(y^2-z^2)+y(z^2-x^2)+z(x^2-y^2)}$,

[Art 14, Chap XIX]

Thus we have $\dfrac{ax^2+by^2+cz^2}{x^3(y^2-z^2)+y^3(z^2-x^2)+z^3(x^2-y^2)}$

$$= \frac{a+b+c}{x(y^2-z^2)+y(z^2-x^2)+z(x^2-y^2)}.$$

Hence, $\dfrac{ax^2+by^2+cz^2}{a+b+c}$

$$= \frac{x^3(y^2-z^2)+y^3(z^2-x^2)+z^3(x^2-y^2)}{x^2(z-y)+y^2(x-z)+z^2(y-x)}$$

$$= \frac{(y-z)(x-z)(x-y)(xy+yz+zx)}{-(y-z)(x-z)(x-y)}$$ [See Example 10, page 125 and formula 19, page 96]

$$= -(xy+yz+zx),$$

whence, $ax^2+by^2+cz^2+(a+b+c)(xy+yz+zx) = 0$

Example 17. If $\dfrac{x}{a} = \dfrac{y}{b}$, show that

$$\frac{x^3+a^3}{x^2+a^2}+\frac{y^3+b^3}{y^2+b^2} = \frac{(x+y)^3+(a+b)^3}{(x+y)^2+(a+b)^2}.$$

Let each of the given ratios $= k$. Then we have $x = ak$ and $y = bk$.

Hence, $\dfrac{x^3+a^3}{x^2+a^2} = \dfrac{y^3+b^3}{y^2+b^2} = \dfrac{a^3(k^3+1)}{a^2(k^2+1)}+\dfrac{b^3(k^3+1)}{b^2(k^2+1)}$

$$= \frac{a(k^3+1)}{k^2+1}+\frac{b(k^3+1)}{k^2+1}$$

$$= \frac{(k^3+1)(a+b)}{k^2+1}$$

$$= \frac{(k^3+1)(a+b)^3}{(k^2+1)(a+b)^2}$$

$$= \frac{k^3(a+b)^3+(a+b)^3}{k^2(a+b)^2+(a+b)^2}$$

$$= \frac{(ka+kb)^3+(a+b)^3}{(ka+kb)^2+(a+b)^2}$$

$$= \frac{(x+y)^3+(a+b)^3}{(x+y)^2+(a+b)^2}.$$

Example 18. Show that $(bcd+cda+dab+abc)^2$
$-abcd(a+b+c+d)^2 = (bc-ad)(ca-bd)(ab-cd)$

We have $(bcd+cda+dab+abc)^2$

$$= \{cd(a+b)+ab(c+d)\}^2$$

$$= c^2d^2(a+b)^2+2abcd(a+b)(c+d)+a^2b^2(c+d)^2,$$

and $(a+b+c+d)^2 = (a+b)^2+2(a+b)(c+d)+(c+d)^2$

Hence, the given expression

$$= c^2d^2(a+b)^2+a^2b^2(c+d)^2-abcd(a+b)^2-abcd(c+d)^2$$

$$= ab(c+d)^2(ab-cd)-cd(a+b)^2(ab-cd)$$

$$= (ab-cd)\{ab(c+d)^2-cd(a+b)^2\}$$

$$= (ab-cd)\{ab(c^2+d^2)-cd(a^2+b^2)\}$$

$$= (ab-cd)\{ac(bc-ad)-bd(bc-ad)\}$$

$$= (ab-cd)(bc-ad)(ac-bd).$$

Example 19. Show that the following expression is an exact square —

$$(x^2-yz)^3+(y^2-zx)^3+(z^2-xy)^3-3(x^2-yz)(y^2-zx)(z^2-xy).$$

Putting a for x^2-yz, b for y^2-zx and c for z^2-xy, we have the given expression $= a^3+b^3+c^3-3abc$

$$= (a+b+c)(a^2+b^2+c^2-bc-ca-ab)$$

(Formula 18 page 96.)

$$= \tfrac{1}{2}(a+b+c)\{(a-b)^2+(b-c)^2+(c-a)^2\} \quad \ldots \quad (1)$$

Now, $\quad a-b = (x^2-yz)-(y^2-zx)$

$= (x^2-y^2)+z(x-y)$

$= (x-y)(x+y+z).$

Similarly, $\quad b-c = (y-z)(x+y+z),$

and $\quad c-a = (z-x)(x+y+z)$,

whence $\quad (a-b)^2+(b-c)^2+(c-a)^2$

$= (x+y+z)^2\{(x-y)^2+(y-z)^2+(z-x)^2\}$

$= 2(x+y+z)^2(x^2+y^2+z^2-yz-zx-xy). \qquad (2)$

Also, $\quad a+b+c = x^2+y^2+z^2-yz-zx-xy. \qquad (3)$

Therefore from (1), (2) and (3), the given expression

$= \frac{1}{2}(x^2+y^2+z^2-yz-zx-xy)$

$\times\{2(x+y+z)^2(x^2+y^2+z^2-yz-zx-xy)\}$

$= \{(x+y+z)(x^2+y^2+z^2-yz-zx-xy)\}^2$

$= (x^3+y^3+z^3-3xyz)^2.$

Example 20. If $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} = \frac{1}{a+b+c}$, show that

$$\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^{2n+1} = \frac{1}{a^{2n+1}+b^{2n+1}c^{2n+1}},$$

where n is any positive integer

From the given relation, we have

$$\frac{bc+a(b+c)}{abc}-\frac{1}{a+b+c} = 0$$

$\therefore \quad \{a(b+c)+bc\}\{a+(b+c)\}-abc = 0.$

Now, the left-hand expression

$= a^2(b+c)+a(b+c)^2+bc(b+c)$

$= (b+c)\{a^2+a(b+c)+bc\}$

$= (b+c)(a+b)(a+c)$;

$\therefore \quad (b+c)(a+b)(a+c) = 0.$

Hence, either $b+c = 0$, or, $a+b = 0$, or, $a+c = 0$.

Taking $b+c = 0$ we have $c = -b$.

Hence, $\quad \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^{2n+1} = \left(\frac{1}{a}\right)^{2n+1}$

$\left[\because \quad \frac{1}{b}+\frac{1}{c} = 0\right.$

$$= \frac{1}{a^{2n+1}}$$

$$= \frac{1}{a^{2n+1}+b^{2n+1}-b^{2n+1}}$$

$$= \frac{1}{a^{2n+1}+b^{2n+1}+c^{2n+1}}$$

[$\because\ c^{2n+1} = (-b)^{2n+1} = -b^{2n+1}$, see foot note, page 109]

The same result would follow if either $a+c$ or $a+b$ were taken equal to zero

Example 21. Having given $x=by+cz+du$, $y=ax+cz+du$, $z=ax+by+du$, and $u=ax+by+cz$, show that

$$\frac{a}{1+a}+\frac{b}{1+b}+\frac{c}{1+c}+\frac{d}{1+d} = 1$$

Putting P for $ax+by+cz+du$, we have

$x+ax = (by+cz+du)+ax$

$$= P, \text{ or, } x(1+a) = P, \quad \therefore \frac{1}{1+a} = \frac{x}{P} \quad \ldots (1)$$

$y+by = (ax+cz+du)+by$

$$= P, \text{ or, } y(1+b) = P, \quad \therefore \frac{1}{1+b} = \frac{y}{P} \quad \ldots (2)$$

$z+cz = (ax+by+du)+cz$

$$= P, \text{ or, } z(1+c) = P, \quad \therefore \frac{1}{1+c} = \frac{z}{P} \quad \ldots (3)$$

$u+du = (ax+by+cz)+du$

$$= P, \text{ or, } u(1+d) = P, \quad \therefore \frac{1}{1+d} = \frac{u}{P} \quad \ldots (4)$$

Hence, from (1), (2), (3) and (4), we have

$$\frac{a}{1+a}+\frac{b}{1+b}+\frac{c}{1+c}+\frac{d}{1+d} = \frac{ax}{P}+\frac{by}{P}+\frac{cz}{P}+\frac{du}{P}$$

$$= \frac{ax+by+cz+du}{P}$$

$$= 1$$

Miscellaneous Exercises (4).

I.

1. Find the value of $\sqrt{(x^3+y^2+z)(x-y-3z)}-\sqrt[3]{xy^2z^2}$, when $x=-1$, $y=-8$, $z=1$.
2. Simplify $3a-2(b-c)-\{2(a-b)-3(c+a)\}-\{9c-4(c-a)\}$.
3. Resolve into factors $3(a+b)^2-2(a^2-b^2)-a(a+b)$.
4. Divide $2x^4-10x^3y+25x^2y^2-31xy^3+20y^4$ by $x^2-3xy+4y^2$.
5. Simplify $\dfrac{b}{a+b}-\dfrac{ab}{(a+b)^2}-\dfrac{ab^2}{(a+b)^3}$.
6. Solve the equation $\dfrac{x+a}{x-a}-\dfrac{x-b}{x+b}=\dfrac{2(a+b)}{x}$.
7. If $\left(x+\dfrac{1}{x}\right)^2=3$, prove that $x^3+\dfrac{1}{x^3}=0$.
8. Simplify $\dfrac{3\sqrt{3}-2\sqrt{2}}{\sqrt{3}-\sqrt{2}}$.

II

1. Find the value of $(2a+b)(a-b)+(2b+c)(b-c)+(2c+a)(c-a)$, when $a=1$, $b=2$, $c=-3$.
2. Divide $1+3x-24x^2+8x^4$ by $2x^2+3x-1$.
3. If x^2+7x+c is exactly divisible by $x+4$, what is the value of c?
4. Simplify $\dfrac{1}{2\sqrt{7}-3\sqrt{2}}-\dfrac{1}{2\sqrt{7}+3\sqrt{2}}$.
5. Find the H. C. F. of $x^4-3x^3-2x^2+12x-8$ and x^3-7x+6.
6. Simplify $\left(1+\dfrac{35}{x-7}-\dfrac{15}{x-3}\right)\left(\dfrac{1}{5}-\dfrac{7}{x+7}+\dfrac{3}{x+3}\right)$.
7. Solve the equation $\dfrac{x+1}{6}+\dfrac{3x-1}{8}-\dfrac{5x-7}{12}+1=\dfrac{7x-5}{24}$.
8. If $x-\dfrac{1}{x}=1$, prove that $x^3-\dfrac{1}{x^3}=4$.

III.

1. Find the value of $\{a^2(b^3-c^3)+b^2(c^3-a^3)+c^2(a^3-b^3)\}\div(bc+ca+ab)$, when $a=3$, $b=-2$, $c=4$.

2. Simplify $\frac{1+x}{1-x}+\frac{1-x}{1+x}-\frac{1+x^2}{1-x^2}-\frac{1-x^2}{1+x^2}$.

3. Resolve into factors $a^2-b^2+6bc-9c^2$.

4. Find the H. C. F. of $x^3+5ax^2-5a^2x-a^3$ and $5x^3-3ax^2-5a^2x+3a^3$.

5. Find the L. C. M. of x^2-5x+6, x^2-4x+3 and x^2-3x+2.

6. Reduce to its lowest terms $\frac{x^5+5x^4+8x^3+4x^2}{x^5+x^4+8x^2+8x}$.

7. Solve $\frac{1}{x+3}+\frac{1}{x-2}=\frac{2}{x-7}$.

8. If $a : b :: x : y$, show that

$$ab : xy :: a^2+b^2 : x^2+y^2.$$

IV.

1. Simplify $\frac{\sqrt{x^2-y^2}+x}{\sqrt{x^2+y^2}+y}-\frac{\sqrt{x^2+y^2}-y}{x-\sqrt{x^2-y^2}}$.

2. If the product of two expressions be $x^8+x^4y^4+y^8$ and one of them be x^2-xy+y^2, find the other.

3. Resolve into factors —

(i) x^3+x^2-x-1; (ii) $a^2b^2-a^2-b^2+1$.

4. Show that $(ax+by)^2+(bx-ay)^2=(a^2+b^2)(x^2+y^2)$.

5. Find the L. C. M. of $8x^3+27$, $16x^4+36x^2+81$ and $6x^2-5x-6$.

6. Solve $\frac{3x-4}{x}+\frac{2}{4x+3}=3$.

7. Find x and y if $\frac{bx+ay}{2ab}=\frac{by-ax}{b^2-a^2}=ab$.

8. If $\frac{x}{a+b-c}=\frac{y}{a-b+c}=\frac{z}{b+c-a}$, show that each of these fractions $=\frac{x+y+z}{a+b+c}$.

V.

1. Simplify $\dfrac{x^2-25y^2}{x^2+3xy-10y^2} \times \dfrac{x^2-4y^2}{x^2-3xy-10y^2}$.

2 Divide $a^3(b-c)+b^3(c-a)+c^3(a-b)$ by $a+b+c$, and find the factors of the quotient

3. Find the value of $\dfrac{x^3-y^3}{x^3+y^3}$, when $x=a+3$, $y=a-3$.

4. Find the square root of

$$24+\frac{x^2}{y}+8\left(\frac{2y}{x^2}-xy^{-\frac{1}{2}}\right)-\frac{32\sqrt{y}}{x}.$$

5. Show that $(a^2+b^2+c^2)(x^2+y^2+z^2)-(ax+by+cz)^2$
$= (ay-bx)^2+(bz-cy)^2+(cx-az)^2$

6 Subtract $\dfrac{7-2\sqrt{5}}{4-\sqrt{5}}$ from $\dfrac{15+6\sqrt{5}}{2+\sqrt{5}}$

7. Solve $\left.\begin{array}{r} 2^x \times 4^y = 32 \\ 3^x - 9^y = 3 \end{array}\right\}$

8. If $a : b :: c : d$, show that $(a^2+c^2)(b^2+d^2)$
$= (ab+cd)^2$

VI

1. Reduce to its simplest form the expression
$$\frac{2a(1-x^2)^2}{yz} - \frac{(1+x)^2(1-x)}{y^2} \div \frac{2ay^2(1-x)}{z}.$$

2. Multiply $a+b+\dfrac{b^2}{a}+\dfrac{a^2}{b}$ by $a-b+\dfrac{b^2}{a}-\dfrac{a^2}{b}$

3. Divide $x^4-2bx^3-(a^2-b^2)x^2+2a^2bx-a^2b^2$ b $x^2-(a+b)x+ab$.

4. If $a=y+z$, $b=z+x$, $c=x+y$, then $a^2+b^2+c^2-bc-ca-ab = x^2+y^2+z^2-yz-zx-xy$.

5. Reduce $\dfrac{5x^3-14x^2+16}{3x^3-2x^2+16x-48}$ to its lowest terms.

6. Solve $\dfrac{2}{x}+\dfrac{7}{y}= 29$, $\dfrac{5}{x}-\dfrac{6}{y}=2$.

7. Solve $2x+3y-8z+35=0$, $7x-4y+z-8=0$, $12x-5y-3z+10=0$

8. If $a : b = c : d = e : f$, prove that

$$a : b :: \sqrt{m^2a^2+n^2c^2-p^2e^2} : \sqrt{m^2b^2+n^2d^2-p^2f^2}.$$

VII.

1. Divide $-2x^5y^{-8}+17x^6y^{-4}-5x^7-24x^8y^4$ by $-x^2y^{-5}+7x^3y^{-1}+8x^4y^3$.

2. Find the H. C. F. of $e^{2x}a^3+e^{2x}-a^3-1$ and $e^{2x}a^2+2e^xa^2-e^{2x}-2e^x+a^2-1$.

3. Show that $1-\dfrac{a^2+b^2-c^2-d^2}{2(ab+cd)}$

$$=\frac{(a+c+d-b)(b+c+d-a)}{2(ab+cd)}.$$

4. Simplify $\dfrac{ab(x^2+y^2)+xy(a^2+b^2)}{ab(x^2-y^2)+xy(a^2-b^2)}$.

5. Solve $\dfrac{x-4}{x-5}-\dfrac{x-5}{x-6}=\dfrac{x-7}{x-8}-\dfrac{x-8}{x-9}$.

6. Show that if each of the expressions x^2+px+q and $x^2+p'x+q'$ be divisible by $x+a$, then $a=\dfrac{q-q'}{p-p'}$.

7. A bill of £120 was paid with guineas and half-crowns, and 48 more half-crowns than guineas were used; find how many of each were paid.

8. If $a : b :: c : d$, prove that

$$4a^6+5b^6 : 4c^6+5d^6 :: a^3b^3 : c^3d^3.$$

VIII.

1. Show that $(ax+by+cz)^3+(cx-by+az)^3$ is divisible by $(a+c)(x+z)$.

2. Resolve into factors:—

(i) $(b+c)^2-6a(b+c)+5a^2$;

(ii) $x^2+2xy+a^2-2ay$.

3. Simplify $\dfrac{(a+b)\{(a+b)^2-c^2\}}{4b^2c^2-(a^2-b^2-c^2)^2}$

4. If $a+b+c=0$, show that
$$a^2-bc=b^2-ca=c^2-ab.$$

5. Solve $3(x+3)^2+5(x+5)^2=8(x+8)^2$

6. Extract the square root of
$$25x^{-2}-12x+16x^{-8}+4x^4-24x^{-5}$$

7. Find the values of x, y, z if $yz=4$, $zx=9$, $zy=25$

8. If $a:b::c.d$, show that
$$a(a+b+c+d)=(a+b)(a+c).$$

IX.

1. Find the value of
$$\{a^2-(b-c)^2\}-\{b^2-(c-a)^2\}-\{c^2-(a-b)^2\},$$
when $a=1$, $b=2$ and $c=-3$

2. Simplify $\dfrac{x}{(x-1)^2}-\dfrac{1}{(x+1)^2}-\dfrac{x(x^2+3)}{(x^2-1)^2}$.

3. Resolve into factors $a^3-b^3+3ab+1$

4. Solve $\dfrac{4x+3}{9}+\dfrac{7x-29}{5x-12}=\dfrac{8x+19}{18}$.

5. Show that $\dfrac{1}{(y-z)^2}+\dfrac{1}{(z-x)^2}+\dfrac{1}{(x-y)^2}$
$$=\left(\frac{1}{y-z}+\frac{1}{z-x}+\frac{1}{x-y}\right)^2.$$

6. Solve $x+y:x-y=5.3$, $x+5y=36$.

7. Find the time between 8 and 9 o' clock, when the hands of a clock are at right angles to each other.

8. If $a:b::b.c$, show that
$$(a+b+c)(a-b+c)=a^2+b^2+c^2.$$

X.

1. Divide $27a^3-8b^3-27c^3-54abc$ by $3a-2b-3c$

2. Find the H C F of x^5+11x^3-54 and $x^5+11x+12$.

3. Resolve into factors $(a^2-b^2)(x^2+y^2)+2(a^2+b^2)xy$.

4. Simplify $\dfrac{\dfrac{a^2}{b^2}+\dfrac{b^2}{a^2}-2}{\dfrac{a^2}{b^2}+\dfrac{b^2}{a^2}+2}-\dfrac{\dfrac{a}{b}\left(1-\dfrac{b^2}{a^2}\right)}{\dfrac{(a+b)^2}{ab}}$

5. Show that $a^3(b+c)+b^3(c+a)+c^3(a+b)+abc(a+b+c)$
$$= (a^2+b^2+c^2)(bc+ca+ab)$$

6. Solve $\sqrt{9+2x}-\sqrt{2x} = \dfrac{5}{\sqrt{9+2x}}$.

7. One man and two boys can do in 12 days a piece of work which would be done in 6 days by 3 men and 1 boy. How long would it take one man to do it?

8. If $a : b :: b : c$, prove that
$$a^4+a^2c^2+c^4 = b^2\left(\frac{b^2}{c^2}-1+\frac{b^2}{a^2}\right)(a^2+b^2+c^2)$$

XI.

1. Show that $(x^2+xy+y^2)^2-4xy(x^2+y^2)$
$$= (x^2-xy+y^2)^2.$$

2. Resolve into factors —

(i) $a^2-b^2-c^2+d^2-2(ad-bc)$,

(ii) $x^2-y^2-z^2+2yz+x+y-z$

3. Extract the square root of
$$\frac{9x^2}{a^2}+\frac{a^2}{9x^2}-\frac{6x}{a}-\frac{2a}{3x}+3$$

4. Solve $x+2y+3z = 6$, $2x+4y+z = 7$,
$$3x+2y+9z = 14$$

5. Find the H. C. F. of $x^4y-x^3y^2-15x^2y^3+38xy^4-14y^5$ and $x^5-7x^4y+21x^3y^2-34x^2y^3+28xy^4$.

6. A man buys 570 oranges, some at 16 for a shilling and the rest at 18 for a shilling, he sells them all at 15 for a shilling and gains three shillings, how many of each sort does he buy?

7. Simplify —
$$\frac{1}{\left(1-\frac{c}{a}\right)\left(1-\frac{b}{a}\right)}+\frac{1}{\left(1-\frac{a}{b}\right)\left(1-\frac{c}{b}\right)}+\frac{1}{\left(1-\frac{b}{c}\right)\left(1-\frac{a}{c}\right)}.$$

8. If $a : b = c : d = e : f$, prove that
$$(a^2+c^2+e^2)(b^2+d^2+f^2) = (ab+cd+ef)^2.$$

XII

1. If $x = a+d$, $y = b+d$ and $z = c+d$, show that $x^2+y^2+z^2-yz-zx-xy = a^2+b^2+c^2-bc-ca-ab$

2. Simplify

$$\frac{y+z}{(y^2-xz)(z^2-xy)}+\frac{z+x}{(z^2-xy)(x^2-yz)}+\frac{x+y}{(x^2-yz)(y^2-xz)}$$

3. Resolve into factors —

(i) $x^2-2ax-b^2+2ab$

(ii) $x^2+(a+b+c)x+ab+ac$

4 Find the H. C F of $6x^4-2x^3+9x^2+9x-4$ and $9x^4+80x^2-9$.

5. Solve $\dfrac{6x+13}{15}-\dfrac{3x+5}{5x-25}-\dfrac{2x}{5}=0$

6 A and B can together do a work in 12 days, A and C in 15 days, B and C in 20 days, find in how many days they will do the work, all working together.

7. Simplify $4\sqrt{147}-3\sqrt{75}-6\sqrt{\frac{1}{3}}+18\sqrt{\frac{1}{27}}$

8. Show that, if $x : y :: a : b$,

then will $\dfrac{x^2+a^2}{x+a}+\dfrac{y^2+b^2}{y+b}=\dfrac{(x+y)^2+(a+b)^2}{x+y+a+b}$.

XIII

1. If $2s = a+b+c$, show that

$a(b-c)(s-a)^2+b(c-a)(s-b)^2+c(a-b)(s-c)^2 = 0$

2. Show that $x^6+x^3a^3+a^6$ is divisible by $x^{\frac{3}{4}}+x^{\frac{3}{8}}a^{\frac{3}{8}}+a^{\frac{3}{4}}$

3. Simplify $\dfrac{x^2-yz}{(x+y)(x+z)}+\dfrac{y^2-zx}{(y+z)(y+x)}+\dfrac{z^2-xy}{(z+x)(z+y)}$

4. Solve $\dfrac{x^2-a^2}{x-a}+\dfrac{x^2-b^2}{x-b}+\dfrac{x^2-c^2}{x-c}=a+b+c-3x$.

5. Find how many gallons of water must be mixed with 80 gallons of spirit which cost 15 shillings a gallon, so that by selling the mixture at 12 shillings a gallon there may be a gain of 10 per cent on the outlay.

6 Simplify $\dfrac{(x^{a+b})^2\,(x^{b+c})^2\,(x^{c+a})^2}{(x^a\,x^b\,x^c)^4}$.

7. Simplify $3\sqrt[3]{128}-4\sqrt[3]{-686}+2\sqrt[3]{54}$

8. If $a : b :: b : c$, prove that

$$a^2+ab+b^2 : b^2+bc+c^2 :: a : c.$$

XIV.

1. If $a+b+c = 2s$, and $a^2+b^2+ab+s^2 = 2s(a+b)$, show that $(a-s)^2+(b-s)^2+(c-s)^2 = s^2$

2 If $x+a$ be a common factor of x^2+px+q and x^2+lx+m, show that $a = \dfrac{m-q}{l-p}$.

3. Simplify $\dfrac{7+3\sqrt{5}}{7-3\sqrt{5}}+\dfrac{7-3\sqrt{5}}{7+3\sqrt{5}}$.

4 Solve $\dfrac{a-b}{x-a}+\dfrac{a-b}{x-b} = \dfrac{a}{x-a}+\dfrac{b}{x-b}$.

5 Solve $\dfrac{y+z-x}{b+c} = \dfrac{z+x-y}{c+a} = \dfrac{x+y-z}{a+b} = 1$.

6. A can do a piece of work in 20 days, which B can do in 12 days. A begins the work, but after a time B takes his place, and the whole is finished in 14 days from the beginning How long did A work?

7. Express $(x+a)(x+2a)(x+3a)(x+4a)$ as the difference of two squares.

8. Show that if $a(y+z) = b(z+x) = c(x+y)$,

then $\dfrac{y-z}{a(b-c)} = \dfrac{z-x}{b(c-a)} = \dfrac{x-y}{c(a-b)}$.

XV.

1. For what value of b will $x^4+2ax^3+(a^2+8)x^2$
$+(4a+ab)x+4b$ be a perfect square?

2. Prove that $(b-c)(1+ab)(1+ac)+(c-a)(1+bc)(1+ba)$ $+(a-b)(1+ca)(1+cb) = (b-c)(c-a)(a-b)$.

3. Simplify

$$\frac{2x^2+2}{x^4+x^2+1}+\frac{1}{x+\sqrt{x}+1}+\frac{1}{x-\sqrt{x}+1}-\frac{1}{x^2-x+1}.$$

4. Find the H. C F of $2x^3+(2a-3b)x^2-(2b+3ab)x$ $+3b^2$ and $2x^2-(3b-2c)x-3bc$

5 Find the value of $\frac{a^n}{2na^n-2nx}+\frac{b^n}{2nb^n-2nx}$, when $x=\frac{a^n+b^n}{2}$

6. Solve $\frac{20x+39}{25}+\frac{5x+20}{3x-16}=\frac{4x}{5}+\frac{86}{25}$.

7. A vessel is filled with a mixture of spirit and water 70 per cent of which is spirit After 9 gallons is taken out and the vessel filled up with water, there remains $58\frac{1}{3}$ per cent of spirit, find the contents of the vessel

8. If $x-z : y-z :: x^2 : y^2$, show that

$$x+z : y+z :: \frac{x}{y}+2 : \frac{y}{x}+2$$

XVI.

1. Find the H. C F of $x^5+2x^4-5x^2-7x+3$ and $3x^6-3x^4-18x^3+x^2+2x+3$

2. Solve $\sqrt{2x-1}+\sqrt{3x-2}=\sqrt{4x-3}+\sqrt{5x-4}$.

3. If $(a+b+c)x=(-a+b+c)y=(a-b+c)z$

$$=(a+b-c)w, \text{ shew that } \frac{1}{y}+\frac{1}{z}+\frac{1}{w}=\frac{1}{x}.$$

4. Solve

$$\left.\begin{array}{l}\frac{5\sqrt{x+y}}{x}+\frac{5\sqrt{x+y}}{y}=10\frac{2}{3}\\[2mm] \frac{3\sqrt{x-y}}{y}-\frac{3\sqrt{x-y}}{x}=\frac{4}{5}\end{array}\right\}.$$

5. Resolve into factors $ax(y^3+b^3)+by(bx^2+a^2y)$.

6. Find the continued product of $\sqrt{a}+\sqrt{b}+\sqrt{c}$, $\sqrt{a}+\sqrt{b}-\sqrt{c}$, $\sqrt{a}-\sqrt{b}+\sqrt{c}$, $\sqrt{b}+\sqrt{c}-\sqrt{a}$.

7. If $\frac{a+b}{a-b}=\frac{c}{d}$, show that $\frac{a^2+ab}{ab-b^2}=\frac{c^2+cd}{cd-d^2}$

8. Each of two vessels contains a mixture of wine and water, a mixture consisting of equal measures from the two vessels contains as much wine as water, and another mixture consisting of four measures from the first vessel and one from the second is composed of wine and water in the ratio of 2 : 3 Find the proportion of wine and water in each of the vessels

XVII

1. Find the H C F of x^5+x^2+2x+2 and x^4+x^2-1

2. Solve $\left.\begin{array}{l}\sqrt{y}-\sqrt{y-x}=\sqrt{20-x}\\ \sqrt{y-x}:\sqrt{20-x}::3:2\end{array}\right\}$

3. Find the value of

$$\left(\frac{x}{x-1}\right)^2+\left(\frac{x}{x+1}\right)^2, \text{ when } x=\sqrt{\frac{n-1}{n+1}}.$$

4. Show that $\frac{a^3(b^2-c^2)+b^3(c^2-a^2)+c^3(a^2-b^2)}{a^2(b-c)+b^2(c-a)+c^2(a-b)}$

$$=ab+bc+ca.$$

5 If $a+b+c=0$, show that

$$4(b^2c^2+c^2a^2+a^2b^2)=(a^2+b^2+c^2)^2.$$

Hence, prove that

$$(y-z)^2(z-x)^2+(z-x)^2(x-y)^2+(x-y)^2(y-z)^2$$
$$=(x^2+y^2+z^2-yz-zx-xy)^2.$$

6 One of the digits of a number is greater by 5 than the other. When the digits are inverted the number becomes $\frac{3}{8}$ of the original number. Find the number.

7. Simplify $\frac{3x^3+x^2-5x+21}{6x^3+29x^2+26x-21}$.

8. If $3(a^2+b^2+c^2)=(a+b+c)^2$, show that $a=b=c$.

XVIII

1. Show that $\{(x-y)^2+(y-z)^2+(z-x)^2\}^2$
$$=2\{(x-y)^4+(y-z)^4+(z-x)^4\}.$$

2. Solve $\dfrac{x+\sqrt{x^2-a^2}}{x-\sqrt{x^2-a^2}}=4\sqrt{2}\left\{\dfrac{\sqrt{x+a}-\sqrt{x-a}}{\sqrt{x+a}+\sqrt{x-a}}\right\}^{\frac{1}{2}}$.

3. Resolve into factors —

(i) $14x^2-37x+5$, (ii) $(1+a)^2(1+c^2)-(1+c)^2(1+a^2)$,

(iii) $m^4-n^4+2n(m^3+n^3)-(m+n)^2(m-n)^2$.

4. A baker charges $9\frac{1}{2}d$ for a loaf which he represents as weighing 4lbs, but which really weighs 3lbs 12 oz After he has sold a certain number of loaves, he is detected and fined £5, and thus loses five shillings more than he has cleared by selling short weight. How many loaves does he sell?

5. Simplify $\dfrac{a^3}{(a-b)(a-c)}+\dfrac{b^3}{(b-c)(b-a)}+\dfrac{c^3}{(c-a)(c-b)}$.

6. If $\dfrac{a-b}{ay+bx}=\dfrac{b-c}{bz+cy}=\dfrac{c-a}{cx+az}=\dfrac{a+b+c}{ax+by+cz}$, then each of these ratios $=\dfrac{1}{x+y+z}$, supposing $a+b+c$ not to be zero

7. Solve $x(x+y+z)=24$, $y(x+y+z)=48$,
$$z(x+y+z)=72.$$

8. Eliminate x from the equations

$$\left.\begin{aligned} a+c&=\frac{b}{x}-dx \\ a-c&=\frac{d}{x}-bx \end{aligned}\right\}.$$

XIX.

1. Solve $(x^2-2ax+3a^2)^{\frac{1}{2}}+(x^2-4ax+5a^2)^{\frac{1}{2}}$
$$=(x^2-5ax+7a^2)^{\frac{1}{2}}+(x^2-7ax+9a^2)^{\frac{1}{2}}.$$

2. Shew that

$$\frac{a(a+b)(a+c)}{(a-b)(a-c)}+\frac{b(b+a)(b+c)}{(b-a)(b-c)}+\frac{c(c+a)(c+b)}{(c-a)(c-b)}=a+b+c$$

3 Simplify $\dfrac{(b-c)a^3+(c-a)b^3+(a-b)c^3}{c^2-bc-ca+ab}$.

4 If m gold coins are equal in weight to n silver coins and p of the former equal in value to q of the latter, compare the values of equal weights of gold and silver.

5 If $x=b+c$, $y=c+a$, $z=a+b$, show that
$$x^3+y^3+z^3-3xyz = 2(a^3+b^3+c^3-3abc).$$

6. If $\dfrac{1}{b^3(a-c)}+\dfrac{1}{a^3(b-c)} = \dfrac{1}{ab(a-c)(b-c)}$,

prove that $\dfrac{1}{a}+\dfrac{1}{b} = \dfrac{1}{c}$, or, $a^2+b^2 = ab$.

7. Simplify $\sqrt{\dfrac{(\sqrt{12}-\sqrt{8})(\sqrt{3}+\sqrt{2})}{5+\sqrt{24}}}$.

8 Eliminate x and y from the equations :—
$(b+c)x+(c+a)y+(a+b)=0$, $(c+a)x+(a+b)y+(b+c)=0$,
$(a+b)x+(b+c)y+(c+a)=0$.

XX.

1 Show that $a(b+c)^2+b(c+a)^2+c(a+b)^2-4abc$
$$= (b+c)(c+a)(a+b).$$

2 If $x+a$ be a factor of $a^2x^3-b^3x^2+ac^3x+3a^3bc$, and if a is not equal to zero, show that $a^3+b^3+c^3 = 3abc$

3. Simplify —
$$\frac{bc}{a(a^2-b^2)(a^2-c^2)}+\frac{ac}{b(b^2-a^2)(b^2-c^2)}+\frac{ab}{c(c^2-b^2)(c^2-a^2)}.$$

4 Divide $a^4(b-c)+b^4(c-a)+c^4(a-b)$ by
$$(a-b)(b-c)(c-a).$$

5 If $a+b+c=0$, show that $a^5+b^5+c^5=5abc(c^2-ab)$.

6 Solve $\dfrac{a}{x+a-c}+\dfrac{b}{x+b-c} = 2$

7. Solve $ax+by+cz = a+b+c$, $\dfrac{ax}{b+c}+\dfrac{by}{a+c} = 1$,
$\dfrac{2x}{b+c}+\dfrac{2y}{a+c} = \dfrac{1}{a}+\dfrac{1}{b}$.

8 A person starts to walk at a uniform speed without stopping from Katak to Jobra and back, at the same time that another starts to walk at a uniform speed without stopping from Jobra to Katak and back They meet a mile and a half from Jobra, and again, an hour after a mile from Katak. Find their rates of walking, and the distance between Katak and Jobra.

XXI.

1 Show that $\{(b+c)^2+(c+a)^2+(a+b)^2\}\times\{a^2(b-c)+b^2(c-a)+c^2(a-b)\} = 2\{a^4(b-c)+b^4(c-a)+c^4(a-b)\}$

2. Show that

$$\frac{(b-c)^2}{(c-a)(a-b)}+\frac{(c-a)^2}{(a-b)(b-c)}+\frac{(a-b)^2}{(b-c)(c-a)} = 3$$

3 If $a+b+c = 0$, show that

$$a^2+ab+b^2 = b^2+bc+c^2 = c^2+ca+a^2.$$

4 If $s = a+b+c$, prove that

$$(s-3a)^2+(s-3b)^2+(s-3c)^2 = 3\{(a-b)^2+(b-c)^2+(c-a)^2\}.$$

5. Resolve into factors $a^2+2ab-2ac-3b^2+2bc$

6. Find the H C F of $x^4-2x^3+5x^2-4x+3$ and $2x^4-x^3+6x^2+2x+3$

7. Find the condition that ax^3+bx+c and $a'x^3+b'x+c'$ may have a common factor of the form $x+f$

8. If $a : b = b : c = c : d$, prove that

$$a : d = \sqrt{a^5+b^2c^2+a^3c^2} : \sqrt{b^4c+d^4+b^2cd^2}$$

XXII.

1. Show that $a(b-c)(1+ab)(1+ac)$

$$+b(c-a)(1+bc)(1+ba)+c(a-b)(1+ca)(1+cb) = abc(a-b)(a-c)(b-c).$$

2. If $a+b+c = 0$, show that

$$a^7+b^7+c^7 = 7abc(c^2-ab)^2.$$

3. Show that if ax^2+bx+c and $a'x^2+b'x+c'$ have a common factor of the form $x+f$, then will $(ac'-a'c)^2 = (bc'-b'c)(ab'-a'b)$

4 A and B run a race, B has 50 yards start, but A runs 20 yards while B runs 19 What must be the length of the course that A may come in a yard ahead of B?

5 Show that

$$\frac{p+q+r}{p-q}-\frac{q(4p+3r)-r(p+r)}{p^2-q^2}=\frac{(p-q+r)^2}{p^2-q^2}.$$

6. Show that $\dfrac{(a^2-b^2)^3+(b^2-c^2)^3+(c^2-a^2)^3}{(a-b)^3+(b-c)^3+(c-a)^3}$

$$=(a+b)(b+c)(c+a).$$

7. Solve $x+y+z=2a+2b+2c$, $ax+by+cz=2bc+2ca+2ab$, $(b-c)x+(c-a)y+(a-b)z=0$.

8. Eliminate x, y, z from the equations—

$$ax+cy+bz=0,\ cx+by+az=0,\ bx+ay+cz=0.$$

XXIII.

1. Show that $(b-c)(1+a^2b)(1+a^2c)$

$$+(c-a)(1+b^2c)(1+b^2a)+(a-b)(1+c^2a)(1+c^2b)$$
$$=abc(a+b+c)(a-b)(a-c)(b-c).$$

2. Find the L C M of $21x^2-13x+2$, $28x^2-15x+2$ and $12x^2-7x+1$.

3 Show that $(x+y)^7-x^7-y^7$ is divisible by $(x^2+xy+y^2)^2$.

4. If $2s=a+b+c$ and $2t^2=a^2+b^2+c^2$, show that

$$(t^2-a^2)(t^2-b^2)+(t^2-b^2)(t^2-c^2)+(t^2-c^2)(t^2-a^2)$$
$$=4s(s-a)(s-b)(s-c).$$

5. If $(1+xx'+yy')^2=(1+x^2+y^2)(1+x'^2+y'^2)$, show that $x=x'$ and $y=y'$.

6. Simplify

$$\frac{ab(a-b)(a^2+b^2)+bc(b-c)(b^2+c^2)+ca(c-a)(c^2+a^2)}{a^2b^2(a-b)+b^2c^2(b-c)+c^2a^2(c-a)}.$$

7. If $a+b+c=0$, prove that

$$\frac{a^5+b^5+c^5}{5}=\frac{a^3+b^3+c^3}{3}\cdot\frac{a^2+b^2+c^2}{2}.$$

8. Eliminate x and y from the equations—

$$ax+by = \sqrt{a^2+b^2},\ \frac{x^2}{p^2}+\frac{y^2}{q^2} = \frac{1}{a^2+b^2},\ x^2+y^2 = 1.$$

XXIV.

1. Solve $\quad x+y+z = a+b+c,$

$$bx+cy+az = cx+ay+bz = ab+bc+ca$$

2. Divide 243 into three parts such that one-half of the first, one-third of the second and one-fourth of the third part shall all be equal to one another

3. If $4(a^2+b^2+c^2+d^2) = (a+b+c+d)^2$, show that $a = b = c = d$.

4. If $2s = a+b+c$, show that

$$a(b-c)(s-a)^2+b(c-a)(s-b)^2+c(a-b)(s-c)^2 = 0.$$

5. If $bz+cy = a$, $az+cx = b$ and $ay+bx = c$,

prove that $\quad \dfrac{a^2}{1-x^2} = \dfrac{b^2}{1-y^2} = \dfrac{c^2}{1-z^2}.$

6. Eliminate x and y from the equations—

$$ax+by = x+y+xy = \ ^2+y^2-1 = 0.$$

7. If ax^2-bx+c and dx^3-bx-c have a common factor, show that $a \ -abd+cd^2 = 0$

8. If $a^3+b^3+c^3 = (a+b+c)^3$, then will $a^{2n+1}+b^{2n+1}+c^{2n+1} = (a+b+c)^{2n+1}$ where n is any positive integer.

XXV.

1. If $x = a^2-bc$, $y = b^2-ca$, $z = c^2-ab$,

prove that $\dfrac{x^2-yz}{a} = \dfrac{y^2-zx}{b} = \dfrac{z^2-xy}{c} = (a+b+c)(x+y+z).$

2. If $2s = a+b+c+d$, show that

$$4(bc+ad)^2-(b^2+c^2-a^2-d^2)^2 = 16(s-a)(s-b)(s-c)(s-d).$$

3. Prove that $(b+c-a)^3+(c+a-b)^3+(a+b-c)^3$

$$-3(b+c-a)(c+a-b)(a+b-c) = 4(a^3+b^3+c^3-3abc).$$

4. Show that, if $a+b+c=0$, then

$$\left(\frac{b-c}{a}+\frac{c-a}{b}+\frac{a-b}{c}\right)\left(\frac{a}{b-c}+\frac{b}{c-a}+\frac{c}{a-b}\right)=9.$$

5. If $x:a=y:b=z:c$, prove that

$$\frac{x^2+a^2}{x+a}+\frac{y^2+b^2}{y+b}+\frac{z^2+c^2}{z+c}=\frac{(x+y+z)^2+(a+b+c)^2}{x+y+z+a+b+c}.$$

6. Prove that,

if $ax+by+cz=0$ and $\dfrac{a}{x}+\dfrac{b}{y}+\dfrac{c}{z}=0$, then will

$$ax^3+by^3+cz^3+(a+b+c)(y+z)(z+x)(x+y)=0.$$

7. Eliminate x, y, z from the equations—

$$\text{(i)}\ \left.\begin{aligned} ax+hy+gz&=0\\ hx+by+fz&=0\\ gx+fy+cz&=0\end{aligned}\right\} \qquad \text{(ii)}\ \left.\begin{aligned} a(y+z)&=x\\ b(z+x)&=y\\ c(x+y)&=z\end{aligned}\right\}$$

8. Eliminate l, m, n from the equations—

$$\left.\begin{aligned} al&=bm=cn,\\ l^2+m^2+n^2&=1,\\ a^2l^3+b^2m^3+c^2n^3&=a'^2l+b'^2m+c'^2n\end{aligned}\right\}.$$

CHAPTER XXI.

QUADRATIC EQUATIONS

1. **Definition.** Any equation which contains the square of the unknown quantity, but no higher power, is called a *quadratic equation* or an *equation of the second degree*

If an equation contains *only* the second power of the unknown quantity (and *not the first*) it is called a *pure* quadratic; if it contains the second *as well as* the first power it is called *an adfected* quadratic.

Thus $3x^2=45$ is a pure quadratic;

and $3x^2-7x=6$ is an adfected quadratic

2 Solution of a Pure Quadratic. In solving a pure Quadratic we have to find the *square of the unknown quantity* just in the same way as simple equations are solved and then to extract the square root of the value so found.

Example 1. Solve $5(x^2+1)-2 = 3(x^2+7)$

We have $5x^2+3 = 3x^2+21$,

hence, $2x^2 = 18$ (by transposition),

$\therefore$ $x^2 = 9$;

now, since the unknown quantity is one of which the square is 9, it must be *either* $+3$ or -3 (Thus there are *two* values of x satisfying the given equation, as the student can easily verify)

Note The student should carefully observe that the last step of the above solution amounts to answering the following question —"What quantity is that of which the square is 9?"

Example 2 If $\dfrac{35-2x}{9}+\dfrac{5x^2+7}{5x^2-7} = \dfrac{17-\frac{2}{3}x}{3}$, find x.

By transposition, we have

$$\frac{5x^2+7}{5x^2-7} = \frac{51-2x}{9}-\frac{35-2x}{9} = \frac{16}{9};$$

$$\therefore \frac{5x^2}{7} = \frac{16+9}{16-9} = \frac{25}{7} \quad \text{(Componendo and dividendo)}$$

$$\therefore x^2 = 5, \quad \therefore x = \pm\sqrt{5}$$

Example 3. Solve $3\left(\dfrac{x^2-9}{x^2+3}\right)+4\left(\dfrac{22\frac{1}{2}+x^2}{x^2+9}\right) = 7.$

By transposition, we have

$$4\left(\frac{22\frac{1}{2}+x^2}{x^2+9}\right)-4 = 3-3\left(\frac{x^2-9}{x^2+3}\right)$$

$$\text{or, } 4\left\{\frac{22\frac{1}{2}+x^2}{x^2+9}-1\right\} = 3\left\{1-\frac{x^2-9}{x^2+3}\right\}$$

$$\text{or, } 4\times\frac{13\frac{1}{2}}{x^2+9} = 3\times\frac{12}{x^2+3},$$

$$\therefore \quad \frac{3}{x^2+9} = \frac{2}{x^2+8}. \quad \text{(removing the factor 18 from both sides)}$$

$$\therefore \quad 3x^2+9 = 2x^2+18,$$

$$\therefore \quad x^2 = 9, \quad \therefore \quad x = \pm 3$$

Example 4. If $a+b = \dfrac{2a\sqrt{1+x^2}}{x+\sqrt{1+x^2}}$, find x.

We have $(a+b)(x+\sqrt{1+x^2}) = 2a\sqrt{1+x^2}$.

$$\therefore \quad (a+b)x = (a-b)\sqrt{1+x^2}$$

$$\text{or,} \quad (a+b)^2x^2 = (a-b)^2(1+x^2);$$

$$\therefore \quad x^2\{(a+b)^2-(a-b)^2\} = (a-b)^2,$$

$$\text{or,} \quad x^2.4ab = (a-b)^2,$$

$$\therefore \quad x^2 = \frac{(a-b)^2}{4ab},$$

$$\therefore \quad x = \pm\frac{a-b}{2\sqrt{ab}}.$$

Example 5 If $\dfrac{1+\sqrt{x^2-1}}{1+2a\sqrt{x^2-1}} = \dfrac{\sqrt{x^2-1}-1}{x^2-2}$, find x.

Put y for $\sqrt{x^2-1}$ and $\therefore$ y^2-1 for x^2-2.

Thus we have $\dfrac{1+y}{1+2ay} = \dfrac{y-1}{y^2-1} = \dfrac{1}{y+1}$.

$$\text{Therefore} \quad (1+y)^2 = 1+2ay,$$

$$\text{or,} \quad 1+2y+y^2 = 1+2ay,$$

$$\therefore \quad y+2 = 2a, \text{ or, } y = 2(a-1);$$

$$\text{i.e.,} \quad \sqrt{x^2-1} = 2(a-1),$$

$$\therefore \quad x^2-1 = 4(a-1)^2,$$

$$\therefore \quad x = \pm\sqrt{1+4(a-1)^2}.$$

Example 6. Solve $(a+x)^{\frac{1}{3}}+(a-x)^{\frac{1}{3}} = b$.

Since $\left\{(a+x)^{\frac{1}{3}}+(a-x)^{\frac{1}{3}}\right\}^3$

$$= (a+x)+(a-x)+3(a^2-x^2)^{\frac{1}{3}}\left\{(a+x)^{\frac{1}{3}}+(a-x)^{\frac{1}{3}}\right\}$$

$$= 2a+3(a^2-x^2)^{\frac{1}{3}}\times b \quad [\text{because } (a+x)^{\frac{1}{3}}+(a-x)^{\frac{1}{3}} = b],$$

therefore, cubing both sides of the equation, we get

$$2a+3(a^2-x^2)^{\frac{1}{3}}\times b = b^3, \text{ or, } 3b(a^2-x^2)^{\frac{1}{3}} = b^3-2a$$

$$\therefore \quad a^2-x^2 = \left\{\frac{b^3-2a}{3b}\right\}^3.$$

$$\therefore \quad x^2 = a^2-\left\{\frac{b^3-2a}{3b}\right\}^3.$$

$$\therefore \quad x = \pm\left\{a^2-\left(\frac{b^3-2a}{3b}\right)^3\right\}^{\frac{1}{2}}.$$

Example 7 Solve $\dfrac{a+x}{a^{\frac{1}{2}}+(a+x)^{\frac{1}{2}}}+\dfrac{a-x}{a^{\frac{1}{2}}+(a-x)^{\frac{1}{2}}} = a^{\frac{1}{2}}$

Since $(a+x)\left\{a^{\frac{1}{2}}+(a-x)^{\frac{1}{2}}\right\} = a^{\frac{1}{2}}(a+x)+(a+x)^{\frac{1}{2}}(a^2-x^2)^{\frac{1}{2}}$,

and $(a-x)\left\{a^{\frac{1}{2}}+(a+x)^{\frac{1}{2}}\right\} = a^{\frac{1}{2}}(a-x)+(a-x)^{\frac{1}{2}}(a^2-x^2)^{\frac{1}{2}}$;

therefore, clearing the equation of fractions, we have

$$2b^{\frac{2}{3}}+(a^2-x^2)^{\frac{1}{2}}\left\{(a+x)^{\frac{1}{2}}+(a-x)^{\frac{1}{2}}\right\}$$

$$= a^{\frac{1}{2}}\left\{a^{\frac{1}{2}}+(a+x)^{\frac{1}{2}}\right\}\left\{a^{\frac{1}{2}}+(a-x)^{\frac{1}{2}}\right\}$$

$$= a^{\frac{1}{2}}\left[a+a^{\frac{1}{2}}\left\{(a+x)^{\frac{1}{2}}+(a-x)^{\frac{1}{2}}\right\}+(a^2-x^2)^{\frac{1}{2}}\right]$$

$$= a^{\frac{3}{2}}+a\left\{(a+x)^{\frac{1}{2}}+(a-x)^{\frac{1}{2}}\right\}+a^{\frac{1}{2}}(a^2-x^2)^{\frac{1}{2}}.$$

Hence, removing $a^{\frac{3}{2}}$ from both sides and transposing, we get

$$a^{\frac{1}{2}}\left\{a-(a^2-x^2)^{\frac{1}{2}}\right\} = \left\{(a+x)^{\frac{1}{2}}+(a-x)^{\frac{1}{2}}\right\}$$

$$\times\left\{a-(a^2-x^2)^{\frac{1}{2}}\right\} \qquad \text{(A)}$$

whence $$a^{\frac{1}{2}} = (a+x)^{\frac{1}{2}} + (a-x)^{\frac{1}{2}},$$

squaring both sides, $$a = 2a + 2(a^2 - x^2)^{\frac{1}{2}};$$

$$\text{or,} \quad -a = 2(a^2-x^2)^{\frac{1}{2}};$$

$$\therefore \quad a^2 = 4(a^2-x^2),$$

$$\therefore \quad 4x^2 = 3a^2,$$

$$\therefore \quad x = \pm\frac{a\sqrt{3}}{2}.$$

Note It must be observed that the above equation admits of another solution which has been overlooked, for $a-(a^2-x^2)^{\frac{1}{2}}$ being a factor common to both sides of (A), if this be taken equal to zero the given equation is evidently satisfied Hence $(a^2-x^2)^{\frac{1}{2}} = a$ or $x=0$ is another solution The same remark applies to example 5, which the student will very easily see for himself

Exercise (99).

Find the values of x in each of the following equations —

1. $8x+\dfrac{7}{x} = \dfrac{65}{7}x.$ 2. $\dfrac{2x^2+10}{15} = 7-\dfrac{80+x^2}{25}.$

3. $\dfrac{14x^2+16}{21}-\dfrac{2x^2+8}{8x^2-11} = \dfrac{2x^2}{3}.$

4. $\dfrac{x+7}{x(x-7)}-\dfrac{x-7}{x(x+7)} = \dfrac{7}{x^2-73}.$

5. $\dfrac{x^3-1}{(x-1)^2}-\dfrac{x^3+1}{(x+1)^2} = 6.$

6. $\dfrac{1}{\sqrt{1-x}+1}+\dfrac{1}{\sqrt{1+x}-1} = \dfrac{1}{x}.$

(Rationalise both the terms of the left-hand side and then proceed.)

7. $(1+x+x^2)^{\frac{1}{2}} = a-(1-x+x^2)^{\frac{1}{2}}.$

8. $\dfrac{(x-a)(x-b)}{(x-ma)(x-mb)} = \dfrac{(x+a)(x+b)}{(x+ma)(x+mb)}.$

9. $\dfrac{ax+1+(a^2x^2-1)^{\frac{1}{2}}}{ax+1-(a^2x^2-1)^{\frac{1}{2}}} = \dfrac{b^2x}{2}.$

10. $(a+x)^{\frac{2}{3}}+(a-x)^{\frac{2}{3}} = 3(a^2-x^2)^{\frac{1}{3}}$.

11. $\dfrac{5x^2+17}{x^2-11}+\dfrac{14x^2-117}{2x^2-9} = 12$

12. $\dfrac{x^2-1}{x^2-4}-\dfrac{x^2-5}{x^2-8} = \dfrac{x^2-2}{x^2-5}-\dfrac{x^2-6}{x^2-9}$.

13. $\left\{a+(a^2-x^2)^{\frac{1}{2}}\right\}^{\frac{1}{2}}+\left\{a-(a^2-x^2)^{\frac{1}{2}}\right\}^{\frac{1}{2}}$

$$= n\left\{\frac{a+x}{a+(a^2-x^2)^{\frac{1}{2}}}\right\}^{\frac{1}{2}}.$$

[Since $a+(a^2-x^2)^{\frac{1}{2}} = \dfrac{(a+x)+(a-x)+2(a^2-x^2)^{\frac{1}{2}}}{2}$

$$= \frac{\left\{(a+x)^{\frac{1}{2}}+(a-x)^{\frac{1}{2}}\right\}^2}{2}$$

and similarly, $a-(a^2-x^2)^{\frac{1}{2}} = \dfrac{\left\{(a+x)^{\frac{1}{2}}-(a-x)^{\frac{1}{2}}\right\}^2}{2}$

$\therefore$ the left-hand side $= \dfrac{2(a+x)^{\frac{1}{2}}}{\sqrt{2}} = \sqrt{2}(a+x)^{\frac{1}{2}}$

Hence, squaring both sides, &c.]

14. $\dfrac{(1+2x)^{\frac{1}{2}}-1}{(1-2x)^{\frac{1}{2}}+1}+\dfrac{(1-2x)^{\frac{1}{2}}+1}{(1+2x)^{\frac{1}{2}}-1} = 2\sqrt{2}$.

3. The ordinary method of solving an Adfected Quadratic.—Bring the terms containing the unknown quantity to the left-hand side of the equation, and the known quantities to the right-hand side; if the co-efficient of x^2 be negative, change the sign of every term of the equation and *then* divide every term by the co-efficient of x^2; thus the equation is reduced to the form $x^2+px = q$. Now add $\dfrac{p^2}{4}$ (*i.e.*, square of half the co-efficient of x) to both sides, on which the left-hand side becomes a complete square and we get $\left(x+\dfrac{p}{2}\right)^2 = q+\dfrac{p^2}{4}$, whence $x+\dfrac{p}{2}$ $= \pm\sqrt{q+\dfrac{p^2}{4}}$, and therefore $x = -\dfrac{p}{2} \pm \sqrt{q+\dfrac{p^2}{4}}$.

Exercise (100).

Solve the following equations —

1. $70x-63=7x^2$.

By transposition, we have $-7x^2+70x=63$

Since the co-efficient of x^2 is negative, changing the sign of every term, we get $\quad 7x^2-70x=-63$

Dividing both sides by 7, $\quad x^2-10x=-9$

Now adding $\left(\frac{10}{2}\right)^2$ or 25 to both sides,

$$x^2-10x+25=25-9=16,$$
$$\text{or,} \quad (x-5)^2=16$$

Hence, $\quad x-5=\pm 4 \quad$ [because $x-5$ is a quantity of which the square is 16],

$$\therefore \quad x=5+4, \text{ or, } 5-4,$$
$$\text{i.e.,} \quad x=9 \text{ or } 1.$$

2. $2x^2-11x+5=0$

By transposition; $\quad 2x^2-11x=-5$;

dividing both sides by 2, $x^2-\frac{11}{2}x=-\frac{5}{2}$.

Adding $\left(\frac{11}{4}\right)^2$ to both sides,

$$x^2-\tfrac{11}{2}x+(\tfrac{11}{4})^2=\tfrac{121}{16}-\tfrac{5}{2}$$
$$\text{i.e.,} \quad (x-\tfrac{11}{4})^2=\tfrac{81}{16},$$
$$\therefore \quad x-\tfrac{11}{4}=\pm\tfrac{9}{4},$$
$$\therefore \quad x=\tfrac{11}{4}\pm\tfrac{9}{4}=5 \text{ or } \tfrac{1}{2}$$

3. $87-98x=30x-16x^2$

4. $17x^2-85x+216=65x-8x^2$.

5. $\dfrac{x^2+8}{11}=5x-x^2-5$

6. $4(x^2-3\frac{3}{5}x)=10(x^2-4\frac{2}{5}x-6)+3(\frac{x^2}{5}-\frac{5}{3})$.

7. $4(5x^2-3\frac{7}{8}x)=5(x^2-7x+12)+\dfrac{8(x-9)}{9}$.

8. $2x+.02=2.45x-x^2$

9. $4(x^2+23x-24)=29x^2-8x+1$.

10. $(3x-1)(x-4)+(x-2)(2x-3) = 4x(x-3)-5$

The left-hand side $= (3x^2-13x+4)+(2x^2-7x+6)$
$= 6x^2-20x+10.$

Hence, we have $5x^2-20x+10 = 4x^2-12x-5,$

$\therefore \quad x^2-8x = -15$ (by transposition),

$\therefore \quad x^2-8x+(4)^2 = 16-15,$

or, $\quad (x-4)^2 = 1,$

$\therefore \quad x-4 = \pm 1,$

$\therefore \quad x = 4 \pm 1,$ i e, $x = 4$ or $3.$

11. $(2x-5)(3x-7)-(x-1)(4x-5) = x^2-3(x+14)$

12. $(3x-11)(x-2)+(2x-3)(x+4)+13x$
$= 10(2x-1)^2+12.$

13. $(x-\frac{1}{2})(x-\frac{1}{3})+(x-\frac{1}{4})(x-\frac{1}{4}) = (x-\frac{1}{4})(x-\frac{1}{6}).$

14. $\dfrac{x}{15}+\dfrac{40}{3(10-x)} = \dfrac{3(10+x)}{95}.$

[By transposition, $\dfrac{40}{3(10-x)} = \dfrac{3(10+x)}{95}-\dfrac{x}{15} =$ &c]

15. $\dfrac{2x}{15}+\dfrac{3x-50}{3(10+x)} = \dfrac{12x+70}{190}$ **16.** $\dfrac{x+4}{x-4}+\dfrac{x-4}{x+4} = \dfrac{10}{3}.$

Subtracting 2 from both sides, we have

$$\left(\frac{x+4}{x-4}-1\right)+\left(\frac{x-4}{x+4}-1\right) = \frac{4}{3},$$

or, $$\frac{8}{x-4}-\frac{8}{x+4} = \frac{4}{3},$$

or, $$2\left(\frac{1}{x-4}-\frac{1}{x+4}\right) = \frac{1}{3},$$

or, $$\frac{2\times 8}{x^2-16} = \frac{1}{3},$$

$$\therefore \quad x^2-16 = 48,$$

$$x^2 = 64,$$

$$\therefore \quad x = \pm 8$$

17. $\dfrac{x}{x+1}+\dfrac{x+1}{x} = \dfrac{13}{6}.$ [Proceed as in the last example]

18. $\frac{x+3}{x+2}+\frac{x-3}{x-2}=\frac{2x-3}{x-1}$. [Proceeding as in example 16, we get $x^2-4x=0$, whence $(x-2)^2=4$; $\therefore$ $x=2\pm 2=4$, or, 0.]

19. $\frac{x-2}{x+2}+\frac{x+2}{x-2}=\frac{2(x+3)}{x-3}$.

20. $\frac{x+2}{x-2}-\frac{x-2}{x+2}=\frac{5}{6}$.

[We have $\left(\frac{x+2}{x-2}-1\right)-\left(\frac{x-2}{x+2}-1\right)=\frac{5}{6}$ or &c &c]

21. $\frac{x-6}{x-12}-\frac{x-12}{x-6}=\frac{5}{6}$.

22. $\frac{2x-9}{2x-7}-\frac{2x-7}{2x-9}=\frac{7}{12}$.

23. $\frac{x+6}{x+7}-\frac{x+1}{x+2}=\frac{1}{3x+1}$. (C U Entr Paper, 1878)

24. $\frac{2x}{x-4}+\frac{2x-5}{x-3}=8\frac{1}{3}$.

[We have $\left(\frac{2x}{x-4}-2\right)+\left(\frac{2x-5}{x-3}-2\right)=4\frac{1}{3}$]

25. $\frac{x}{x+5}-\frac{11x}{11x-8}+\frac{7}{6-4x}=0$

26. $\frac{1}{x+a}+\frac{1}{x+2a}+\frac{1}{x+3a}=\frac{3}{x}$.

[We have $\left(\frac{1}{x+a}-\frac{1}{x}\right)+\left(\frac{1}{x+2a}-\frac{1}{x}\right)+\left(\frac{1}{x+3a}-\frac{1}{x}\right)=0$,

whence $\frac{1}{x+a}+\frac{2}{x+2a}-\frac{3}{x+3a}=0$,

or, $\frac{1}{x+a}+\frac{1}{x+3a}=-2\left(\frac{1}{x+3a}+\frac{1}{x+2a}\right)$,

whence $\frac{x+2a}{x+a}=-\frac{2x+5a}{x+2a}$, &c. &c]

4. The general expression for the roots of a quadratic.

N B The roots of any equation are those values of the unknown quantity that satisfy the equation

As every quadratic equation can be written in the form $ax^2+bx+c=0$ (after suitable reduction if necessary) we must regard this equation as the *general type* of all quadratics. Let us solve it.

By transposition, $ax^2+bx = -c.$

Dividing both sides by a, $x^2+\frac{b}{a}x = -\frac{c}{a}$

Adding $\left(\frac{b}{2a}\right)^2$ to both sides,

$$x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}-\frac{c}{a},$$

$$\text{or,}\quad \left(x+\frac{b}{2a}\right)^2 = \frac{b^2-4ac}{4a^2},$$

$$\therefore\ x+\frac{b}{2a} = \pm\frac{\sqrt{b^2-4ac}}{2a},$$

$$\therefore\quad x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Thus the roots of the quadratic $ax^2+bx+c = 0$, are $\frac{-b+\sqrt{b^2-4ac}}{2a}$ and $\frac{-b-\sqrt{b^2-4ac}}{2a}$, and therefore we must regard the expression $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ as the *general expression* sought

By the application of this formula we can find out the *roots of a quadratic equation without going through the* process explained in Art. 3.

Example 1. Write down the roots of $2x^2-13x+15 = 0$

Comparing this with the equation $ax^2+bx+c = 0$, we have $a = 2$, $b = -13$, $c = 15$.

Hence the roots of the given equation are

$$= \frac{-(-13)\pm\sqrt{(-13)^2-4\times2\times15}}{2\times2}$$

$$= \frac{13\pm\sqrt{169-120}}{4}$$

$$= \frac{13\pm\sqrt{49}}{4} = \frac{13\pm7}{4}$$

That is, $x = 5$ or $\frac{3}{2}$.

Example 2. Write down the roots of $-3x^2 = 11x-4$.

Bringing all the terms to one side, we have

$$-3x^2-11x+4 = 0$$

Here $a=-3,\ b=-11,\ c=4$.

Hence,
$$x = \frac{-(-11)\pm\sqrt{(-11)^2-4\times(-3)\times 4}}{2\times(-3)}$$
$$=\frac{11\pm\sqrt{121+48}}{-6}$$
$$=\frac{11\pm\sqrt{169}}{-6}$$
$$=\frac{11\pm 13}{-6}=-4 \text{ or } \tfrac{1}{3}$$

Exercise (101).

Write down the roots of the following equations —

1. $3x^2-17x+24 = 0$.
2. $x^2+9x+20 = 0$
3. $6x^2 = 20-7x$.
4. $-9x^2+25 = 6x-10$.
5. $8x^2 = 14x+15$.
6. $-3x^2+20 = 25$
7. $5+x-4x^2 = 0$.

4. Sreedharacharyya's method of solving a Quadratic.—Reduce the equation to the form $px^2-qx = r$ multiply both sides of this by $4p$ (*i e*, by four times the co-efficient of x^2) and *then* add q^2 to both sides we thus get $4p^2x^2+4pqx+q^2 = 4pr+q^2$, the left-hand side of which is evidently a complete square, being equal to $(2px+q)^2$

Example 1. Solve $5x^2-17x+6 = 0$

By transposition, $5x^2-17x=-6$,

Multiplying both sides by 4×5,

$$4\times(5x)^2-4\times(5x)\times 17=-120,$$

Adding $(17)^2$ to both sides, we have

$$4+(5x)^2-4\times(5x)\times 17+(17)^2 = 289-120,$$

$$\text{or,}\quad (2\times 5x-17)^2 = 169,$$

$$\therefore \qquad 10x-17 = \pm\ 13,$$

$$x = \frac{17\pm13}{10} = 3 \text{ or } \tfrac{2}{5}$$

Example 2. Solve $-8x^2+10x = 3$.

Multiplying both sides by $4\times(-8)$,

$$4\times64x^2-4\times8\times10x = -96.$$

Adding $(10)^2$ to both sides,

$$4\times64x^2-4\times8\times10x+(10)^2 = 100-96$$

$$\text{or,} \quad (2\times8x-10)^2 = 4,$$

$$\therefore \quad 16x-10 = \pm\ 2,$$

$$\therefore \qquad x = \frac{10\pm2}{16} = \tfrac{3}{4} \text{ or } \tfrac{1}{2}.$$

Example 3. $6x^2+23x = 12x+10$

By transposition,

$$6x^2+11x = 10.$$

Multiplying both sides by 4×6,

$$4\times(6x)^2+4\times(6x)\times11 = 240.$$

Adding $(11)^2$ to both sides

$$4\times(6x)^2+4\times(6x)\times11+(11)^2 = 121+240,$$

$$\text{or,} \quad (2\times6x+11)^2 = 361,$$

$$\therefore \quad 12x+11 = \pm19,$$

$$\therefore \qquad x = \frac{-11\pm19}{12}$$

$$= \tfrac{2}{3} \text{ or } -\tfrac{5}{2}$$

Exercise (102).

Solve the following equations by Sreedharacharyya's method :—

1. $2x^2+9x = 18.$
2. $15x^2-28 = x.$
3. $16x^2+100x = 3x^2+x+40.$
4. $x^2+50a = 102-15x-x$
5. $17x^2+19x = 1848.$
6. $2cx^2-acx = 3(2x-a).$
7. $x^2+ax = ab(3x+a)-2x^2.$

6. Solution of a Quadratic by the method of Resolution into factors.—Reducing a Quadratic to the form $ax^2+bx+c=0$, if we know the factors of which the left-hand side is the product, then by equating to zero either of these factors, we get a soluton of the Quadratic

Example 1. Solve $x^2-5x+6=0$

Evidently the left-hand side $=(x-2)(x-3)$.

Hence, we have

$$(x-2)(x-3)=0,$$

$$\therefore \text{ either } \left.\begin{aligned} x-2&=0 \\ \text{and } \therefore \quad x&=2 \end{aligned}\right\} \quad \text{or, } \left.\begin{aligned} x-3&=0 \\ \text{and } \therefore \quad x&=3 \end{aligned}\right\}$$

Thus 2 and 3 are the roots of the equation, as the student can easily verify.

Example 2 Solve $2x^2-10x=3x-15$.

We have $\quad 2x(x-5)=3(x-5) \quad \ldots \ldots \quad (1)$

$$(2x-3)(x-5)=0$$

$$\text{Hence, either } \left.\begin{aligned} 2x-3&=0 \\ \text{and } \therefore \quad x&=\tfrac{3}{2} \end{aligned}\right\} \quad \text{or, } \left.\begin{aligned} x-5&=0 \\ \text{and } \therefore \quad x&=5 \end{aligned}\right\}.$$

Thus $\frac{3}{2}$ and 5 are the roots of the equation

Note The solution also at once follows from equation (1), for $x-5$ being a factor common to both sides, the equation evidently holds good when this factor is zero, *i.e.* when $x=5$, and, evidently also the equation is satisfied when $2x=3$ or $x=\frac{3}{2}$, therefore 5 and $\frac{3}{2}$ are the roots of the equation The student will thus observe that it is not always necessary to transpose all the terms to the left-hand side of the equation

Example 3. Solve $10(2x+3)(x-3)+(7x+3)^2$
$$=20(x+3)(x-1).$$

We have $\quad 10(2x^2-3x-9)+(49x^2+42x+9)$
$$=20(x^2+2x-3),$$

$$\therefore \quad 49x^2-28x-21=0,$$

$$\therefore \quad 7x^2-4x-3=0,$$

$$\text{or,} \quad (7x^2-7x)+(3x-3)=0,$$

$$\text{or,} \quad (7x+3)(x-1)=0.$$

$$\left.\begin{aligned}\text{Hence, either } 7x+3 &= 0\\ \text{and } \therefore \quad x &= -\tfrac{3}{7}\end{aligned}\right\} \quad \text{or,} \quad \left.\begin{aligned} x-1 &= 0\\ \text{and } \therefore \quad x &= 1\end{aligned}\right\}$$

Thus $-\frac{3}{7}$ and 1 are the roots of the equation

Example 4. Solve $(7-4\sqrt{3})x^2+(2-\sqrt{3})x = 2$

Since $\quad 7-4\sqrt{3} = (2-\sqrt{3})^2,$

we have $\quad (2-\sqrt{3})^2x^2+(2-\sqrt{3})x = 2$

Hence, putting z for $(2-\sqrt{3})x$, we have

$$z^2+z-2 = 0, \quad \text{or,} \quad (z+2)(z-1) = 0$$

$$\left.\begin{aligned}\text{Hence, either } z+2 &= 0\\ \text{and } \therefore \quad z &= -2\end{aligned}\right\} \quad \text{or,} \quad \left.\begin{aligned} z-1 &= 0\\ \text{and } \therefore \quad z &= 1\end{aligned}\right\}.$$

$$\left.\begin{aligned}\text{Thus,} \quad x &= \frac{-2}{2-\sqrt{3}} = -2(2+\sqrt{3})\\ \text{or,} \quad &= \frac{1}{2-\sqrt{3}} = 2+\sqrt{3}\end{aligned}\right\}.$$

Example 5. Solve $\dfrac{1}{(x-b)(x-c)}+\dfrac{1}{(a+c)(a+b)}$

$$= \frac{1}{(a+c)(x-c)}+\frac{1}{(a+b)(x-b)}$$

By transpositon,

$$\frac{1}{x-c}\left\{\frac{1}{x-b}-\frac{1}{a+c}\right\} = \frac{1}{a+b}\left\{\frac{1}{x-b}-\frac{1}{a+c}\right\}.$$

Therefore, either $\dfrac{1}{x-b}-\dfrac{1}{a+c} = 0.$ [See note Ex. 2]

whence $x = a+b+c$, or, $\dfrac{1}{x-c} = \dfrac{1}{a+b}$

whence also $x = a+b+c$

Thus the equation has got two *equal roots.*

Example 6. Solve $\dfrac{a+c(a+x)}{a+c(a-x)}+\dfrac{a+x}{x} = \dfrac{a}{a-2cx}.$

Since $\dfrac{a+c(a+x)}{a+c(a-x)} = \dfrac{a}{a+c(a-x)}+\dfrac{c(a+x)}{a+c(a-x)},$

we have by transposition,

$$(a+x)\left\{\frac{c}{a+c(a-x)}+\frac{1}{x}\right\} = a\left\{\frac{1}{a-2cx}-\frac{1}{a+c(a-x)}\right\}$$

or, $$(a+x)\frac{a(1+c)}{x\{a+c(a-x)\}} = a\,\frac{c(a+x)}{(a-2cx)\{a+c(a-x)\}},$$

or, $$\frac{(a+x)(1+c)}{x} = \frac{c(a+x)}{a-2cx}.$$

Hence, either $a+x=0$, and $\therefore\ x=-a$,

or, $\dfrac{1+c}{x}=\dfrac{c}{a-2cx}$, whence $x=\dfrac{a(1+c)}{c(3+2c)}$.

Thus $-a$ and $\dfrac{a(1+c)}{c(3+2c)}$ are the roots of the equation.

Exercise (103).

Solve the following equations —

1. $3x^2-12x+1=6x-23$
2. $4x^2-4x=80.$
3. $x+2-\dfrac{6}{x+2}=1.$
4. $x^2+9x-52=0.$
5. x^2- [illegible] $x-4=0.$
6. $6x^2+5x-4=0.$
7. $3(x-2)^2=18+(8x+1).$
8. $x-\dfrac{x^3-8}{x^2+9}=2.$
9. $\dfrac{21x^3-16}{3x^2-4}-7x=5$
10. $x^2-(a+b)x+ab=0.$
11. $(a-b)x^2-(a+b)x+2b=0.$
12. $\dfrac{x^2}{a^{\frac{1}{2}}+b^{\frac{1}{2}}}-\left(a^{\frac{1}{2}}-b^{\frac{1}{2}}\right)x=\dfrac{1}{\left(ab^2\right)^{-\frac{1}{2}}+\left(a^2b\right)^{-\frac{1}{2}}}.$
13. $\dfrac{2x(a-x)}{3a-2x}=\dfrac{a}{4}.$
14. $\dfrac{16}{x^{\frac{3}{2}}}+\dfrac{x^{\frac{1}{2}}}{2}=\dfrac{6}{x^{\frac{1}{2}}}.$
15. $\dfrac{a}{x-a}+\dfrac{b}{x-b}=\dfrac{2c}{x-c}.$
16. $\dfrac{a-\sqrt{2ax-x^2}}{a+\sqrt{2ax-x^2}}=\dfrac{x}{a-x}.$

ANSWERS.

Exercises.

1 [Pages 2, 3.]

1.	65	2.	18	3.	12 miles	4.	$7\frac{1}{2}$ miles.
5.	9.	6.	12.	7.	45 minutes	8	15 minutes.
9.	32	10.	$\frac{1}{64}$, $20\frac{1}{4}$	11.	5 sq yds	12.	7*s* 6*d*
13	20	14.	9	15	28	16.	4
17.	4480	18.	900	19.	1952	20.	720

2 [Pages 8, 9.]

1.	34	2.	0	3.	4	4.	1.	5.	$5\frac{1}{3}$.
6.	$1\frac{1}{3}$	7.	6	8.	4	9.	12	10.	8.
11.	2	12.	$\frac{1}{2}$	13.	5	14.	80.	15.	29.
16.	325	17.	63.	18	0	19.	40.	20.	14.
21	114	22	4	23.	64	24.	69.	25	19.

3 [Pages 11, 12.]

1.	24.	2.	$37\frac{1}{3}$.	3	4	4.	720.	5.	$3\frac{1}{3}$.
6.	$\frac{2}{3}$	7.	1	8	40.	9.	$\frac{2}{3}$	10.	$\frac{8}{5}$
11.	0	12.	50	13.	1	14.	25	15.	100.
16.	200	17.	1520.	18	14 625			19.	22680
20.	845000								

4 [Pages 14, 15.]

1.	4	2.	2	3.	6	4.	18.	5.	8
6.	16	7.	32.	8.	256	9.	11	10.	21.
11.	11.	12.	9	13.	3	14.	162.	15.	18
16.	9	17.	0	18.	21.	19.	28.	20.	1.
21.	98	22.	50.	23.	9	24.	42	25.	51.
26.	2805.	27.	7.	28.	171	29.	2401.	30.	192.
31.	1029.	32.	1218	33	48	34.	143		
35.	18750	36.	16.	37.	160	38.	78.		

5. [Page 19.]

1. A's loss = £100. 2. -70 3. -25 4. -100.
5. -30. 6. 4, -3, 5. 7. 15, -10, -20, 30.
8. -15, 10, 20, -30

6 [Pages 21, 22.]

1. -22. 2. -18 3. -31; -41 4. -19.
5. -1180. 6. -222. 7. -2034. 8. -658.
9. -7128 10. -220416417.

7 [Pages 23, 24.]

1. 9. 2. -5 3. -4. 4. 0. 5. -14.
6. 16 7. -51. 8. -8 9. 16. 10. -32.

8 [Pages 26, 27.]

1. $-a-2b+3c$ 2. $-a^2-2b^2+c^2$. 3. $-2a+4ab+5c$.
4. $-3mnp^2-2abr-3c^2$. 5. $2a^3b-9b^2c^2-2df$.
6. $-6x^4y-11xyz-10x^2y^2$. 7. $-11a^2bc+11ab^2c-8abc^2$.
8 $-25x^3mn+16m^3nx$. 9. -14 10. -234. 11. 92.
12. 5. 13. 177. 14 -4653 15. -12015

9 [Pages 28, 29.]

1. $-6a+b-3c$. 2. $2x-z$. 3. $2x^3+9x^2+7$. 4 $-a+2b-3d$ 5. $2x^2-3y^2$. 6. $-5x^2-2xy-y^2-2x-y-2$.
7. $4a^2b$ 8. $-m^3n^2-mnp-m^2n^2$. 9. $a^2b^2x^2$.
10 $a^4b^4c^4$. 11 $a^3b^3-b^3c^3+c^3a^3-a^2b^2c^2$. 12. $a^3+b^3+c^3-3abc$ 13. 153. 14. -125 15. 200
16. 120 17 400 18 1280. 19. 1280.
20. 0

10 [Page 32.]

1. -10 2. 10 3. -6. 4. -22 5. 0.
6 -201 7. -77 8. 83. 9. 17. 10. 177.

11 [Page 33.]

1. $a+4b-2c$. 2. $-3a+3b+4c$ 3. $3x+2y-3z$.
4. $-2m^2-3mn-1$. 5. $2x^2+y^2-z^2$. 6. $-ax+5xy-11y^2$.

7. $4a^2-7ab-b^2$ 8 $7bc-7c^2+10xy$ 9. $-x^3+x^2-x+2$. 10. $x^4-4x^3y-2xy^2$. 11 $-m^4-6m^2n+7mn^2-24n^4$. 12. $-3p^4-5p^3q-3p^2q^2-10pq^3-2q^4$ 13 $10x^5-11x^4y+10x^3y^2+6x^2y^3-3y^4$. 14 $2m^3nx-7n^3xm+12x^3mn+7m^2n^2x+8n^2x^2m$ 15. $11x^6-3x^5y-50x^4y^2+15x^3y^3+26x^2y^4-19xy^5+40y^6$.

12 [Pages 34–36.]

1. $-a+7b-10c$ 2. $10m-7n+12r$. 3 $3x^2-4xy+5y^2$. 4. x^3+5x^2-8x-1 5. $a^2+2ab-4bc-2b^2$. 6. $8x^3-8x^2y+12xy^2-12y^3$ 7. $6-9x+12x^2-16x^3$. 8. $2x^4-2x^3-x^2-2$ 9 $6x^5-8x^4+x^3-10x^2-9x-3$ 10. $-x^3-x^2y-15xy^2+8y^3-a^3-2ab$ 11. $14x^4y-7x^3y^2+5xyz^2-2x^2yz$ 12. $-3a^2+b^2+8c^2+ab+5bc-9ac$. 13. $5x^2-5xy+y^2-x-y-3$ 14 $-a^3-a^2b+12ab^2-2b^3+8a^2+7ab-3b^2$. 15. $5ax^4-8a^2x^3+8yzbc^3+2y^2zbc+4yz^2bc$ 16 $-2-x^3y^5z+2xy^3z^5+2x^3z^5y+6x^2y^2z^2-3xyz^4$. 17. $4x^4y^3z^2-80x^3y^4z^2+28x^2y^3z^4-22x^3y^2z^4-102x^4y^2z^3+155x^2y^4z^3$. 18. $-12x^3y^4z^5-100x^3y^5z^4+58x^4y^5z^3+92x^4y^3z^5+39x^5y^3z^4-38x^5y^4z^3$ 19. $-4x^2+5xy-7y^2-8yz$. 20. $6x^3-12x^2y^2+2a^2bx-7xby^2-9xyab$. 21. $-2x^4+6x^3y-2x^2y^2+8xy^3+7y^4$. 22. $-2x^5+3x^4y-10x^3y^2-4x^2y^3-13xy^4+50y^5$ 23. $a^2+5ab-8b^2$. 24. $4x^2-8xy+y^2-12x-15y+9$. 25. $2a^3-4a^2b+7ab^2-15b^3$ 26. $-12x^3y+7x^2y^2-8x^2+17y-29$ 27. $5a^2-4ab-5bc+11b^2$ 28. $-2x^3-3y^2-5xy-3x-2$. 29. $-3a^3-11b^2c-6ac^2-5b^3$. 30. $-4x^3-22xy^2-45y^3-11x^2-24xy-15$

13 [Pages 38, 39.]

1. $-4a+8b$. 2. $7x-4y$. 3. $-2x$. 4. $-4a+2b$. 5. $5a+2b$ 6. $2b$ 7. 6 8 8 9. $-2a+7b$. 10. 0. 11. $-2x+5y+7z$ 12. $-2c$ 13. $15x-15y$ 14. $8a-8b$. 15. $11m-7n$. 16. $6a-6b-18c$. 17. $6x-6y-20z$. 18. $x-y-18z$. 19. $-3x-y-z$.

20. $a-11b+17c$ 21. $2x-12y+20z$ 22. $5a-b+11c$. 23. $x-3y+2z$ 24. $11a-2b-16c$. 25. $a-(b+c-d)+(-m+n-x)+y-z$ 26. $a-\{b+c-d+m+(-n+x-y+z)\}$. 27. $\{a-b-(c-d+m)\}-\{-n-(-x+y-z)\}$ 28. $-\{-a-(-b-c)\}-\{-d-\{-m+n\}\}-\{x-(y-z)\}$.

Miscellaneous Exercises (1).

[Pages 39–44.]

I.

1. 10, $\frac{1}{2}$. 2. 8 3 15, $2a$, $7ab^2$; $16m^2pq$. 4. 6 5. $-\frac{1}{2}$ 8. $9, 7, 5, 2, -1, -3, -4, -8, -12$.

II.

1. $0, 46, 25, 45$. 2. 16 3. $(\sqrt[3]{a})\times(\sqrt[3]{a})\times(\sqrt[3]{a}) = a$, &c.; 35 4. $-7x^2y$, -560 6 $16x^4-8xy^3+24x^2y^2+y^4-32x^3y$, 81 7. $-23a+30b+13c$. 8. $x-2y+z$.

III.

1. (i) $c(a+b) = x-yz$, (ii) $(x+y)^2 = x^2+y^2+2xy$, (iii) $\sqrt[3]{m-n}-m^3n^3 < \sqrt{x}+\sqrt{y}$, (iv) $\because a>b$, $3a>3b$. 2. $5, -4, -\frac{13}{2}, \frac{13}{2}, -10$ 3. -1000 5. 66, 7. $-6a^2+bc-9x^2+16$. 8. a.

IV.

1. -11, 1 3. $4\frac{13}{15}$ 5. 9. 6. $3x+2a+b$. 7. $a^2+b^2+c^2$ 8. $7x^2-y^2-2xy$.

V.

3. $2, a, b, a+b$ 5. $7\frac{1}{2}$ 6 505 8. y.

VI.

2. 536 6. 60. 8. 3808.

VII.

2. 2, 0 4. $1+x$, $3a+b-5c$. 5. (i) $(a+b)(a-b) = a^2-b^2$, (iii) $(a+b)^2-(a^2+b^2) = 2ab$. 6. 0. 7. $a+b+c$. 8. $2a-\frac{3}{4}b+\frac{7}{8}c-\frac{3}{10}d$

15 [Pages 47, 48.]

1. 54 **2** 99 **3.** 126 **4.** -393 **5.** 33.
6. 30 **7.** 0 **8.** 1186 **9** -280.

17 [Page 53.]

14. $-42x^7y^5$. **15.** $-24x^{11}y^{11}$. **16.** $40x^{17}y^{16}$.
17. $-156x^{10}y^9z^6$. **18.** $140x^6y^7z^{20}$ **19.** $24a^6b^6$.
20. $-70a^{15}b^{12}$. **21.** $48x^{10}y^{10}z^7$. **22.** $24x^7y^8z^7$.

18 [Page 54.]

1. $-10x^7$ **2.** $-20a^4b^6$. **3.** $21m^7n^8$. **4.** $-18x^4y^7$
5. $3a^7b^{10}$. **6.** $-40m^8n^7$. **7.** $50x^2y^3z^3$ **8** $-24x^4y^4z^4$.
9. $48x^5y^5z^5$ **10.** $25a^5b^3c^{13}$ **11.** $-24x^3y^3z^5$
12. $32a^3b^2x^2y^3$ **13.** $35a^3b^3z^4$. **14.** $-60a^6x^7y^9$.
15. $70x^5y^5z$ **16.** $-18a^8b^6c^6$. **17.** $63a^9x^8y^2$.
18. $160x^{14}y^7z^7$ **19.** $65a^{10}b^{14}c^{20}$. **20.** $112a^{13}x^{10}y^9z^7$

19 [Page 57.]

1. $-2a^2b+3ab^2$. **2.** $-5a^2+10ab-15ac$. **3.** $2a^3bc-3ab^3c-abc^3$. **4** $-3x^3y^2+6x^2y^3+3xy^4$. **5.** $7a^2b^3-7ab^4+21a^2b^4-35a^3b^2$. **6.** $-16a^3b^2c+24a^2b^2c^2-32a^3bc^2+40a^3b^2c^2$ **7.** $-6a^4x+8a^3x^2-10a^3x$ **8.** $-8m^4n+12m^3n^3-20m^2n^3$. **9.** $-20x^7+24x^6-28x^5+32x^4$. **10.** $12c^4d^5-18c^3d^7+30c^3d^6+24c^4d^6$ **11.** $-16a^7b^3+12a^6b^4-10a^5b^5+8a^4b^6$ **12.** $a^5b^4c^4-a^4b^5c^4+a^4b^4c^5-a^5b^5c^3+a^3b^5c^5$ **13.** $7x^4-2x^2$. **14.** $-5x^4+5x$. **15.** $9x^6-25y^4$ **16.** x^6+4x^2. **17.** $a^{12}b^6+4a^4b^2$. **18** $4a^{18}b^{12}+81a^6b^4$ **19.** $3a^2y$. **20.** $-2a^3b$. **21.** ax^3.

20 [Pages 58, 59.]

1. $2a^2+5ab+3b^2$. **2.** $2m^2-5mn+3n^2$ **3.** $a^2+b^2+c^2+2ab+2ac+2bc$. **4.** $a^2+b^2+c^2-2ab+2ac-2bc$
5. $a^2+b^2+c^2-2ab-2ac+2bc$. **6.** $2a^2+2b^2+3c^2-5ab-7ac+5bc$ **7.** $2x^2+3y^2+4z^2-5xy-6xz+7yz$. **8.** $5x^2-2a^2-3b^2+3xa-3xb+5ab$ **9.** $x^3-y^3-z^3-x^2y-x^2z+xy^2-y^2z+xz^2-z^2y$. **10.** $x^2y^2-y^2z^2-z^2x^2-2yz^2x$.

21 [Pages 61–64.]

1. x^4+x^2+1 2. a^3+b^3. 3. a^3-b^3. 4. $a^4-2a^2b^2+b^4$. 5. x^8+x^4+1. 6. $x^6-x^4y^4+2x^3y^3+y^6$. 7. m^6+n^6. 8. p^6-q^6. 9. $27a^3-75ab^2+45a^2b-125b^3$. 10. $a^5-26a^3b^2+25ab^4$. 11. $4a^2-9b^2+24bc-16c^2$. 12. $x^5-5x^3+5x^2-1$. 13. $x^6-2a^3x^3+a^6$ 14. $a^6-3a^4b^2+3a^2b^4-b^6$ 15. $x^6+10x-33$. 16. x^6-2x^3+1 17. $a^8+a^6b^2+a^4b^4+a^2b^6+b^8$. 18. $x^3+y^3+z^3-3xyz$ 19. $a^3+b^3+c^3-3abc$ 20. $2a^6-a^5b-14a^4b^2+13a^3b^3-43a^2b^4+23ab^5-20b^6$. 21. a^4-b^4 22. a^6-b^6. 23. $x^{12}-y^{12}$. 24. $x^8+25x^4y^4-24x^2y^6+25y^8$. 25. $a^{18}-b^{18}$ 26. $apx^3+(bp-aq)x^2-(cp+bq)x+cq$. 27. $mnx^4-(n^2+mr)x^2+r^2$ 28. $ax^4-(1+a)bx^3+(c+b^2-ac)x^2-c^2$. 29. $abx^5-(b^2+ac)x^4+(2bc+ad)x^3-(2bd+c^2)x^2+2cdx-d^2$ 30. $mpx^4-(mq-mr+np)x^3+(ms+nq-nr-ps)x^2+(q-r-n)sx-s^2$. 31. $x^3+(a+b+c)x^2+(ab+ac+bc)x+abc$ 32. $x^3-(a+b+c)x^2+(ab+ac+bc)x-abc$.

51. $-6x^2$ 52. $-2y^4$. 53. $6x^2y$ 54. $15x^{\frac{5}{4}}y^{\frac{5}{4}}$.
55. $-3ab^{-2}$ 56. $-ay^{-1}$. 57. $12a^2b^2c^2$.
58. $15xyz$ 59. $-30y^{\frac{4}{5}}bc^{-1}$. 60. $76a^{\frac{4}{3}}x^{-\frac{5}{8}}y^{-2}$.
61. $a+2a^{\frac{1}{2}}b^{\frac{1}{2}}+b$ 62 $a-2a^{\frac{1}{2}}b^{\frac{1}{2}}+b$. 63 $9x^{\frac{4}{3}}-16y^{\frac{2}{3}}$.
64. $a+b$ 65. $x-y$ 66. $a^3+a^{\frac{3}{2}}b^{\frac{3}{2}}+b^3$.
67. $4x^{\frac{8}{3}}-37x^{\frac{4}{3}}y^{\frac{4}{3}}+9y^{\frac{8}{3}}$. 68. a^3-b^2. 69 x^2-y^2.
70. $a-b^2$. 71. $x+y+z-3x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}}$. 72. $a^{3n}+x^{3n}$.
73. $a^{-5}-6a^{-4}b+13a^{-3}b^2-13a^{-2}b^3+6a^{-1}b^4-b^5$
74. $x^{-6}-5x^{-3}y^3+4y^6$. 75. $4a^{-10}+12a^{-\frac{15}{2}}b^{-\frac{3}{2}}+9a^{-5}b^{-3}-25b^{-6}$. 76. $9x^{-\frac{5}{2}}-25x^{-\frac{5}{4}}y^{-\frac{3}{4}}+70x^{-\frac{5}{8}}y^{-\frac{9}{8}}-49y^{-\frac{3}{2}}$.
77. $2x^5-3x^{\frac{7}{2}}a+4x^2a^2-5x^{\frac{1}{2}}a^3+6x^{-1}a^4-7x^{-\frac{5}{2}}a^5+8x^{-4}a^6$.

78 $-3a^{-\frac{5}{6}}-2a^{-\frac{3}{4}}b^{-\frac{2}{5}}+5a^{-\frac{3}{8}}b^{-\frac{4}{5}}-7a^{-\frac{5}{8}}b^{-\frac{2}{7}}+6a^{\frac{5}{4}}b^{-\frac{3}{4}}-5a^{\frac{3}{2}}b^{-4}$ $+8a^{\frac{7}{4}}b^{-2}$.

22 [Pages 66, 67.]

1. $9x^2+24x+16$ **2.** $49x^2+112x+64$. **3** $a^2+10ab+25b^2$. **4.** $4a^2+28ab+49b^2$ **5** $9x^2+48xy+64y^2$ **6.** $25m^2+80mn+64n^2$. **7.** $a^2x^2+6abxy+9b^2y^2$. **8.** $16a^2b^2+8abc^2+c^4$ **9.** $4a^4+20a^2b^2+25b^4$. **10.** $16m^6+8m^3n^2+n^4$. **11.** $a^2+4b^2+9c^2+4ab+6ac+12bc$ **12.** $a^2b^2+b^2c^2+c^2a^2+2ab^2c+2a^2bc+2abc^2$. **13** $4p^2+9q^2+16r^2+12pq+16pr+24qr$. **14** $x^4+y^4+z^4+2x^2y^2+2x^2z^2+2y^2z^2$ **15** $4x^2+9y^2+16z^2+12xy+16xz+24yz$ **16** $x^4+y^6+z^8+2x^2y^3+2x^2z^4+2y^3z^4$. **17.** $x^2+y^2+4a^2+9b^2+2xy+4xa+6xb+4ya+6yb+12ab$. **18.** $9a^2+16b^2+c^2+4d^2+24ab+6ac+12ad+8bc+16bd+4cd$ **19** $4a^2+x^2+16y^2+9z^2+4ax+16ay+12az+8xy+6xz+24yz$. **20** $16m^2+9n^2+9p^2+4q^2+24mn+24mp+16mq+18np+12nq+12pq$ **21.** $4x^2$ **22** $4z^2$. **23** $16a^2$. **24** $a^2+4ab+4b^2$ **25** $x^2+2xy+y^2$. **26** 81. **27.** 4 **28** 0 **29.** 9 **30.** 1. **31** 16. **32** 25.

23 [Pages 69, 70]

1 $4x^2-28x+49$ **2.** $64x^2-80x+25$ **3** $a^2x^2-2abxy+b^2y^2$ **4** $25x^2+30xy+9y^2$ **5** $9m^2-48mn+64n^2$. **6** $a^2b^2+2abcd+c^2d^2$ **7.** $a^4b^2-2a^2bc^2d+c^4d^2$. **8** $x^6-4x^2yz+4y^2z^2$. **9.** $m^2n^2p^2+2mnpqr+q^2r^2$ **10.** $4a^6-20a^3b^3+25b^6$. **11.** $a^2+4b^2+9c^2-4ab-6ac+12bc$ **12.** $4x^2+9y^2+16z^2-12xy-16xz+24yz$. **13.** $9m^2+16n^2+25q^2-25mn-30mq+40nq$ **14.** $a^4+9b^4+25c^4-6a^2b^2-10a^2c^2+30b^2c^2$. **15.** $x^2+y^2+a^2+b^2-2xy-2xa-2xb+2ya+2yb+2ab$ **16.** $a^2+4x^2+9b^2+16y^2-4ax-6ab-8ay+12xb+16xy+24by$. **17.** 7921 **18.** 13689 **19.** 248004. **20.** 986049. **21.** $36b^2$. **22.** $64b^2$. **23.** $49a^2$. **24.** $121z^4$.

25. $25b^2c^2+10bc^2a+c^2a^2$. **26.** 4 **27.** 81. **28.** 16. **29.** 25 **30.** 144.

24 [Pages 71, 72.]

1. $a^2x^2-b^2y^2$. **2.** $25x^2-64$ **3** $c^2x^2-d^4$ **4.** $a^2b^2-b^2c^2$. **5.** $49a^2-9b^2$. **6.** $16a^4-25b^4$. **7.** m^4-9n^2. **8.** $4a^2b^2-49b^2c^2$. **9.** x^4-1. **10.** a^8-b^8. **11.** $a^2+2ab+b^2-c^2$. **12.** $a^2-b^2-2bc-c^2$ **13.** $m^4+m^2n^2+n^4$. **14.** $a^2x^2-b^2y^2+2bycz-c^2z^2$. **15.** $a^2-4b^2+12bc-9c^2$. **16.** x^4+x^2+1. **17.** x^8+x^4+1. **18.** x^4+4y^4 **19.** a^4+b^4. **20.** x^8-2x^4+1. **21.** $4a(b-c)$ **22.** $4a(3c-2b)$. **23.** $4xy(x^2+y^2)$. **24.** $4x(y-a+b)$ **25.** $8a(3b-5c+7d)$. **26.** 9376. **27** 1069840. **28.** 4985645 **29.** $(5x+6)(5x-6)$ **30.** $(3a+4c)(3a-4c)$ **31.** $(4m+7n)(4m-7n)$. **32** $(2p+9q)(2p-9q)$ **33.** $(ax+8b)(ax-8b)$. **34.** $(6x^2+11y^2)(6x^2-11y^2)$. **35.** $(7+8d)(7-8d)$. **36.** $(12c+5d)(12c-5d)$ **37.** $(a+b+c)(a+b-c)$. **38.** $(a+2b+5c)(a+2b-5c)$. **39.** $(2x+3a-4b)(2x-3a+4b)$. **40.** $(a+2b-3c)(a-2b+3c)$ **41.** $(a^2+9b^2)(a+3b)(a-3b)$. **42** $(x-y+a-b)(x-y-a+b)$ **43.** $(9x^2+25y^2)(3x+5y)(3x-5y)$. **44.** $(2a+x)(2a-11x)$. **45.** $(9x+1)(x-1)$. **46.** $(7a-b)(a+15b)$ **47.** $(5x-2y)(x+12y)$. **48** $(2a+3b-4c)(b-2c)$. **49.** $(2m+5n-2p)(2m+n-8p)$. **50** $(5x-7y+12z)(x-y+2z$

25 [Pages 73, 74.]

1. $8x^3+12x^2+6x+1$ **2.** $27x^3+27x^2y+9xy^2+y^3$. **3.** $8x^3+36x^2a+54xa^2+27a^3$ **4.** $a^6+6a^4b+12a^2b^2+8b^3$ **5.** $a^3b^3+3a^2b^3c+3ab^3c^2+b^3c^3$. **5.** $a^3+8x^3+b^3+6a^2x+12x^2a+3a^2b+3ab^2+12x^2b+6xb^2+12abx$ **7.** $8m^3+27n^3+p^3+36m^2n+54mn^2+12m^2p+6mp^2+27n^2p+9np^2+36mnp$. **8.** $x^3y^3+y^3z^3+z^3x^3+3x^3y^3z+3xy^3z^2+3x^3y^2z+3x^3yz^2+3y^2z^3x+3z^3x^2y+6x^2y^2z^2$. **9.** $125m^3$. **10** $x^3+3x^2y+3xy^2+y^3$. **11.** $27b^3$. **12.** x^3+3x^2+3x+1.

13. $x^3+6x^2+12x+8$ **14.** $8a^3$. **15.** 90. **16.** 175. **18.** 52 **19.** -64. **20.** 37. **21.** 0. **22.** 10.

26 [Pages 75, 76.]

1. $1-6a+12a^2-8a^3$. **2.** $8-36x+54x^2-27x^3$. **3.** $27-108x+144x^2-64x^3$. **4.** $125m^3-300m^2n+240mn^2-64n^3$. **5.** $8p^3-60p^2q+150pq^2-125q^3$ **6.** $8x^3-y^3-z^3-12x^2y+6xy^2-12x^2z+6xz^2-3y^2z-3yz^2+12xyz$. **7.** $8m^3-27n^3-p^3-36m^2n+54mn^2-12m^2p+6mp^2-27n^2p-9np^2+36mnp$. **8.** $l^6-m^6-n^6-3l^4m^2+3l^2m^4-3l^4n^2+3l^2n^4-3m^4n^2-3m^2n^4+6l^2m^2n^2$. **9.** $64b^3$. **10** $x^3-3x^2y+3xy^2-y^3$. **11.** $8x^3$. **12.** 0 **13.** 343. **14.** -505 **15.** 27. **16.** 36 **17.** 140

27 [Page 77.]

1. a^3+1. **2.** $8x^3+1$. **3.** $125m^3+1$. **4.** $64x^3+343y^3$. **5.** $27m^3+512n^3$ **6** $a^3b^3+64c^3$ **7.** $a^3x^3+125b^3$. **8.** $125a^3+729b^3$. **9.** $(x+2)(x^2-2x+4)$ **10.** $(2a+1)(4a^2-2a+1)$. **11.** $(m+3)(m^2-3m+9)$. **12.** $(3x+1)(9x^2-3x+1)$ **13.** $(z+4)(z^2-4z+16)$ **14.** $(5m+1)(25m^2-5m+1)$ **15.** $(2a+7x)(4a^2-14ax+49x^2)$. **16.** $(4ax^2+3y)(16a^2x^4-12ax^2y+9y^2)$.

28 [Page 78.]

1. $1-27a^3$ **2.** $64x^3-1$. **3.** $125m^3-27n^3$. **4.** $x^3-8y^3z^3$ **5.** $27a^3-b^3c^3$ **6.** $(5a-1)(25a^2+5a+1)$. **7.** $(7x-2y^2)(49x^2+14xy^2+4y^4)$ **8.** $(6k-5l)(36k^2+30kl+25l^2)$ **9.** $(1-8k)(1+8k+64k^2)$. **10.** $(9m-4an^2)(81m^2+36man^3+16a^2n^4)$.

29 [Page 79.]

1. $x^2+10x+16$ **2.** $x^2+8x+15$. **3.** $x^2+17x+66$. **2.** $m^2+16m+63$ **5.** $x^2-4x-12$ **6.** $m^2+6m-16$. **7.** $a^2-7a+12$ **8.** $x^2+2x-15$ **9.** $x^2+5x-36$.

10 $x^2-15x+50$. 11. $x^2-7x-60$ 12. $k^2-11k-26$.
13. $a^2+19a+70$ 14. $m^2-8m-84$ 15. $x^2-18x+65$
16. $x^2+19x+84$ 17. $a^2-14a+33$ 18. $x^2-9x-52$.
19. $m^2-11m-80$ 20. $x^2-18x+80$. 21. $a^2-6a-72$
22. $m^2+6m-91$ 23. $x^2-26x+160$. 24. $x^2-13x-90$.
25. $x^2-6x-160$

30 [Page 81.]

1. $x^3+6x^2+11x+6$ 2. $x^3+14x^2+59x+70$. 3. $x^3-x^2-24x-36$ 4. $x^3-x^2-70x-200$ 5. $x^3-4x^2-29x-24$. 6 $x^3+x^2-46x+80$ 7. $x^3-37x+84$ 8. $x^3-6x^2-37x+210$ 9. $x^3-23x^2+167x-385$. 10. $x^3-18x^2+99x-162$. 11. $x^3-13x^2-8x+240$ 12. $x^3+25x^2+199x+495$ 13. $x^3-52x+96$ 14. $x^3-23x^2+151x-273$
15 x^3+13x^2-144 16. $x^3-7x^2-138x+1080$.
17. $x^3-3x^2-73x+315$ 18. $x^3+35x^2+396x+1440$.
19. $x^3-148x-672$ 20. $x^3-31x^2+290x-800$.

31 [Page 83.]

1. $x^2+y^2+z^2+2xy-2xz-2yz$ 2. $x^2+y^2+z^2-2xy+2xz-2yz$ 3. $x^2+y^2+z^2-2xy-2xz+2yz$ 4. $x^2+y^2+z^2+2xy-2xz-2yz$ 5 $x^2+y^2+z^2-2xy-2xz+2yz$
6 $a^2+x^2+y^2+z^2-2ax+2ay-2az-2xy+2xz-2yz$
7. $a^2+x^2+y^2+z^2-2ax-2ay-2az+2xy+2xz+2yz$
8. $m^2+n^2+p^2+q^2+r^2+2mn+2mp+2mq+2mr+2np+2nq+2nr+2pq+2pr+2qr$. 9 $p^2+q^2+r^2+x^2+y^2-2pq+2pr-2px-2py-2qr+2qx+2qy-2rx-2ry+2xy$ 10. $a^2+b^2+c^2+x^2+y^2+z^2-2ab+2ac-2ax+2ay+2az-2bc+2bx-2by-2bz-2cx+2cy+2cz-2xy-2xz+2yz$ 11 $a^2+4x^2+9y^2+16z^2-4ax-6ay-8az+12xy+16xz+24yz$. 12. $4a^2+b^2+4c^2+d^2-4ab+8ac-4ad-4bc+2bd-4cd$. 13. 49.
14. 9. 15. 0 16. 144. 17. 1635. 18. 1.
19. 63 20. 0 21. 47. 22. 69.

32 [Page 86.]

1. $x^5+5x^4+10x^3+10x^2+5x+1$ 2. $x^6+6x^5+15x^4+20x^3+15x^2+6x+1$ 3 $a^8+8a^7b+28a^6b^2-56a^5b^3+70a^4b^4+56a^3b^5+28a^2b^6+8ab^7+b^8$ 4. $a^9+9a^8b+36a^7b^2+84a^6b^3+126a^5b^4+126a^4b^5+84a^3b^6+36a^2b^7+9ab^8+b^9$. 5. $x^5-5x^4y+10x^3y^2-10x^2y^3+5xy^4-y^5$ 6 $m^7-7m^6n+21m^5n^2-35m^4n^3+35m^3n^4-21m^2n^5+7mn^6-n^7$. 7. $x^4+8x^3+24x^2+32x+16$ 8. $x^5+10x^4+40x^3+80x^2+80x+32$ 9. $x^8+8x^7+28x^6+56x^5+70x^4+56x^3+28x^2+8x+1$. 10. $x^4+12x^3+54x^2+108x+81$. 11. $x^5-5x^4+10x^3-10x^2+5x-1$. 12. $64-192z+240z^2-160z^3+60z^4-12z^5+z^6$. 13. $16x^4-32x^3+24x^2-8x+1$ 14. $x^9-9x^8y+36x^7y^2-84x^6y^3+126x^5y^4-126x^4y^5+84x^3y^6-36x^2y^7+9xy^8-y^9$ 15. $243x^5-810x^4+1080x^3-720x^2+240x-32$. 16 $1-8a+28a^2-56a^3+70a^4-56a^5+28a^6-8a^7+a^8$ 17. $1-7c+21c^2-35c^3+35c^4-21c^5+7c^6-c^7$. 18. $1-18x+135x^2-540x^3+1215x^4-1458x^5+729x^6$. 19. $1-14x+84x^2-280x^3+560x^4-672x^5+448x^6-128x^7$. 20. $256x^8-1024x^7a+1792x^6a^2-1792x^5a^3+1120x^4a^4-448x^3a^5+112x^2a^6-16xa^7+a^8$. 21. $x^{10}-10x^9a+45x^8a^2-120x^7a^3+210x^6a^4-252x^5a^5+210x^4a^6-120x^3a^7+45x^2a^8-10xa^9+a^{10}$.
22. $243x^5-810x^4a+1080x^3a^2-720x^2a^3+240xa^4-32a^5$
23 $10x^4+20x^2+2$. 24. $2x^6+30x^4+30x^2+2$
25. $14x^6a+70x^4a^3+42x^2a^5+2a^7$. 26. 16. 27. 32.
28. 64. 29. 128. 30. 256. 31. 30. 32. 8.
33. 0. 34. 16 35. 0.

33 [Page 87, 88.]

1. $x^3+y^3-z^3+3xyz$. 2. $p^3-8q^3-r^3-6pqr$. 3. $8x^3-27y^3-z^3-18xyz$ 4. $a^3-8b^3+27+18ab$ 5. $27a^3-125b^3-64-180ab$. 6. $(x-y-1)(x^2+y^2+1+xy+x-y)$. 7. $(x-y+2)(x^2+y^2+4+xy-2x+2y)$. 8. $(x-2y-3z)(x^2+4y^2+9z^2+2xy+3xz-6yz)$. 9. 0. 10. 0. 11. 0.

34 [Page 89]

3 $8(c-b)(a-b)(a-c)$ 4 $(x-z)(y-z)(y-x)$. 5. 0
6. 0.

35 [Pages 92—95]

11. $27x^3+8y^3+z^3-8xyz$. 14. $2a^2b^2+2a^2c^2+2b^2c^2-a^4-b^4-c^4$ 15. $4(a^2b^2+a^2c^2+b^2c^2)$. 16. $2(a^4+b^4+c^4)$. 19. $8ac$. 20. $8a^2b^2$. 23. $(a+d+b-c)(a+d-b+c)(b+c+a-d)(b+c-a+d)$. 24 x^8-2x^4+1 25. $a^4-2a^2b^2+4abc^2+b^4-c^4$. 27. $-2b^2c$ 28. $2a^2c$. 34. $8x^3$ 44. 247. 45. 36. 46. 84. 47. 349.

36 [Pages 99, 100]

1. $-4a^2$ 2. $-5a^4$. 3. $4x^4a^2$. 4 $3mn^2$. 5 $-4p^2q$. 6 $3a^4b^3$ 7. $5x^6y^4z$. 8 $-8a^3c^2$. 9. $-3m^5n^6p$ 10. $3a^2c^2$. 11. $-5x^4y^2$ 12. $3a^6x^7y^4z^2$. 13. a^{44}. 14 $-7x^{48}$. 15. $-7m^{18}$. 16 $-7a^{41}b^{120}$.

37 [Pages 100, 101.]

1. $a-2b$. 2 $-x^2+2xa$. 3 $xa-2x^2$ 4. $3a^2-4b^2$. 5. $-2ab^2+3a^2b$. 6 $x^2-2ax+3a^2$ 7. $ax-2x^2+3a^2$. 8. $-3x^2+2a^2-5ax$. 9 $2m^2n^2-3m^4-4n^4$. 10 $-p^2+\frac{5}{8}pq+\frac{3}{8}q^2$. 11. $-2xy^2+3x^3-4y^3$. 12. $\frac{3}{4}x^3-\frac{3}{2}a^3-\frac{3}{4}a^2x$ 13. $3xa+\frac{15}{4}a^2-4x^2$. 14. $5m^2n-7mn^2-8p^3$. 15. $ax^2y-a^3x+x^2y^2-a^4$.

38 [Pages 105—108]

1. $x-2$. 2. $x-5$. 3 $3x+4$. 4. $5x-7$. 5. $2a-3b$. 6. x^2-xy+y^2. 7. $2x-3a$. 8 x^2-ax+a^2. 9. $a^2+2ab-b^2$. 10 $m^2-3mn+2n^2$. 11. $a^2-3ab+b^2$. 12. $2x^2-3xy-2y^2$. 13 $a^2-4ax-2x^2$. 14 $3+2x-2x^2+x^3$. 15. x^2-2x+3. 16. $2a^2+3ab-4b^2$. 17. $a^2-2ax+4x^2$. 18. $a^2-2ab+2b^2$. 19. $2x^2-3x-8$. 20. $x^3+3x^2+9x+27$.

21. $a^4+2a^3+4a^2+8a+16$ 22 $3-x^2+2x^3$. 23. $3x^2-4x+5$ 24 $32+16x+8x^2+4x^3+2x^4+x^5$. 25. $x^4+2x^3+3x^2+2x+1$ 26 $2a^2-3ab+4b^2$ 27 $a^2+3ab-5b^2$. 28 $x^3+2x^2a+2xa^2+a^3$ 29 $a^3-3a^2b-b^3$. 30 $x^4+2yx^3+3y^2x^2+2y^3x+y^4$ 31 $x^4+2ax^3+3a^2x^2+2a^3x+a^4$. 32 $x^4-2x^3y+3x^2y^2-2xy^3+y^4$ 33 $x^2+(a+b)x+ab$. 34 $x-c$ 35 $a+b+c$ 36 $ab+ac+bc$ 37. $ab+ac-bc$. 38 $x^2-(a-b)x-ab$ 39 $a^2+b^2+c^2-ab-ac-bc$. 40 $x^2+y^2+1-xy+x+y$. 41. $x^2+4y^2+9z^2+2xy+3xz-6yz$. 42 $x^2+y^2+z^2+xy-xz+yz$ 43. $2x-3y-z$. 44 $ab-ac-bc+c^2$ 45 $x+c$ 46 $x+a$. 47 $a^2+ab-bc-c^2$ 48 $ab-ac+bc-b^2$. 49 $y^2x+2y^2z+yx^2-2yz^2-x^2z-xz^2$ 50 x^2-ax+a^2 51 $c+a-b$. 52 $2(a+b)x$. 53 $x+y+z+xyz$ 54. $16x^4-8x^2(2y^2+a^2)+(4y^2-a^2)^2$. 59. a^3b 60. $ab^{-1}c^{\frac{1}{2}}$. 61 $-3x^{\frac{1}{3}}y^{\frac{2}{5}}z^{-\frac{1}{3}}$. 62 $3x^{\frac{2}{3}}-4y^{\frac{1}{3}}$. 63 $a^{\frac{2}{3}}-a^{\frac{1}{3}}b^{\frac{1}{3}}+b^{\frac{2}{3}}$ 64 $a^{\frac{3}{2}}-a^{\frac{3}{4}}b^{\frac{3}{4}}+b^{\frac{3}{2}}$ 65 $2x^{\frac{4}{3}}-5x^{\frac{2}{3}}y^{\frac{2}{3}}-3y^{\frac{4}{3}}$ 66 $a^{\frac{3}{4}}+a^{\frac{1}{2}}b^{\frac{1}{2}}+a^{\frac{1}{4}}b+b^{\frac{3}{2}}$ 67 $2a^{-5}+3a^{-\frac{5}{2}}b^{-\frac{3}{2}}+5b^{-3}$ 68 $3x^{-\frac{5}{4}}-5x^{-\frac{5}{8}}y^{-\frac{3}{8}}+7y^{-\frac{3}{4}}$ 69 $a^{\frac{5}{2}}+a^2b^{\frac{1}{3}}+a^{\frac{3}{2}}b^{\frac{2}{3}}+ab+a^{\frac{1}{2}}b^{\frac{4}{3}}-b^{\frac{5}{3}}$ 70 $x^{\frac{2}{3}}+y^{\frac{2}{3}}+z^{\frac{2}{3}}-x^{\frac{1}{3}}y^{\frac{1}{3}}-y^{\frac{1}{3}}z^{\frac{1}{3}}-z^{\frac{1}{3}}x^{\frac{1}{3}}$.

39 [Pages 109, 110.]

13 x^3+x^2+x+1 14 $x^3-x^2y+xy^2-y^3$ 15 $x^4+x^3+x^2+x+1$ 16 $x^4-x^3y+x^2y^2-xy^3+y^4$. 17. $x^5+x^4+x^3+x^2+x+1$. 18 $x^5-x^4y+x^3y^2-x^2y^3+xy^4-y^5$. 19 $x^6+x^5+x^4+x^3+x^2+x+1$. 20. $x^6-x^5y+x^4y^2-x^3y^3+x^2y^4-xy^5+y^6$

40 [Page 111.]

1 $a(b+c)$ 2 $a^2b^2(b+a)$. 3. $x^3y^3(y-2x)$. 4. $2xyz(x+2y-3z)$. 5 $2a^3b(2a^2-3ab-4b^2)$ 6 $ax^2(y-5axy^2+3x)$. 7. $3x^2y^2z^2(x^2y-4y^2z+7xz^2)$.

8. $14a^3b^2(2a^3-3b^2)$ 9. $36x^ny^n(2x^2+3y^n)$.
10. $13a^2bc(3b^2c^2-5c^2a^2-7a^2b^2)$.

41 [Page 111.]

1. $(8a+4b)(8a-4b)$. 2. $a(2a+5x)(2a-5x)$.
3. $(6x^2+1)(6x^2-1)$. 4. $(4x^2+1)(2x+1)(2x-1)$.
5. $x(4x^2+3)(4x^2-3)$ 6. $x(4x^2+9)(2x+3)(2x-3)$
7. $(1+4a^2)(1+2a)(1-2a)$ 8. $x^2(1+9x^2)(1+3x)(1-3x)$.
9. $(6+x^2a)(6-x^2a)$ 10 $(8a^2+7x^3)(8a^2-7x^3)$.
11. $(11+m^2)(11-m^2)$ 12. $(7x^3a+9)(7x^3a-9)$.
13. $(ab+5cd)(ab-5cd)$ 14. $(9x^6+8a^3)(9x^6-8a^3)$
15. $p^2(q^2+10)(q^2-10)$. 16. $x^3(12x^2+5a^2)(12x^2-5a^2)$.
17. $3a^2(8a^2+9x^3)(8a^2-9x^3)$ 18. $2ax(7ax^2+8)(7ax^2-8)$.
19. $4xa^2(9x^4a^2+11)(9x^4a^2-11)$. 20. $5m^3n^3(7m^2n^3+11)(7m^2n^3-11)$ 21. $(a+3b+5c)(a+3b-5c)$
22 $(a+3b-5c)(a-3b+5c)$ 23 $4xy$. 24. $(5a+3x)(a+x)$. 25. $(2a-2b+3c-3d)(2a-2b-3c+3d)$.
26. $(7x+5y-3z)(7x-5y+3z)$. 27. $12(5x-1)(x+2)$.
28. $4a(b-c)$ 29. $(3a+b-c)(a-7b+9c)$.
30. $(14a+21x-23y)(2a+27x-11y)$.
31. $-9(x+a)(x-a)(x^2+a^2)$. 32. $28a(5a-8)$.

42 [Pages 112, 113.]

1. $(x^2+x+1)(x^2-x+1)$. 2. $(x^2+x+1)(x^2-x+1)(x^4-x^2+1)$. 3. $(a^2+ax+x^2)(a^2-ax+x^2)$. 4. $(a^2+ax+x^2)(a^2-ax+x^2)(a^4-a^2x^2+x^4)$. 5. $(x^2+4x+8)(x^2-4x+8)$.
6. $(2x^2+6x+9)(2x^2-6x+9)$. 7. $9(x^2+2x+2)(x^2-2x+2)$.
8. $(a^2+2a+3)(a^2-2a+3)$ 9. $(x^2+x-3)(x^2-x-3)$
10. $(2x^2+2x+3)(2x^2-2x+3)$. 11. $(2x^2+2x-3)(2x^2-2x-3)$ 12. $(2x^2+3x+3)(2x^2-3x+3)$. 13. $(2a^2+5a-3)(2a^2-5a-3)$. 14. $(2a^2+10a+25)(2a^2-10a+25)$.
15. $(3x^2+x+4)(3x^2-x+4)$. 16. $(3a^2+a-4)(3a^2-a-4)$
17. $(3x^2+3x-4)(3x^2-3x-4)$. 18. $(3a^2+5a+1)(3a^2-5a+1)$. 19. $(4x^2-6xa+5a^2)(4x^2+6xa+5a^2)$ 20. $(3a^2+$

$7ax+5x^2)(3a^2-7ax+5x^2)$ **21.** $(x^2+4x+12)(x^2-4x+12)$. **22.** $(a^2+5ab-5b^2)(a^2-5ab-5b^2)$ **23.** $(6a^2+2ab-b^2)(6a^2-2ab-b^2)$. **24.** $(7m^2+2mn-4n^2)(7m^2-2mn-4n^2)$. **25.** $(8a^2+12ax+9x^2)(8a^2-12ax+9x^2)$ **26.** $(2x^2+14xa+49a^2)(2x^2-14xa+49a^2)$. **27.** $(x+y-z)(x-y+z)$. **28.** $(2a+b-3c)(2a-b+3c)$ **29.** $(3x+2y-3z)(3x-2y+3z)$ **30** $(a+2b-5c)(a-2b+5c)$ **31.** $(4y+3x-5z)(4y-3x+5z)$ **32.** $(a-2b+3c-2d)(a-2b-3c+2d)$. **33.** $(x-2y+z)(x-z)$ **34.** $(2x+3a+5b+1)(2x+3a-5b-1)$ **35.** $(3x+2y-7z-5)(3x-2y+7z-5)$ **36.** $(4a+3b-4c-3)(4a-3b+4c-3)$. **37.** $(x-7y+5z-2)(x-7y-5z+2)$ **38.** $(4x+5a+3y-7b)(4x+5a-3y+7b)$. **39.** $(7x-4y+8z-1)(7x-4y-8z+1)$. **40.** $(a+b-c-d)(a-b+c-d)$.

43 [Pages 114, 115.]

1. $(a-2b)(a^2+2ab+4b^2)$ **2.** $a(a-3x)(a^2+3ax+9x^2)$. **3.** $(2x+1)(4x^2-2x+1)(64x^6-8x^3+1)$ **4.** $(a-2b)(a^2+2ab+4b^2)(a^6+8a^3b^3+64b^6)$. **5.** $(3a^2+5x^2)(9a^4-15a^2x^2+25x^4)$. **6.** $(m+n)(m-n)(m^2-mn+n^2)(m^2+mn+n^2)$. **7.** $(7x+8y)(49x^2-56xy+64y^2)$ **8.** $(2x^2-1)(2x^2+1)(4x^4+2x^2+1)(4x^4-2x^2+1)$. **9.** $(a-2x^2)(a+2x^2)(a^2+2ax^2+4x^4)(a^2-2ax^2+4x^4)$. **10.** $(5x^3-6a^3)(25x^6+30x^3a^3+36a^6)$. **11.** $ab(4a^4+7b^4)(16a^8-28a^4b^4+49b^8)$. **12.** $x^2y^2(3x^3+2y^3)(3x^3-2y^3)(9x^6-6x^3y^3+4y^6)(9x^6+6x^3y^3+4y^6)$ **13.** $(a+b)^2(a^4-2a^3b+6a^2b^2-2ab^3+b^4)$. **14.** $2(x+y)(x-y)(4x^4-14x^2y^2+13y^4)$. **15.** $2(a-b)(a^2+ab+b^2)(4a^6-2a^3b^3+b^6)$.

44 [Pages 118, 119]

1. $(x+3)(x+1)$. **2.** $(x+2)(x+3)$. **3.** $(x+3)(x+4)$. **4.** $(x+4)(x+5)$. **5.** $(x+3)(x+6)$. **6.** $(x+4)(x+7)$. **6.** $(x-4)(x-6)$ **8.** $(x-3)(x-5)$ **9.** $(x-5)(x-6)$.

10. $(x-4)(x-8)$. 11. $(x-12)(x-2)$. 12. $(x-2)(x-20)$.
13. $(x+10)(x-8)$. 14. $(x+8)(x-6)$. 15. $(x+18)(x-2)$.
16. $(x+12)(x-8)$. 17. $(x+14)(x-3)$. 18. $(x+18)(x-4)$.
19. $(x-8)(x+5)$. 20. $(x-16)(x+5)$. 21. $(x-32)(x+3)$.
22. $(x-11)(x+4)$. 23. $(x-7)(x+6)$. 24. $(x-9)(x+8)$.
25. $(x+10)(x+12)$. 26. $(x+20)(x-4)$. 27. $(x-24)(x+3)$.
28. $(x+13)(x-7)$. 29. $(x-12)(x-8)$. 30. $(x+26)(x-3)$.
31. $(x-12)(x+6)$. 32. $(x-21)(x-1)$. 33. $(x-22)(x-4)$.
34. $(x+15)(x-8)$. 35. $(x-10)(x+8)$. 36. $(x+14)(x-6)$.
37. $(a-8)(a+7)$. 38. $(m-15)(m+6)$. 39. $(a+20)(a-3)$.
40. $(a-9)(a-6)$. 41. $(p-24)(p+2)$. 42. $(m+9)(m-8)$.
43. $(m+20)(m-3)$. 44. $(a-24)(a-5)$.
45. $(x+13)(x-6)$. 46. $(a-51)(a+2)$. 47. $(a-15)(a-4)$.
48. $(x+16)(x-1)$. 49. $(a-30)(a+4)$. 50. $(x+15)(x-7)$.
51. $(x-7y)(x+6y)$. 52. $(a-8b)(a-4b)$.
53. $(m+6n)(m-5n)$. 54. $(a+4b)(a-3b)$. 55. $(a-5b)(a+3b)$. 56. $(x-8y)(x+y)$. 57. $(x+8y)(x-5y)$. 58. $(p-6q)(p-8q)$. 59. $(p+10q)(p-8q)$. 60. $(x+21y)(x-4y)$. 61. $(a+1)(a-1)(a^2+5)$. 62. $(x^2+5)(x^2-3)$. 63. $(x+2)(x-2)(x^2+7)$. 64. $(x-1)(x^2+x+1)(x^3+3)$. 65. $(a-2)(a^2+2a+4)(a^3-2)$. 66. $(x-1)(x+3)(x^2+x+1)(x^2-3x+9)$. 67. $(a-1)(a+2)(a^2+a+1)(a^2-2a+4)$. 68. $(x^2+2)(x^2-2)(x+2)(x-2)(x^2+4)$. 69. $(a+2)(a-2)(a^2+4)(a^4+5)$. 70. $(x^2+1)(x^2-2)(x^4-x^2+1)(x^4+2x^2+4)$.
71. $(a+1)^2(a^2+2a-2)$. 72. $(x+1)(x+2)(x^2+3x+1)$.
73. $(x-1)^2(x+1)(x-3)$. 74. $(a+1)(a-4)(a^2-3a+1)$.
75. $(x+1)(x-5)(x^2-4x+1)$. 76. $(x+1)(x-2)(x+2)(x-3)$.
77. $(x-2)(x-3)(x-1)(x-4)$. 78. $(a-2)(a+9)(a+2)(a+5)$.
79. $(a-1)(a+10)(a+2)(a+4)$. 80. $(x+1)(x-9)(x+2)(x-10)$.
81. $2(x+y)(7x+8y)$. 82. $(5a+b)(a-7b)$. 83. $(5m+17n)(3m-8n)$. 84. $4(5a+1b)(2a+b)$. 85. $(x+11y)(7x+5y)$.
86. $(2a+23b)(9a-5b)$. 87. $(3a+11b)(10a-7b)$.
88. $7n^2(9m^2-11n^2)$. 89. $(x+2y)(x+3y)(x^2+4xy+12y^2)$.

90 $(a-2b)(a-6b)(a-3b)(a-8b)$ 91. $(2x-5)(x+3)$. 92. $(3a-5)(2a+3)$ 93. $(4m+3)(2m-3)$ 94. $(2x-3y)(3x+8y)$ 95. $(5a-3b)(2a-7b)$ 96 $(3m-4n)(4m+5n)$. 97. $(2x+5y)(6x-y)$ 98. $(4a+5b)(5a-6b)$ 99 $(3x-5)(6x-7y)$ 100. $(4x-3y)(3x+8y)$.

45 [Pages 122, 123.]

1. $(x+3)(x+1)$ 2. $(x+5)(x+1)$ 3 $(x+5)(x+3)$. 4. $(x-7)(x-3)$. 5. $(x-8)(x+6)$ 6 $(x-9)(x+5)$. 7 $(x-8)(x-4)$ 8. $(x-11)(x+5)$ 9. $(a+2b-c)(a+c)$. 10. $(x+y)(x-y+2)$ 11. $(x+y+1)(x-y+5)$. 12 $(a+5b-c)(a-b+c)$ 13 $(x-y-z)(x-5y+z)$. 14. $(x-2y-2z)(x-8y+2z)$ 15. $(a+b-3c)(a-13b+3c)$. 16. $(x+12y-3z)(x+3z)$ 17. $(x+y-5z)(x-15y+5z)$ 18 $(2x+1)(x-3)$ 19. $(3x+1)(x-2)$ 20 $(3x+2)(x+4)$ 21 $(4x-1)(x+2)$ 22 $(2x-1)(3x+2)$ 23 $(2x+1)(3x-4)$ 24. $(3x-1)(2x+3)$ 25. $(2x+3)(4x-5)$ 26. $(2x-5)(2x+7)$ 27 $(2x-3)(3x+4)$ 28. $(3x+2)(x-6)$. 29. $(2x+5)(x-7)$. 30. $(2x-7)(x+6)$. 31. $(3x-5)(x+6)$ 32. $(3x-2)(4x+3)$. 33. $(a+5b)(2a-3b)$. 34. $(2x-3y)(3x-2y)$. 35. $(3m+2n)(2m-5n)$ 36. $(3p-4q)(p+3q)$. 37. $(2a-5b)(4a+3b)$. 38. $(5m-2n)(2m+3n)$. 39. $(4x-y)(3x+4y)$ 40. $(3a-4b)(5a+3b)$. 41. $(2a-b)(a-2b)$. 42. $(a-3b)(3a+b)$. 43. $(x+3y)(3x-y)$. 44. $(a+4)(4a-1)$ 45. $(a-4b)(4a-b)$. 46. $(x-5)(5x+1)$ 47. $(x-5y)(5x-y)$. 48. $(x+6)(6x+1)$. 49. $(a+6b)(6a-b)$ 50. $(a-6b)(6a+b)$. 51 $(a-7b)(7a-b)$. 52. $(a+7b)(7a-b)$ 53 $(a-7b)(7a+b)$ 54. $(8x-y)(x+8y)$. 55. $(9x-y)(x-9y)$. 56. $(10x-y)(x+10y)$. 57. $(2a+2b-1)(a+b+2)$. 58. $(x-y)^2(2x^2+2y^2+xy)$. 59. $(a+b)^2(2a^2+2b^2+ab)$. 60. $(x-4y)(4x-y)(x^2+y^2)$. 61. $(x+2)(x-2)(2x^2+3)$. 62. $(2a+3b)(2a-3b)(2a^2+b^2)$. 63. $(3a+4b)(3a-4b)$

(a^2+2b^2). 64. $(x-2)(2x-1)(x^2+2x+4)(4x^2+2x+1)$. 65. $(2a^2+b^2)(2a^2-b^2)(a^2+2b^2)(a^2-2b^2)$.

46 [Pages 126, 127]

1. $(x+1)(x^2+1)$. 2. $(x+1)^2(x-1)$. 3. $(x+1)(x-1)^2$. 4. $(ab+c)(ac+b)$. 5. $(x-a)(x+b)(x^2-bx+b^2)$. 6. $(ax+by)(bx+ay)$ 7. $(x-z)(x+y+z)$. 8. $(x+a)(b-c)$. 9. $(x^2-ab)(2a-3b)$. 10. $(a-b)(a+b+c)$. 11. $(2a-3b)(2a+3b+4c)$. 12. $(ax-by)(ax+by+cz)$ 13. $(x^2-yz)(x^2+yz+y^2)$. 14. $(4x-5a)(4x+5a+3b)$. 15. $(a+b)(a^2+ab+b^2)$. 16. $(m-n)(m^2-mn+m^2)$ 17. $(a-b)(a+b)^2$. 18. $(x+y)(x-y)^3$ 19. $(a+2)(a^2+3a+4)$ 20. $(x-5)(x^2-12x+25)$. 21. $(2a-3b)(4a+3b)(a+3b)$. 22. $(x-y)(x-y-1)$. 23. $(2a-b)(2a-b-3)$. 24. $(x^2+a^2)(x-a)^2$. 25. $(a^2+2b^2)(a-b)(a-2b)$ 26. $(a+2b)(a+b+c)$ 27. $(x-3y)(x-y+z)$. 28. $(m-2n)(m-3n+2p)$ 29. $(a-3b)(a-7b-5c)$. 30. $(x+4a)(2x-3a+4b)$ 31. $(a-4b)(a-2b+3)$. 32. $(3x+y)(x-3y+2)$. 33. $(a-b-c)(a+b+c+1)$. 34. $(x-2y+3z)(x+2y-3z+4)$ 35. $(3x-4y-2z)(3x-4y+2z-5)$ 36. $(a+b)(a-b)(x+a)(x-a)(2x^2-3a^2)$ 37. $(2x-3b)(x^2+ax-b)$. 38. $(x+a)(x-a)(a+b)^2$. 39. $(b-c)(a-c)(a-b)$ 40. $(b+c)(a+c)(a+b)$. 41. $(b-c)(a-c)(a-b)(ab+bc+ca)$. 42. $(b-c)(a-c)(a-b)(a^2+b^2+c^2+ab+bc+ca)$. 43. $(b+c)(a+c)(a+b)(b-c)(a-c)(a-b)$. 44 $(b-c)(a-c)(a-b)(a^3+b^3+c^3+a^2b+ab^2+a^2c+ac^2+b^2c+bc^2+abc)$.

47 [Pages 132—134.]

1 $a^2+b^2+c^2-ab+ac+bc$. 2. $x^2+y^2+1+xy+x-y$. 3 $x^2+4y^2+9z^2+2xy-3xz+6yz$ 4. $2a-3b-c$. 5. x^2-2x-3 6 a^2+2bc. 7. $5-2a+b$. 8 $2x^2-3x+4$. 9 x^3-2x^2+3x-5 10 x^3-6x^2+5x-2. 11 $x^2+3x+13$. 12 $a^2-6ab+12b^2$. 13. $a^3-2a^2b+2ab^2-b^3$. 14. x^3-3x^2+2x-1. 15. $x^2+4xy+19y^2$. 16. $(x+1)$

$(x+3)(x+4)$. **17.** $(x+2)(x+3)(x+4)$ **18.** $(x-1)(x-2)(x-3)$ **19.** $(x-2)(x+3)(x+4)$ **20** $(x-1)(x^2-3x-2)$. **21.** $(x+1)(x^2+4x-6)$ **22.** $(x-2)(x^2-4x+5)$ **23.** $(x-2)(x+2)(x^2-3x-5)$ **24.** $(x-1)(x+2)(x^2-4x+5)$ **25.** $(x+1)(x-3)(x^2-3x-2)$ **26.** $(x-2)(x+3)(x^2+4x-6)$ **27.** $(x+2)(x-4)(x^2-5x+7)$ **28.** $(x-5)(x^2-2x+3)$ **29** $(x+3)(x^2-3x+4)$. **30.** $(x+2)(x-4)^2$ **31.** $(x-2)(2x^2+x+2)$ **32.** $(x+2y)(x^2-2xy-5y^2)$ **33.** $(a+3b)(a^2+ab-3b^2)$ **34.** $(a-2b)(5a^2+7ab+14b^2)$ **35.** $(2x-1)(4x^2+2x+3)$ **36.** $(x-1)(x+3)(2x+1)$ **37.** $(x+1)^2(x-2)$ **38** $(a-b)(2a^2+ab+b^2)$ **39.** $(x-1)(3x^2+11x+3)$ **40.** $(x+3y)(x^2-3xy+3y^2)$. **41.** $(x+a-b)(x-a+2b)$ **42.** $\{x^2+(a+b)^2y^2\}\{x+(a-b)y\}\{x-(a-b)y\}$. **43** $\{a^2+(x+y)^2b^2\}\{a^2+(x-y)^2b^2\}$ **44.** $(a+2x-y)(a-x+2y)$ **45.** $(x+2a+b)(x-a+2b)$. **46** $(x+3y-z)(x+y+z)$ **47.** $(2a+b-3c)(2a-3b+3c)$ **48** $(x^2+4x-3)(x^2+2x+3)$ **49.** $(a^2+ab-b^2)(a^2-5ab+b^2)$ **50.** $(2x^2-4x-3)(2x^2-6x+3)$. **51.** $(x-1)^2(x^2+1)$ **52.** $(a^2+3a-5)(a^2-3a+5)$ **53.** $(a-bx)(a-bx-cx^2)$ **54.** $(x^2y^2+xy-z+1)(x^2y^2-xy+z+1)$ **55.** $\{(y+z)x-y+z\}\{(y-z)x+y+z\}$ **56** $\{(a+b)x+(a-b)y\}\{(a-b)x+(a+b)y\}$. **57.** $(x^2-2x-1)(x^2-2x-4)$. **58.** $(a^2-3a+5)(a^2-3a+1)$ **59.** $(2x^2+3x-3)(2x^2+3x-4)$ **60.** $(x^2-xy+y^2)(x^2-4xy+y^2)$ **61.** $(x^2-2x+4)(x^2-3x+4)$ **62.** $(a^2-2ab+2b^2)(a^2-5ab+2b^2)$ **63.** $(x^2-3x+5)(x^2+7x+5)$ **64.** $(a-b)^2(a^2+6ab+b^2)$ **65.** $(x^2+4x+10)(x^2+4x-2)$ **66.** $(x^2-3x-5)(x^2-3x-17)$ **67.** $(x-1)(x+8)(x^2+7x+30)$. **68.** $(x-3)(2x+3)(2x^2-3x+7)$ **69.** 0. **70.** 0 **71.** 0. **72.** 300 **73.** 0. **74.** 5 **75.** $a(a^4-5a^2b^2+5b^4)$. **76.** 392000.

48. [Page 136]

1. a^2b^2 **2** $4a^2$ **3** $3xy^2$. **4.** $5a^2y^3$. **5.** $9m^2n^3$. **6.** $4ax$. **7.** $12mp$. **8.** $15x^2y^2z^2$ **9.** $18a^2c^2$.

10. $24c^3$. 11. $12z^2$. 12. $15m^3n^3p^3q^3$. 13. $18a^2b^2c^2d^2$.
14. 6. 15. $8x^2y^2$.

49 [Pages 137, 138.]

1. $a(a+b)$. 2. $x^3y^3(x+y)$ 3. $3(x+3)$. 4. $4a^2(a^2+bc)$.
5. $m^3n^2(m-n)^2$ 6. $ax(2a+3x)$. 7. $2a^2b^2(3a+4b)$.
8. $3x^2y^2(x-2y)$. 9. $2ab(a+2b)$. 10. $16x^3a^3(x^2-a^2)$.
11. $8(x^2+ax+a^2)$ 12. $8xa^2(x^2+a^2)$. 13. $6(a+3b)$.
14. $4x(x-5)$ 15. $xy(x+6y)$ 16 $a^2x^2(a+2x)$.
17 $2x+3$ 18 $a-2b$ 19. $x-2$ 20 $18(x+2a)$.
21. $a-b$ 22 $x+2$. 23 $4ab(3a+b)$ 24. x^2+5x+6.

50 [Pages 144, 145.]

1. $2x-1$. 2. $3x-2$. 3 $2x+5a$ 4. $x(3x+4)$.
5. $3a-1$ 6 $2a-3b$ 7. x^2+x+1. 8 $x-xy+y^2$.
9 $x(2x^2+x+1)$ 10. x^2+3x+1. 11 x^2+4x+1.
12 $x^2+2ax+3a^2$ 13. $x^2+3ax+5a^2$ 14 $2a^2-3ax+7x^2$.
15 $2-3x+5x^2$. 16. $1+4x-7x^2$ 17. $x^2(2x^2+3ax+4a^2)$
18 $2(a^2+5a+2)$. 19. x^2+3x-2 20 x^2-3x+5.
21 x^2+5x+1 22 x^2+2x+4 23 x^2+3x+5.
24 $2(x^2-2ax+2a^2)$. 25. $3x^2+2xy+4y^2$. 26 x^2+2x+3.
27 $4a^2+2a-5$. 28 x^2+2x+3

51 [Page 148.]

1 x^2-5x+6. 2 $2x^3-17x+12$ 3. x^3+3x+4.
4 $3x^3-5x^2+7$, 5. $6x^2-11x+4$. 6. $2x^2+15x-8$.
7 $3x^2+5x-1$. 8. $5x^2-3x-1$. 9 $2x^2+3x-1$.
10. $3x^3-2x+1$. 11 x^2+x+2. 12. x^2+3x-2.

52 [Pages 150, 151.]

1. $x+4$. 2 $2x-1$. 3. $2x-3$ 4 $2x^2+1$.
5 $3a-2b$. 6. $3a-5b$. 7. $3x-4$. 8. $2x^2-3$.

53 [Pages 153, 154.]

1. a^2b^2 2 a^3b^2c 3 $30x^2y^4$. 4. $28m^4n^3p$. 5 $24x^3y^3z^2$ 6 $140a^2b^2c^2$ 7 $120a^3b^3c^3$. 8. $180x^4y^3z^2a^2$. 9 $a^2b^2(a^2-b^2)$ 10 $24(x^2-y^2)^2$. 11. $(x-1)(x-3)(x-2)$. 12 $a^2(a-x)(a+3x)(a-2x)$. 13. $a^2(a-2)(a+2)(a+4)$ 14. $12a^3x^2(x^2-a^2)(x^2-ax+a^2)$. 15. $48(x-2)(x+5)(x+6)$ 16. $(x-3)(x+5)(x+4)(x+7)$. 17. $3a^2(4a^2-9b^2)(a^2-b^2)$ 18. $(8a^3+27b^3)(8a^3-27b^3)$. 19. $12x^2(4x^2-25y^2)(2x-y)$ 20 $(2x-3a)^2(9x^2-a^2)$. 21. $2x(4x^4+81)$ 22. $6(3a-x)^2(a^2-4x^2)$. 23 $(2x-1)^2(4x^2-1)(x+3)$. 24 $(x-2y)(x-4y)(x-3y)(x+5y)$. 25 $(x+2)(2x-1)(3x+1)$ 26 $(1-4x^2)(1+2x+4x^2)(1+2x-4x^2)$ 27 $(x^2-3)^2(9x^2-1)(9x^4-1)$.

54 [Page 156]

1 $9x^4+30x^3-17x^2-76x+32$ 2 $18x^4+3x^3-109x^2-84x+32$ 3 $48x^5-65x^4-120x^3+160x^2+27x-36$ 4. $45x^4-24x^3-123x^2+40x+80$ 5 $12x^4-14x^3-94x^2+63x+180$ 6 $12x^6+8x^5+25x^4+34x^3+15x^2+18x+8$. 7. $32x^6-24x^5-8x^4+18x^3-48x^2+27x-18$. 8. $12x^6+24x^5+95x^4+118x^3+249x^2+144x-216$

55 [Page 158]

1. $12x^4-100x^3+195x^2+70x-72$ 2 $6x^4-79x^3+273x^2-188x-96$. 3. $48x^4-92x^3-128x^2+157x-30$ 4 $16x^8+40x^7+20x^6+38x^5-20x^4-39x^3-15x^2-9x+9$.

Miscellaneous Exercises (2).

[Pages 158—164]

I.

2. (i) $y^3(x-2z)-y^2xz+y(xz^2-x^3-2z^3)+(x^3z-xz^3)$;
(ii) $(xy^3-x^3y)+z(x^3-xy^2-2y^3)+z^2xy-z^3(x+2y)$.

4. $a^4+x^3y+x^2y^2+xy^3+y^4$. 5. $8ab$, 128.
6. $(a+c)^2-(b-d)^2$. 7. $(2x+3y)(2x+3y-4)$. 8. $x-3$.

II.

2. $x-2x^{\frac{1}{2}}+1$. 4. $x^6-6x^5+15x^4-20x^3+15x^2-6x+1$.
5. $x^4-(p-1)x^3+(q-p+1)x^2-(p-1)x+1$.
6. $(a-b)(b-c)$, $(b+2a+3c)(b-2a-3c)$ 7. $x-2$.
8. $x^4+(b-a)x^3+(a-ab)x^2+(ab-a^2)x-a^2b$.

III.

1. 392. 2. 16 3. $x^2-xy-xz+yz$.
4. $(4a-1)(16a^2+8a+3)$ 7. x^2-1.
8. $3x^5-29x^4+104x^3-164x^2+96x$.

IV.

1. a^3-64b^3. 2. (i) $a^3-3abc+(b^3+c^3)$, (ii) $a^2(b-c)$ $-a(b^2-c^2)+(b^2c-bc^2)$; (iii) $a^4(b-c)-a(b^4-c^4)+(b^4c-bc^4)$.
3. -11; -68 6 $a+2b+3c$. 7. $12a^3$.
8. $x^5-x^3y^2-x^2y^3+y^5$.

V.

1. 121. 3. $8a^3+12a^2c+6ac^2+c^3$. 4. $b^{\frac{3}{4}}+a^{\frac{1}{2}}b^{\frac{1}{4}}+a^{\frac{1}{4}}b^{\frac{1}{2}}+b^{\frac{3}{4}}$. 5. (i) $a^2x(a-x)(6ax+1)$, (ii) $(x+yz)(y+zx)$.
6. $2a+3b+c$. 8. $18x^4-45x^3+37x^2-19x+6$.

VI.

1. 87659405 2. -125 5. $(a+b+c+d)(a-b-c+d)$ $(a+b-c-d)(a-b+c-d)$. 6. (i) $(a-b)(a-b+2)$, (ii) $(2a-b)(3a+b+3)$; (iii) $(5x+2y)(3x-2y+2)$.
7. $(2x-y)a^2+(x+y)ax-x^3$ 8 $x^2-(a+b)x+ab$.

VII.

1. 7 3. $x+a$ 4 (i) $(2x-3)(3x+5)$; (ii) $(5x-5y-3)(7x-7y-4)$; (iii) $(x-3y^2)(11x-21y^2)$.
7. $(a-b)(a-c)(b-c)$. 8. $280x^3-123x^2-37x+6$.

VIII.

1. 55 4. $(x-3)(2x-1)(3x-2)$ 5 $(a-b)(x+a)$.
6. $2(a+b)x$ 7. a^2-2, a^3-3a; a^4-4a^2+2.

IX.

2. $x^5+10x^3+40x+\frac{80}{x}+\frac{80}{x^3}+\frac{32}{x^5}$. 3. x^2-2x+3.
4. $a^2(b-c)-a(b^2-c^2)+bc(b-c)$.
5. $(x+2y)(x+7y)(x-3y)(x-5y)$. 8. 0

X.

2. 242. 3 $a^{\frac{2}{3}}+b^{\frac{2}{3}}+c^{\frac{2}{3}}+2a^{\frac{1}{3}}b^{\frac{1}{3}}+2b^{\frac{1}{3}}c^{\frac{1}{3}}+2c^{\frac{1}{3}}a^{\frac{1}{3}}$
4. $(a^2+b^2)(a+b)(a-b)(2a-b)(a+2b)$ 5. (i) $(3a^2-4ab+3b^2)(2a^2+17ab+2b^2)$, (ii) $(3x^2-7x+3)(4x^2-3x+4)$, (iii) $(ax^2+bx+a)(bx^2+cx+b)$ 7. 2528000.

56 [Pages 168, 169.]

1. $\frac{1}{2b}$ 2. $\frac{3x}{4y}$ 3. $\frac{2a}{5x}$ 4. $\frac{3xz}{5y^2}$ 5. $\frac{2d}{3ab}$.
6. $\frac{2az}{5xy}$. 7. $\frac{2d^5}{3a}$ 8. $\frac{3npq}{5m}$. 9. $\frac{x-a}{x}$.
10. $-\frac{1}{x+3}$. 11. $\frac{2x-3a}{2x}$ 12. $-\frac{a}{a+4b}$ 13. $\frac{3a}{x+4a}$.
14. $\frac{2x^2}{x^2-2a^2}$ 15. $\frac{4x}{x+3}$ 16. $\frac{x-2}{x-3}$. 17. $\frac{x-3}{x+4}$.
18. $\frac{a-4b}{a-5b}$ 19. $\frac{a^2}{a+b}$ 20. $\frac{1-4x}{1-5x}$. 21. $\frac{x-y}{x+7y}$.
22. $\frac{1-2a^2}{1+3a^2}$ 23. $\frac{x^2-13}{x^2-4}$ 24. $\frac{3ax}{a-3x}$. 25. $\frac{2x+3}{3x+4}$.
26. $\frac{x-a}{x+a}$. 27. $\frac{x+5a}{x+7a}$. 28. $\frac{2x-5}{3x-2}$. 29. $\frac{2x-5a}{3x-7a}$
30. $\frac{2-3ax}{1-5ax}$. 31. $\frac{x-a}{x^2+a}$. 32. $\frac{3a+5b}{3c-1}$. 33. $\frac{2x+3a}{3x+2a}$.

34. $\frac{2a-b}{a^2-1}$. 35. $x+3$. 36. $\frac{a-b-c}{a+b-c}$. 37. $\frac{x+3}{x+5}$.

38. $\frac{a+3b}{a-4b}$. 39. $\frac{x^2-ax+b^2}{x^2+ax-b^2}$. 40. $\frac{3x-2y}{2x+5y}$.

41. $\frac{1+2x-3x^2}{1-2x+3x^2}$. 42. $\frac{(x-1)^2}{x^2-3x+1}$. 43. $\frac{x^2+3x+5}{x^2+3x-5}$.

44. $\frac{x^2+3ax+7a^2}{2x^2-3ax+5a^2}$. 45. $\frac{2x+3}{3x+4}$. 46. $\frac{3x^2-ax-2a^2}{3x^2+ax-2a^2}$.

47. $\frac{2(a^2-5ab+7b^2)}{3(a^2+5ab+7b^2)}$. 48. $\frac{3(3x^2+4x+5)}{4(2x^2+3x+4)}$.

49. $\frac{a(3a^2-b^2)}{2a^2-b^2}$. 50. $\frac{4x(2x^2-3y^2)}{5y(3x^2-2y^2)}$.

57 [Page 171.]

1. $\frac{2adf}{4bdf}, \frac{3bcf}{4bdf}, \frac{4bde}{4bdf}$. 2. $\frac{6ax^2}{12abc}, \frac{4by^2}{12abc}, \frac{3cz^2}{12abc}$.

3. $\frac{15x^2ab}{60x^3y^2}, \frac{10xybc}{60x^3y^2}, \frac{6y^2ca}{60x^3y^2}$. 4. $\frac{a^2(a+b)}{a(a^2-b^2)}, \frac{ab(a-b)}{a(a^2-b^2)}, \frac{c(a-b)}{a(a^2-b^2)}$.

5. $\frac{x^2(a-2b)}{a(a^2-4b^2)}, \frac{ay^2(a+2b)}{a(a^2-4b^2)}$. 6. $\frac{2a^2}{a(a-b)}, \frac{c-a}{a(a-b)}$.

7. $\frac{2a(a+b)}{a^2-b^2}, \frac{-3b(a+b)}{a^2-b^2}, \frac{4c(a-b)}{a^2-b^2}$.

8. $\frac{2b^2c^2x(a-x)}{a^2b^2c^2(a^2-x^2)}, \frac{3c^2a^2y(a+x)}{a^2b^2c^2(a^2-x^2)}, \frac{4a^2b^2z}{a^2b^2c^2(a^2-x^2)}$.

9. $\frac{a^2x(2x+3y)}{xy(4x^2-9y^2)}, \frac{b^2y(2x-3y)}{xy(4x^2-9y^2)}, \frac{c^2}{xy(4x^2-9y^2)}$.

10. $\frac{a^2(x^2-x+1)}{x^4+x^2+1}, \frac{b^2(x^2+x+1)}{x^4+x^2+1}$. 11. $\frac{3(x+3)}{x^3+2x^2-5x-6}, \frac{4(x+1)}{x^3+2x^2-5x-6}$.

12. $\frac{a^2-4b^2}{a^4+8ab^3}, \frac{abc}{a^4+8ab^3}$.

13. $\frac{a(a^2+3ab+9b^2)}{a^3-27b^3}, \frac{b(a-3b)}{a^3-27b^3}, \frac{c}{a^3-27b^3}$.

14. $\frac{a^2(a-b+c)}{ab(a^2+b^2-c^2-2ab)}, \frac{b^2(a-b-c)}{ab(a^2+b^2-c^2-2ab)}$,

$\frac{abc}{ab(a^2+b^2-c^2-2ab)}$. 15. $\frac{(c-a)^2}{(a-b)(b-c)(c-a)}$.

$\frac{(a-b)^2}{(a-b)(b-c)(c-a)}$, $\frac{(b-c)^2}{(a-b)(b-c)(c-a)}$

58. [Pages 174, 175]

1. $\frac{a^2+b^2}{ab}$. 2. 0. 3. 1. 4. $\frac{4ab}{a^2-b^2}$.

5. $\frac{a+b}{2(a-b)}$. 6. $\frac{12xy}{4x^2-9y^2}$. 7. $\frac{a^2-2ab-b^2}{(a+b)^2(a-b)}$.

8. $\frac{2a^3}{a^2-b^2}$. 9. $\frac{1}{(a-b)(b-c)}$. 10. $\frac{2}{x^2-4x+3}$.

11. $\frac{2}{x^2+10x+16}$ 12. $\frac{6xy}{8x^3+27y^3}$. 13. $\frac{2ab}{a^2-b^2}$.

14. 0. 15. $\frac{8x^2y^2}{x^4-y^4}$. 16. $\frac{-64ax^3}{a^4-16x^4}$.

17. $\frac{x^2}{6(x^2-9)}$. 18. $\frac{2b}{1-16a^2b^2}$. 19. $\frac{4x^4}{x^4-16a^4}$.

20. $\frac{8a^7b}{a^8-b^8}$. 21. $\frac{108x^4}{81x^4-y^4}$. 22. $\frac{9ax(x+a)}{x^4-81a^4}$.

23. $\frac{4ab}{(a-b)^2}$. 24. $\frac{6x^2-12}{x^4-5x^2+4}$ 25. $\frac{6a^2x}{4x^4-5a^2x^2+a^4}$.

26. $\frac{48a^3}{x^4-10a^2x^2+9a^4}$. 27. $\frac{2x}{x^4-1}$. 28. $\frac{x-c}{(x-a)(x-b)}$.

29. $\frac{4}{x^2-6x+5}$. 30. $\frac{4}{x^2+14ax+13a^2}$. 31. $\frac{2}{x+3}$.

32. 0. 33. $\frac{4x^3}{1+x^4+x^8}$ 34. $\frac{12x^4}{x^6-64}$.

35. $\frac{96ax^5}{16x^8-6561a^8}$.

59. [Pages 177, 178.]

1. $\frac{1}{3}$. 2. $\frac{a^2}{9}$. 3. xyz. 4. $\frac{x^2y^2z^2}{9a^2b^2c^2}$.

5. $\frac{5n^2x^2}{8my}$ 6. $\frac{x+2}{x}$. 7. 3 8. $\frac{a^2-b^2}{a}$.

9. $\frac{a^2-4x^2}{a^2}$. 10. $\frac{x^2-1}{x^2-x-6}$. 11. 1. 12. 1.
13. $\frac{a^2+b^2}{a}$. 14. $\frac{x}{2}$. 15. $\frac{x+2}{x+3}$ 16. $\frac{a^2(a-b)}{x}$.
17. $\frac{x^4}{a^4}+\frac{x^2}{a^2}+1$. 18. $\frac{8ab}{9x^2}+2+\frac{9x^2}{8ab}$. 19 $2\left(\frac{bc}{ad}+\frac{ad}{bc}\right)$.
20. 1. 21. $\frac{a+b-c}{a+b+c}$. 22. 1

60 [Pages 180, 181.]

1. $\frac{5ax}{6by}$ 2. $\frac{(a+b)^2}{b}$. 3. $\frac{x-7}{x-5}$. 4. $a-b$.
5 $\frac{m-n}{m+2n}$ 6 $(m-n)^2$ 7. 1 8 1 9. $\frac{x^2+y^2}{2xy}$.
10. $\frac{x^2-8x+12}{x^2-10x+21}$. 11. $\frac{1}{p^2+q^2}$. 12. a^2-b^2. 13. xy.
14 ab 15. $2x$. 16. 1. 17. $\frac{xy}{x^2+y^2}$.
18. $-\frac{a^4+a^2b^2+b^4}{ab(a-b)^2}$. 19. $\frac{a}{a-b}$. 20. a^2-b^2. 21. $a-b$.

61 [Pages 188—193.]

1. $\frac{adf+ac}{bdf+bc+cf}$. 2. x^2. 3. a^2+b^2. 4. m. 5. $4xy$.
6. 1. 21 0. 24. 3. 25. $\frac{a^4-10a^3b-6ab^3-b^4}{a^4+10a^3b+6ab^3-b^4}$. 26. 1.
27. $\frac{a-b}{a+b}$. 28 0 29 x^6+2. 30 1 31. $\frac{3(a+b)}{a-b}$.
32. 1 34. 0 35. 3. 36 $x+y+z$. 37. $a^2+b^2+c^2$. 38. $a+b+c$. 39. 0 42. 0 43. 1. 44. 2.
45. 1. 46. 0 47. $\frac{1}{xyz}$. 48. $\frac{1}{(x-a)(x-b)(x-c)}$.
49. $\frac{x}{(x-a)(x-b)(x-c)}$. 50. $\frac{x^2}{(x-a)(x-b)(x-c)}$.
51. $\frac{x^3+hx+k}{(x-a)(x-b)(x-c)}$. 56. x^2.

62 [Pages 197—200]

1. 1 2. 2 3. 3 4. -4 5. 0 6 7.
7. -2 8. -1. 9. 7 10. 3 11 5 12. 7.
13. -6 14 0 15. -8 16. 9 17. -2.
18. $\frac{1}{3}$. 19. 1. 20. -1 21. 2. 22. 3. 23. -5.
24. 1. 25. 2 26. $\frac{1}{2}$ 27. 3 28 -5. 29 6
30. $a+b$. 31. $2a$ 32. $\frac{1}{2}(a+b)$ 33 $a+b$ 34 $m-n$.
35 $a+b$ 36. $\frac{a+b}{4}$. 37. $\frac{2ab}{a+b}$. 38. $\frac{12ab}{3b+a}$.
39. $c+d$. 40 $\frac{1}{3}(a+b+c)$ 41. $-\frac{1}{3}(a+b+c)$. 42 12.
43 ab. 44 30 45 12. 46 $\frac{1}{ab}$. 47. 13. 48 16
49 20. 50 -3. 51 8 52. 10 53. 9 54. 9.
55. 5 56 8 57 5 58 7. 59. $\frac{8a}{25}$ 60 -24.
61 18 62 $\frac{6a}{7}$. 63. 56. 64 $4\frac{1}{2}$. 65. 6. 66. $10\frac{3}{8}$.
67. $\frac{ac+b^2}{b^2+c^2}$ 68 $-2\frac{39}{73}$ 69 8 70. 11. 71 2
72 $25a+24b$ 73 $\frac{2ab}{a+b}$. 74. 72 75 3 76 2.
77. -1. 78 -2. 79 1 80. $-7\frac{1}{2}$

63 [Pages 201, 202.]

1. 27 2 5. 3. 20. 4. 2 5 10.
6. 5 7. $\frac{2}{3}$ 8. 5. 9 $1\frac{1}{2}$. 10. 5.
11. 7. 12 3. 13. 5

64 [Pages 205—207.]

1. $\frac{15}{16}$. 2. $\frac{3}{2}$. 3. $2\frac{4}{7}$. 4. 4·05 5. 3.
6. 3. 7. $10\frac{7}{8}$ 8. 2. 9. 3. 10. 4
11. 5. 12. $1\frac{1}{2}$. 13. 4. 14 4 15. 7
16. 1. 17. 4. 18. 2. 19 $\frac{12}{73}$. 20. 3.

21. 2. 22. $\frac{3}{5}$. 23. $2\frac{3}{4}$ 24. $26\frac{1}{3}$.
25. 13. 26. $55\frac{1}{5}$. 27. $-\frac{19}{28}$. 28. $-\frac{49}{114}$.
29. $-\frac{11}{60}$. 30. $-\frac{24}{7}$. 31. $2\frac{21}{50}$. 32. 15
33. $\frac{31}{47}$. 34. -1. 35. $\frac{4}{5}$. 36. $\frac{ab(a+b-2c)}{a^2+b^2-ac-bc}$.

37. $\frac{(a^2-ab+b^2)c+ab}{ab(c+1)}$. 38. $\frac{bn-am}{m-n}$. 39. $\frac{a^2-bc}{b+c-2a}$.

40. $\frac{3ab-a^2-b^2}{a+b}$. 41. $3a$. 42. $\frac{2}{5}$.

65 [Pages 209, 210]

1. $\frac{3}{2}$ 2. -3 3. $-\frac{46}{73}$. 4. $-\frac{10}{11}$. 5. $\frac{5}{6}$.
6. $-\frac{5}{7}$. 7. $\frac{7}{8}$ 8. $-\frac{11}{7}$. 9. 2. 10. $\frac{7}{4}$.
11. $\frac{5}{3}$ 12. $1\frac{8}{11}$. 13. $\frac{1}{5}$. 14. $\frac{7}{8}$ 15. $4\frac{1}{2}$.
16. 6 17. 7. 18. $4\frac{1}{2}$. 19. $-\frac{4}{3}$ 20. 1.
21. $-\frac{5}{7}$. 22. $3\frac{1}{2}$.

66 [Page 213.]

1. 24. 2. $-b$. 3. $-\frac{3}{4}$. 4. 2 5. 7.
6. $\frac{a^2+b^2}{a+b}$. 7. $\frac{ab}{a+b}$. 8. $\frac{ab(c+d)-cd(a+b)}{ab-cd}$.

9. 2. 10. $\frac{3}{2a^2}$. 11. 4. 12. $\frac{a^2+b^2}{a+b}$. 13. 3.

14. 2. 15. $\frac{ab}{a-b}$. 16. 25. 17. 3.

18. $\frac{2(a^2+b^2)}{a-b}$. 19. 6. 20. $\frac{1}{2}(a-b)$. 21. 10.

67 [Pages 215–217.]

1. $15-x$. 2. $x-20$ 3. $x+25$. 4. $25-y$.
5. $y-2x$. 6. $\frac{21}{x}$. 7. $100-3x$. 8. $4x-3y$.

9. xy. 10 $\frac{x}{y}$ hours. 11. $(x+20)$ years, $(x-3)$ years.

12. $\frac{60}{x}$ miles. 13. $\frac{44}{x}$. 14. $\frac{7x}{4}$. 15. $x-2$, $x-1$,

x, $x+1$, $x+2$. **16.** $3x$ **17.** $2m+3$ **18.** $2x-2$.

19. $\frac{10x}{y}$ days **20.** $3ab$ **21.** $\frac{3ab}{16}$. **22.** $\frac{x}{3y}$.

23 $\frac{16a}{x}$ hours **24.** $(x+15)$ years, $(x+45)$ years.

25. $10y+x$ **26** $100x+10y+z$ **27.** $100+10y+x$.

68 [Pages 220—223]

1. 40 and 10 **2** 80 **3.** 12 **4.** 60. **5** 40
6. 96 **7** 42, 43, 44 **8.** 33 **9.** 25, 65.
10. 15 and 24 **11** 36 **12** 72 **13.** 10, 11
14. £600, £250 **15** £120, £300 **16.** £3 10*s*
17. 35, 25. **18.** 90 by 180, 100 by 230 **19.** 14 miles per hour **20.** 8 miles **21.** 30, 10. **22.** 15 ft, 12 ft.
23. £83. **24** 50, 30 **25** 20 men, 16 women
26. *A* 84 miles and *B* 70 miles in 56 hours. **27** 28 days.
28. £2 15*s* **29** 24 ft. **30.** Worked for 22 days.
31 4 days **32** £52 52*s*. **33.** A, £162, B, £118; C, £104 **34.** 34 sheep, £70 **35.** $98\frac{2}{3}$ miles from London, $10\frac{2}{3}$ hours **36.** 44 **37** 32 **38.** 72 **39.** 23.
40. 5, 8, 2, 24 **41.** 19, 5, 4, 32. **42** 22, 31, 9, 54

69 [Pages 231—235]

1. $16\frac{4}{11}$ minutes past 3 **2.** $27\frac{3}{11}$ minutes past 5
3. (i) $5\frac{5}{11}$ minutes past 7, (ii) $21\frac{9}{11}$ and $54\frac{6}{11}$ minutes past 7, (iii) $38\frac{2}{11}$ minutes past 7 **4.** (i) at $5\frac{5}{11}$ minutes past 7, (ii) at $16\frac{4}{11}$ minutes past 6 **5.** $\frac{pa}{p+q}$ miles
6. 8 miles from the starting place of the faster walker; 6 hours **8.** 36 minutes. **9.** $3\frac{1}{2}$ anb $4\frac{1}{2}$ miles per hour
10. 160 **11.** £6 **13.** 300. **13.** Greyhound, 960; hare, 1200. **14.** 180. **15.** 20 shillings; 5 shillings.
16. 40 **17.** 76 lbs of gold and 30 lbs. of silver **18.** 4 hours and 6 hours. **19.** 42 years. **20.** $6\frac{2}{3}$ oz from the 1st bar,

$13\frac{1}{3}$ *oz* from the 2nd. **21.** £450 8*s*. 4*d*., £156. 13*s* 4*d*. **22** 11 pice; each man of the 1st set 6 pice, of the 2nd set 5 pice, of the 3rd set 4 pice, and of the 4th set 4 pice.

23. 189 **24** 25 *oz*; 8*s* per *oz*. **25** 7*s*., 11*s*. 8*d*.
26. 2080. **27** $\frac{3}{4}$*d*. each, 512 **28** 12 **29.** 654.
30. 1504. **31** 80 **32** 736. **33.** 4550.

70 [Page 237.]

1. $\left.\begin{aligned}x&=2\\y&=3\end{aligned}\right\}$. **2.** $\left.\begin{aligned}x&=5\\y&=2\end{aligned}\right\}$. **3.** $\left.\begin{aligned}x&=7\\y&=6\end{aligned}\right\}$. **4.** $\left.\begin{aligned}x&=4\\y&=7\end{aligned}\right\}$.

5. $x=\dfrac{ac+b^2}{a^2+b},\ y=\dfrac{ab-c}{a^2+b}$. **6** $\left.\begin{aligned}x&=2\\y&=3\end{aligned}\right\}$. **7** $\left.\begin{aligned}x&=40\\y&=16\end{aligned}\right\}$.

8 $\left.\begin{aligned}x&=8\\y&=5\end{aligned}\right\}$. **9.** $\left.\begin{aligned}x&=6\\y&=4\end{aligned}\right\}$. **10** $\left.\begin{aligned}x&=6\\y&=8\end{aligned}\right\}$.

71 [Page 239.]

1. $\left.\begin{aligned}x&=3\\y&=2\end{aligned}\right\}$ **2.** $\left.\begin{aligned}x&=2\\y&=3\end{aligned}\right\}$. **3.** $\left.\begin{aligned}x&=7\\y&=2\end{aligned}\right\}$ **4.** $\left.\begin{aligned}x&=3\\y&=7\end{aligned}\right\}$.

5. $\left.\begin{aligned}x&=4\\y&=4\end{aligned}\right\}$ **6.** $\left.\begin{aligned}x&=13\\y&=3\end{aligned}\right\}$ **7.** $\left.\begin{aligned}x&=6\\y&=5\end{aligned}\right\}$. **8.** $\left.\begin{aligned}x&=5\\y&=5\end{aligned}\right\}$.

9. $\left.\begin{aligned}x&=2\tfrac{1}{2}\\y&=1\tfrac{1}{2}\end{aligned}\right\}$. **10.** $\left.\begin{aligned}x&=21\\y&=20\end{aligned}\right\}$.

72 [Pages 242—244.]

1. $\left.\begin{aligned}x&=3\\y&=2\end{aligned}\right\}$ **2.** $\left.\begin{aligned}x&=4\\y&=1\end{aligned}\right\}$. **3.** $\left.\begin{aligned}x&=7\\y&=4\end{aligned}\right\}$. **4.** $\left.\begin{aligned}x&=2\\y&=3\end{aligned}\right\}$.

5. $\left.\begin{aligned}x&=4\\y&=2\end{aligned}\right\}$. **6.** $\left.\begin{aligned}x&=6\\y&=4\end{aligned}\right\}$. **7.** $\left.\begin{aligned}x&=2\\y&=1\end{aligned}\right\}$. **8.** $\left.\begin{aligned}x&=-2\\y&=3\end{aligned}\right\}$

9. $\left.\begin{aligned}x&=5\\y&=2\end{aligned}\right\}$. **10.** $\left.\begin{aligned}x&=1\\y&=3\end{aligned}\right\}$. **11.** $\left.\begin{aligned}x&=1\\y&=2\end{aligned}\right\}$. **12** $\left.\begin{aligned}x&=3\\y&=-1\end{aligned}\right\}$.

13. $\left.\begin{aligned}x&=1\\y&=4\end{aligned}\right\}$. **14.** $\left.\begin{aligned}x&=-5\\y&=2\end{aligned}\right\}$. **15.** $\left.\begin{aligned}x&=-2\\y&=1\end{aligned}\right\}$. **16** $\left.\begin{aligned}x&=5\\y&=11\end{aligned}\right\}$.

17. $x=\dfrac{bc-c^2}{ba-a^2},\quad y=\dfrac{ac-c^2}{ab-b^2}$. **18.** $\left.\begin{aligned}x&=7\\y&=9\end{aligned}\right\}$.

19. $\left.\begin{aligned}x&=\tfrac{1}{4}\\y&=\tfrac{1}{5}\end{aligned}\right\}$. **20.** $\left.\begin{aligned}x&=3\\y&=2\end{aligned}\right\}$. **21.** $\left.\begin{aligned}x&=7\\y&=9\end{aligned}\right\}$. **22.** $\left.\begin{aligned}x&=7\\y&=4\end{aligned}\right\}$.

23. $\left.\begin{array}{l} x = 10 \\ y = 5 \end{array}\right\}$. 24. $\left.\begin{array}{l} x = 4 \\ y = 10 \end{array}\right\}$. 25. $\left.\begin{array}{l} x = 2 \\ y = 3 \end{array}\right\}$.

26. $x = \dfrac{a^2 - b^2}{am - bn}$, $y = \dfrac{a^2 - b^2}{an - bm}$. 27. $x = \frac{2}{3}$, $y = \frac{2}{5}$.

28 $x = \frac{1}{5}$, $y = \frac{1}{2}$ 29. $x = 4$, $y = 2$ 30. $x = \frac{1}{15}$, $y = 18$.

73 [Pages 248—250.]

1. $x = 1$, $y = 2$. 2. $x = 2$, $y = 3$. 3 $x = 3$, $y = 4$.
4. $x = 4$, $y = 5$. 5 $x = 5$, $y = 6$. 6. $x = 6$, $y = 7$.
7. $x = 7$, $y = 8$. 8. $x = 8$, $y = 9$ 9. $x = 4$, $y = 2$.
10. $x = 5$, $y = 3$ 11. $x = 7$, $y = 4$. 12 $x = 5$, $y = 8$.
13. $x = 8$, $y = 12$ 14. $x = 6$, $y = 14$ 15 $x = 8$, $y = 18$. 16. $x = 8$, $y = 9$. 17. $x = 12$, $y = 16$.
18. $x = 21$, $y = 12$. 19. $x = 21$, $y = 24$. 20. $x = 18$, $y = 28$. 21. $x = 99$, $y = 15$ 22. $x = 10$, $y = 8$.
23 $x = 3$, $y = 7$. 24. $x = 4$, $y = 7$.
25 $x = 3$, $y = 5$ 26. $x = 1$, $y = 2$, $z = 3$
27. $x = 2$, $y = -3$, $z = 1$ 28. $x = 3$, $y = 4$, $z = 2$.
29. $x = 2$, $y = 3$, $z = 4$. 30. $x = 1$, $y = 3$, $z = 5$.
31. $x = 2$, $y = 3$, $z = 4$. 32. $x = 3$, $y = 6$, $z = 9$.
33. $x = 4$, $y = 10$, $z = 14$. 34. $x = 8$, $y = 12$, $z = 20$.
35. $z = 3$, $y = 4$, $z = 5$.

74 [Pages 253, 254]

1. $x = 1$, $y = 2$, $z = 3$. 2. $x = 2$, $y = 3$, $z = 4$.
3. $x = 2$, $y = 3$, $z = 4$. 4. $x = 2$, $y = 3$, $z = 4$.
5. $x = 3$, $y = 2$, $z = 1$. 6. $x = 3$, $y = 2$, $z = 1$.
7. $x = 4$, $y = 3$, $z = 2$. 8. $x = 4$, $y = 5$, $z = 6$.
9. $x = 7$, $y = 5$, $z = 3$. 10. $x = 1$, $y = -2$, $z = 3$.
11. $x = 3$, $y = 2$, $z = 5$. 12. $x = 3$, $y = \frac{1}{2}$, $z = \frac{2}{5}$.
13. $x = 10$, $y = 20$, $z = 5$ 14. $x = 2$, $y = -3$, $z = 4$.
15. $x = 5$, $y = 6$, $z = 7$. 16. $x = 2$, $y = 4$, $z = 6$.
17. $x = 2$, $y = 5$, $z = 10$. 18. $x = 12$, $y = 12$, $z = 12$.
19. $x = 6$, $y = 12$, $z = 8$. 20. $x = \frac{1}{2}$, $y = \frac{1}{3}$, $z = \frac{1}{4}$.

21. $x = 7, y = 10, z = 9$. **22.** $x = 1, y = -2, z = 3$.

23. $x = \dfrac{b^2+c^2-a^2}{2bc}, y = \dfrac{c^2+a^2-b^2}{2ca}, z = \dfrac{a^2+b^2-c^2}{2ab}$.

24. $x = 1, y = 2, z = 3$. **25.** $x = -28, y = 10, z = 9$.

75 [Pages 257, 258.]

1. $x = \dfrac{a}{2}, y = \dfrac{b}{2}, z = \dfrac{c}{2}$. **2.** $x = \dfrac{2}{a+b-c}, y = \dfrac{2}{a-b+c}$,

$z = \dfrac{2}{b+c-a}$. **3.** $x = \dfrac{2abc}{ab+ac-bc}, y = \dfrac{2abc}{bc+ab-ac}$,

$z = \dfrac{2abc}{ac+bc-ab}$. **4.** $x = \dfrac{a^2+b^2}{a^2-b^2}c, y = \dfrac{a^2+b^2}{2ab}c$. **5.** $x = 2$,

$y = 4, z = 6$. **6.** $x = 5, y = 3, z = 1$. **7.** $x = 12$, $y = 10, z = 8$. **8.** $x = 13, y = 8, z = 9$. **9.** $x = 4$,

$y = 5, z = 7$. **10.** $x = \dfrac{b(2a-b)}{a-b}, y = \dfrac{a(2b-a)}{b-a}$.

11. $x = \dfrac{bc}{(b-a)(c-a)}.A, y = \dfrac{ca}{(c-b)(a-b)}A, z = \dfrac{ab}{(a-c)(b-c)}.A$,

12. $x = \dfrac{1}{(b-c)(a-c)}, y = \dfrac{1}{(c-b)(a-b)}, z = \dfrac{1}{(a-b)(a-c)}$.

13. $x = \dfrac{a^2bc}{(a-b)(a-c)}, y = \dfrac{b^2ca}{(b-c)(b-a)}, z = \dfrac{c^2ab}{(c-a)(c-b)}$.

14. $x = abc, y = ab+bc+ca, z = a+b+c$ **15.** $x = b-c$, $y = c-a, z = a-b$. **16.** $a_3(b_1c_2-b_2c_1)+b_3(c_1a_2-c_2a_1)+c_3(a_1b_2-a_2b_1) = 0$. **17.** $a = 6$ **18.** $w = 4, x = 12$, $y = 5, z = 7$. **19.** $x = 5, y = 4, z = 3, w = 2, t = 1$. **20.** $x = ab, y = bc, z = ac$

76 [Pages 262–265.]

1. $\frac{4}{5}$. **2.** 7, 9. **3.** 6, 2 **4.** 60, 15 **5.** 24, 15. **6.** $\frac{6}{13}$. **7.** $\frac{2}{3}$ **8.** $\frac{3}{5}$. **9.** Rs. 15, Rs 24 **10.** 3, 5. **11.** 6 miles and 3 miles per hour. **12.** 8, 16. **13** 20 days. **14.** 480 sq yards **15.** Tea 2*s*. 8*d*. and coffee 1*s* 6*d*. per lb.

16. 3 miles, $4\frac{2}{7}$ miles per hour **17.** 22 and 26.
18. A, As 500, B, Rs 400, C, Rs 200 **19.** 75 **20** 65.
21. 21, 40. **22.** A horse, £24, a cow, £12 **23** 5s, 3s.
24. A, 24 days, B, 48 days **25.** $\frac{8}{15}$. **26.** 15 miles.
27. 72. **28.** 75s., 35s. **29.** 34 sheep, £70. **30.** 27

77 [Pages 273—276.]

1. 375 **2.** 50lb.; 28s. per lb **3** A, 14s, B, 19s.

4. 20, 30, 60 **5.** 3s 6d, 4s 2d. **6.** A, $\frac{pm}{p+n-m}$,

B, $\frac{pm}{m-n}$ days **7.** A, Rs 980, B, Rs 1540, C, Rs 2380.

8. 8 hours **9** 720 miles **10.** 4 and 3 gallons.
11. 253. **12** 3 half-crowns, 8s, 9 six-pences.
13. 20 persons, 6s **14.** Each of the equal cocks in 32 hours and the other in 24. **15** 8s and 5s respectively.
16. 75 and 25 quarts **17.** 6 qrs of wheat, 10 qrs of barley.
18. 45 and $22\frac{1}{2}$ miles an hour **19.** 20 bushels of rye and 52 of wheat. **20** 21 guineas and 21 crowns, 9 guineas and 12 crowns **21.** $2\frac{1}{2}$ miles per hour. **22.** A, 5, B, 6 minutes.
23. 10 and 12 miles an hour. **24.** $\frac{b(n-1)}{a-c}$ miles per hour

24. 100 miles.

Miscellaneous Exercises (3).

[Pages 276—286.]

I.

1. $a^4+a^2x^2+x^4$ **3.** $(x+y-1)(x^2+y^2+1-xy+y+x)$.
5. x^4+2. **6.** $-\frac{1}{2}$. **7.** £125 each **8** £2700, 9 men.

II.

1. $\dfrac{1}{m^2-m+1}$. 3. $2(a+m)(c+n)+2bd$. 4. $\frac{3}{2}$.

5. $x=\dfrac{b}{a^2-ab+b^2}$, $y=\dfrac{a}{a^2-ab+b^2}$. 6. $x=-3$, $y=3$, $z=1$. 7. $\dfrac{ab}{a+b}$ hours; $\dfrac{abc}{ab-bc-ac}$ hours. 8. $\dfrac{9}{20}$.

III.

1. $x^{\frac{2}{3}}-x^{\frac{1}{3}}(y^{\frac{1}{6}}+z^{\frac{1}{9}})+(y^{\frac{1}{3}}-y^{\frac{1}{6}}z^{\frac{1}{9}}+z^{\frac{2}{9}})$ 2. $(a-b)(x+a)$.

3. 0. 4. $\dfrac{(a-b)^2}{ab}$. 5. 6. 6 $x=\frac{1}{3}$, $y=\frac{1}{2}$.

7. $27\frac{3}{11}$ minutes past 5. 8. 345.

IV.

2. (i) $(x-a)$, (ii) $x-y-z$. 3. $\dfrac{a+b+c}{a-b-c}$, 1.

4. $\dfrac{a^2+b^2}{2a^{\frac{1}{2}}b^{\frac{1}{2}}(a+b)}$. 5. (i) 3; (ii) $x=\dfrac{2(b-1)}{2ab-a-b}$, $y=\dfrac{2(a-1)}{2ab-a-b}$. 7. 48 minutes past 10 8 £2800 and £1200.

V.

1. $(5x-2)(3x-7)$. 3. $3(x^2+y^2+z^2)$.
4 $x^2+(2m-3)x-6m$. 5. $x^5+ax^4-a^4x-a^5$. 6. (i) 7; $x=1$, $y=-1$, $z=0$ 7. 1800, 8. 150 yds.; A's speed = 30 yds per minute and B's speed = 20 yds. per minute.

VI.

1. 16, $(x-y)(x^2-xy+y^2)$. 2. $5xy(x-y)(x^2-xy+y^2)$; $(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$, $(p+q)^3+(p+q)^3-a=0$.

3. $\dfrac{x^2+y^2}{x^2-y^2}$. 4. x^6-1 5 17 6. $x=2$, $y=0$, $z=-2$. 7. $d-(a-b)t$, $\dfrac{d}{a-b}$. 8. 6 hours $16\frac{4}{11}$ mins. and 6 hours $49\frac{1}{11}$ minutes.

VII.

1. $a^{\frac{3}{2}}+b+c^{\frac{3}{4}}-3a^{\frac{1}{2}}b^{\frac{1}{3}}c^{\frac{1}{4}}$, $\frac{x^2}{2}-\frac{3x}{4}+6$ 2. $\frac{z(x+y-z)}{x(y+z-x)}$.

3. $\frac{x^2-2x+1}{x^2-3x+1}$. 4. $x-3$. 6. $x=1$, $y=4$, $z=27$.

7. $\frac{abn}{b-a}$ feet 8. 4 days

VIII.

1. $\frac{a^2+y^2}{a-y}$. 2. $3x^2+4ax-a^2$. 3. $(2x+1)(x^2-1)$.

4. $\frac{1}{a+c}$. 6. 0 7. $x=\frac{b^2+c^2-a^2}{2a}$, $y=\frac{c^2+a^2-b^2}{2b}$.

8. £40.

IX.

2. 1 3. 1. 4 $\frac{x^2-y^2}{x^2+y^2}$. 6. $x=1, y=2,$
$z=3$. 7. 12*s* 8. 12, $10\frac{10}{11}$ and $10\frac{10}{23}$ miles an hour.

X

1. $a^{x+2}-a^xb^2+a^2b^y-b^{y+2}$, $x^5+x^4-5x^3-5x^2+4x+4$
2. 2. 6 -1; $x=a$, $y=2a$, $z=3a$. 8. 50 miles, 30 miles.

XI.

1. $(x+y)^4+z^4$. 2. $2x^2-3x+5$ 6. $(9x+32)(3x-16)$; $4(a-d)(b-c)$, $(m^2+mn+n^2)(m^2-mn+n^2)$ 7. $x=4$, $y=5$, $z=6$ 8. 480

XII.

1. $\frac{(x-y)^4}{x}$. 3 $2x^2-3x+1$, $2x^6-3x^5-7x^4+28x^3$
$-36x^2+20x-4$. 5. $\frac{b}{a}$. 7. $x=5$, $y=-3$, $z=\frac{1}{4}$.
8. 50 miles.

78 [Pages 289, 290.]

1. $\sqrt[7]{a^5}$. 2. $\frac{1}{\sqrt{x^3}}$. 3. $3\sqrt[5]{x^4}$. 4. $\frac{3}{\sqrt[5]{x^2}\times\sqrt{a}}$.

5. $\frac{8}{\sqrt[3]{m^8}}$ 6. $\frac{\sqrt[4]{a^5}}{3\sqrt[5]{x^4}}$ 7. $\frac{1}{2\sqrt[6]{x}}$. 8. $\sqrt[5]{x^{a+2}}$.

9. $\sqrt[2m]{a^{11}}$ 10. $\sqrt[a]{x^4}$ 11. $x^{\frac{7}{3}}$.

12. $\frac{1}{a^{\frac{1}{2}}}$ 13. $x^{\frac{2}{3}}$. 14. $a^{\frac{2}{5}}$. 15. $x^{\frac{3}{2}}$. 16. $a^{\frac{3}{4}}$

17. $\frac{1}{8}$. 18. 4 19. 27. 20 32 21. $\frac{1}{27}$.
22. 36. 23. $\frac{1}{16}$. 24. 81. 25. 36. 26. x^{-m}.

79 [Page 293.]

1. a^{-6}. 2. $a^{-\frac{1}{2}}b^{\frac{r}{3}}$ 3. ab^6. 4. $a^{-8}b^{-\frac{r}{3}}$

5. a^8b^6. 6. $x^{-\frac{9}{2}}y^4$. 7. $x^{\frac{5}{32}}$. 8. a^{-1}.
9. y. 10. $\frac{4}{9}x^2a^2$. 11. $\frac{9}{16}x^{-2}a^{-2}$.
12. $a^{\frac{1}{2}}b^{\frac{1}{3}}c^{\frac{2}{3}}$ 13. $a^{-1}b^{\frac{2}{3}}c^{\frac{1}{6}}$. 14. $a^{\frac{3}{2}}b^{-\frac{1}{3}}c^{\frac{1}{2}}$
15. a^4b^2.

80 [Pages 299—301.]

1. $x-2x^{\frac{1}{2}}+1$. 2. $a-27b$. 3. $1+a^2b^{-2}+a^4b^{-4}$.
4. $x^2+6xz^{\frac{1}{3}}-4y+9z^{\frac{2}{3}}$. 5. $x^{-2}+x^{-1}y^{-1}+y^{-2}$.
6. $a+a^{\frac{1}{3}}-1+a^{-\frac{1}{3}}+a^{-1}$. 7. $x+y+z-3x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}}$.
8. $a^{2m}-9b^{2n}+12b^nc^p-4c^{2p}$. 9. a^3-64b^2.
10. $a+a^{\frac{3}{4}}x^{-\frac{1}{4}}-a^{\frac{1}{4}}x^{-\frac{3}{4}}-x^{-1}$. 11. $x-x^{\frac{1}{2}}$.
12. $2x^{-2}+4x^{-1}+2$ 13. $y+x^{\frac{1}{2}}y^{\frac{1}{2}}+x$
14. $a+a^{\frac{1}{2}}b^{\frac{1}{2}}-b$ 15. $x^{2n}-1+x^{-2n}$.
16. $4x-2x^{\frac{1}{2}}y^{-\frac{1}{2}}+2x^{\frac{1}{2}}z^{\frac{1}{3}}+y^{-1}+y^{-\frac{1}{2}}z^{\frac{1}{3}}+z^{\frac{2}{3}}$.
18. $x^{2^n}-a^{2^n}$. 19. $x^{2^{n-1}}-y^{2^{n-1}}$. 20. a^{m-1}.

21 $x^{\frac{3}{4}}+3x^{\frac{1}{2}}-1$ 22. $x^{\frac{3}{2}}+xy^{-\frac{1}{2}}-2x^{\frac{5}{4}}y^{-\frac{1}{4}}-2x^{\frac{1}{2}}y^{\frac{1}{4}}+2x^{\frac{3}{4}}y^{\frac{1}{2}}+y$. 23. $x^{\frac{5}{n}}+x^{\frac{n}{2}}a^{\frac{n}{2}}+a^n$ 24. $x^{\frac{2}{3}}-4x^{\frac{5}{6}}+4x+2x^{\frac{7}{6}}-4x^{\frac{4}{3}}+x^{\frac{5}{3}}$. 25. $a^{\frac{2}{3}}x^{-\frac{2}{3}}+a^{\frac{1}{3}}x^{-\frac{1}{3}}+a^{-\frac{1}{3}}x^{\frac{1}{3}}+a^{-\frac{2}{3}}x^{\frac{2}{3}}$

26. $\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}$. 27. $\frac{2x+36x^{\frac{1}{3}}y^{\frac{2}{3}}}{x-27y}$. 28. $\frac{x+a}{x^2+3xa+a^2}$.

29. 1. 30. $x^{\frac{1}{4}}y^{\frac{1}{4}}-x^{-\frac{1}{4}}y^{\frac{3}{4}}$ 31. 1 32. 3

33. 5 34. 2 35. 4 36. $x=1,\ y=2$.

37. $x=2,\ y=3$ 38. $x=1,\ y=3$.

39. $x=1,\ y=2,\ z=3$ 40. $x=2,\ y=4,\ z=6$

81 [Page 302.]

1. $\sqrt{45}$ 2 $\sqrt[3]{24}$ 3. $\sqrt[4]{96}$ 4. $\sqrt[4]{1280}$

5. $\sqrt[n]{a_n b}$ 6. $\sqrt[a]{x^{3a}y}$ 7. $\sqrt[5]{a^{10}b^2}$.

82 [Page 303.]

1. $3\sqrt{2}$ 2. $4\sqrt{5}$ 3. $5\sqrt[3]{2}$ 4. $2\sqrt[5]{4}$. 5. $3\sqrt[4]{5}$
6. $7\sqrt[3]{4}$ 7 $5\sqrt[4]{3}$. 8. $a^2\sqrt[3]{b}$ 9. $x^4\sqrt[n]{a}$ 10. $-8\sqrt[3]{5}$.
11. $-4ab\sqrt[3]{3b}$ 12. $5a^2x\sqrt[3]{4ax}$

83 [Page 304.]

1 $7\sqrt{3}$ 2. $7\sqrt{2}$. 3. $8\sqrt{5}$. 4. $2\sqrt{2}$ 5. $\sqrt[3]{2}$.
6 $5\sqrt[4]{5}$. 7. $\sqrt[4]{3}$. 8. $3\sqrt{3}$ 9. $6\sqrt{5}$ 10. 0 11. 0.
12. $17\sqrt[3]{2}$. 13. $(7x+y)\sqrt{5x}$ 14. $(x^2-2y^2+3z^2)\sqrt[3]{a}$.
15. $4a\sqrt[4]{2x}$

84 [Page 305.]

1. $\sqrt[6]{27}$ and $\sqrt[6]{4}$ 2. $\sqrt[12]{256}$ and $\sqrt[12]{125}$ 3. $\sqrt[15]{8}$ and $\sqrt[15]{243}$ 4. $\sqrt[12]{27}$ and $\sqrt[12]{25}$ 5. $\sqrt[24]{256}$ and $\sqrt[24]{216}$.
6. The latter. 7. The former. 8. The former.
9. $\sqrt[3]{4}$, $\sqrt[4]{6}$, $\sqrt{2}$. 10. $\sqrt[3]{10}$, $\sqrt[4]{3}$, $\sqrt[12]{25}$.

85 [Page 307.]

1. $5\sqrt{2}$. 2. $4\sqrt{3}$. 3. 9. 4. $3\sqrt{10}$ 5. 30. 6. 5.
7. $3ax\sqrt[3]{6x}$. 8. $\sqrt[6]{864}$. 9. $\sqrt[6]{288}$. 10 $4\sqrt[6]{2}$.
11. $9\sqrt[6]{3}$. 12. $\sqrt[12]{72}$. 13. $\sqrt[8]{27}$. 14 $\sqrt[12]{32}$.
15. $\sqrt[12]{1024}$ 16 $40\sqrt{3}$. 17. $288\sqrt{2}$. 13 $480\sqrt[3]{3}$.
19. $210abx\sqrt[3]{x}$. 20 $2\sqrt{\frac{2}{3}}$. 21. $\frac{1}{3}$. 22. $\sqrt[3]{\frac{3}{4}}$.
23 $\sqrt[6]{\frac{2}{3}}$ 24. ·577. 25. 1·341. 26 3·535.
27. 26·832.

86 [Page 308.]

1. $a\sqrt{b}+b\sqrt{a}$ 2 $a-b$ 3 $6a-10\sqrt{a}$. 4. $16x-9y$.
5. $6x-54$. 7. $6+\sqrt{10}$. 7 $7+4\sqrt{6}$ 8 $6-6\sqrt{5}$.
9. $2+6\sqrt{2}$. 10. $5+3\sqrt[3]{15}+3\sqrt[3]{12}$. 11 $2x-2\sqrt{x^2-a^2}$
12. $182+80\sqrt{3}$. 13. $83+12\sqrt{35}$ 14. $2a^2-2\sqrt{a^4-4b^4}$.
15. $29x^2-21y^2+20\sqrt{x^4-y^4}$.

87 [Page 311]

1. $\frac{23-3\sqrt{21}}{10}$. 2. $5+2\sqrt{6}$ 3. $24+17\sqrt{2}$
4 $9+2\sqrt{15}$. 5. $\frac{a+\sqrt{a^2-x^2}}{x}$. 6. $x^2-\sqrt{x^4-1}$
7. $\frac{1}{4}(2+\sqrt{2}-\sqrt{6})$ 8 5·828. 9. 6·464 10 5·414.
11. 3·650 12 6·854. 13 ·504 14. $2x$ 15 $\sqrt{5}(1+\sqrt{2})$.
16. $2+\sqrt{3}$. 17. $\frac{1}{3}(\sqrt{30}+2\sqrt{3}-3\sqrt{2})$ 18. 198.
19. $4x\sqrt{x^4-1}$ 20 $2x^2$. 21. $\frac{\sqrt[3]{9}-\sqrt[3]{6}+\sqrt[3]{4}}{5}$.
22 $\sqrt[3]{16}+\sqrt[3]{12}+\sqrt[3]{9}$

88 [Page 316.]

1. $\sqrt{3}-1$. 2. $\sqrt{3}+2$. 3. $3-\sqrt{2}$. 4. $\sqrt{5}+\sqrt{3}$.
5. $3-\sqrt{5}$ 6. $5+\sqrt{3}$ 7. $4-\sqrt{5}$ 8. $3+2\sqrt{2}$.
9. $6+\sqrt{5}$ 10 $5-2\sqrt{3}$. 11 $2\sqrt{7}+\sqrt{3}$. 12. $3\sqrt{5}-2\sqrt{7}$ 13. $2\sqrt{11}+\sqrt{3}$. 14. $\sqrt{\frac{7}{2}}-\sqrt{\frac{1}{2}}$. 15. $\sqrt{\frac{7}{2}}-\sqrt{\frac{5}{2}}$.
16. $\sqrt[4]{2}(\sqrt{2}-1)$ 17 $\sqrt[4]{2}(\sqrt{3}-1)$ 18 $\sqrt[4]{3}(1+\sqrt{2})$.
19. $\sqrt[4]{5}(\sqrt{3}+\sqrt{2})$ 20 $\sqrt{2}$. 21. 1, or $\frac{4}{3}\sqrt{3}-2$.

22. b 23. $x+\sqrt{a^2-x^2}$. 24. $\sqrt{a+b}+\sqrt{a-b}$

25. $\sqrt{a+\frac{1}{2}x}+\sqrt{\frac{1}{2}x}$. 26. $\sqrt{x+2}+\sqrt{x-3}$

27. $\sqrt{x+y}+\sqrt{z}$

89 [Pages 320, 321.]

1. 9 2. 3. 3. 16. 4. $\frac{3}{4}$ 5. $\frac{9}{20}$ 6. 25

7. 8. 8. 25. 9. 2 10. $\frac{a}{4}$. 11. $\frac{(b-a)^2}{2b}$.

12. 5. 13. 9 14. 7 15. 5 16. 6 17 3

18. $\frac{5}{8}$ 19. 81 20. $x=\frac{81}{a}$. 21 $\frac{1}{a}\left(b+\frac{c^2}{c-1}\right)$.

22. 5 23 $\frac{17a}{8}$. 24. $\frac{4}{5}a$ 25 36

26 $\frac{2a^2-2ab+b^2}{2(b-a)}$. 27. 4. 38. $\frac{16}{25}$. 29. $1\frac{1}{2}$

30 $\frac{24}{25}$ 31. $\frac{ab^{\frac{2}{3}}}{a^{\frac{2}{3}}-b^{\frac{2}{3}}}$ 32. [illegible] 33. 1 or -1 34 $\frac{2a}{\sqrt{5}}$.

35. $\frac{41a^2}{40b}$. 36. 7 37. 5 38. $\frac{b^2-4a^2}{4a}$.

39. 5. 40. 4.

90 [Pages 326, 327.]

1. $2xz+3y$. 2 x^2-2x+3. 3. x^2-x+1

4. $2x^2-3x+4$ 5. $2x^2+2ax+4b^2$. 6. $3x^2-\frac{1}{3}xy+3y^2$.

7. $x^2-x+\frac{1}{4}$ 8. $7x^2-\frac{x}{5}+3$ 9. $x^2-\frac{x}{2}+\frac{2}{x}$.

10. $\frac{a^2}{2}+\frac{a}{x}-\frac{x}{a}$. 11. $\frac{a}{2b}-1-\frac{2b}{a}$ 12. $\frac{3a}{x}-\frac{1}{5}+\frac{2x}{3a}$.

13. $2x^2-2xy^2-y^4$ 14. $\frac{7x}{y}-3-\frac{y}{7x}$. 15. $\frac{x}{y}-\frac{1}{2}-\frac{y}{x}$.

16. $\frac{2x}{7y}-5+\frac{3y}{4x}$. 17. $x-x^{\frac{1}{2}}+1$. 18. $x^{\frac{5}{6}}-2x^{\frac{1}{2}}-x^{\frac{1}{6}}$.

19. $ax^{-1}+1+a^{-1}x$.

20. $x^{\frac{3}{4}}-x^{\frac{1}{2}}y^{-\frac{1}{4}}+y^{\frac{1}{2}}$.

21. $\dfrac{3x^{\frac{3}{2}}}{2}-\dfrac{5xy^{\frac{1}{2}}}{3}+\dfrac{2x^{\frac{1}{2}}y}{5}$.

22. a^m-2a^n.

23. $3a^m+a^{2m+1}-5c^{m-2}$.

91 [Pages 330, 331.]

1. $5xy-4$
2. $7ax^2-3b^2$.
3. $7a^3b^4+9a^4b^3$.
4. $\frac{1}{2}x^4y^2-\frac{1}{5}x^3y^5$.
5. $\dfrac{5ab}{2}-\dfrac{c^2}{3}$.
6. $a+b+c$.
7. $a-b+c$
8. $2a-b-3c$.
9. $a^2+2b^2-3c^2$.
10. $2a^2-3b^2+5c^2$.
11. $x+\dfrac{a}{3}-\dfrac{b}{2}$.
12. $x-2-\dfrac{1}{x}$.
13. $x^2+1+\dfrac{1}{x^2}$.
14. $\dfrac{a}{b}+1+\dfrac{b}{a}$.
15. $\dfrac{x}{y}-\dfrac{1}{\sqrt{2}}+\dfrac{y}{x}$.
16. $\dfrac{3x}{a}-1+\dfrac{a}{3x}$.
17. $x+2+\dfrac{1}{x}$.
18. $a^{\sqrt{2}}-a^{-\sqrt{2}}$.
19. $a-b+c-d$
20. a^2+b^2.
21. $a^2-b^2+c^2-d^2$
22. $a^2+a-\frac{1}{2}$.
23. $2a(b+c)+2bc$

92 [Page 334.]

1. $x+9$.
2. $3x-8$
3. $4a-3b$.
4. x^2-3x+2.
5. $2x^2+x-3$
6. $1-3x^2+2x^4$.
7. $2x^2-3cx+c^2$.

93 [Pages 338–340.]

1. The latter
2. The latter.
3. The former
4. The former.
5. The latter.
6. $a : d$.
7. 1 : 4.
8. 1 : 1
9. 75 : 8.
10. 28 : 27.
11. 5 : 7.
12. 3 : 4.
13. 63 and 72
14. 85 and 51.
15. 28 and 35
16. 42 and 54
17. −15.
18. 35.
19. −17.
20. $\dfrac{ad-bc}{c-d}$.
23. 76 75
24. 1772 : 1771.
25. B.

94 [Page 341.]

1. 4 2. 18. 3. $37\frac{1}{2}$ 4. 36 5. 20.
6. 60 7. 20. 8. 6 9. 14 10. 18

95 [Pages 345, 346]

1. $x=9, y=6$ 2. $x=25, y=9$. 3. $x=56, y=30$.
4. $\frac{1}{3}a$. 5. $\frac{r}{8}$ 6. $\frac{3}{4}$. 7. $\frac{4}{81}$. 8. $\frac{1}{6}$ 9. $2\frac{7}{9}$.
10. $\sqrt{2ab-b^2}$ 11. $a\left\{1-\frac{16b^2}{(1+b)^4}\right\}$. 15. 2.

97 [Pages 352—354.]

26. 0.

98 [Pages 359, 360.]

1. $b^2c^2=a^2d^2$. 2. $b^3c^2=a^3d^2$. 3. $p^3n^4=q^3m^4$.
4. $ad^2-bd+c=0$ 5. $lb^2-mab+a^2n=0$
6. $(bn-cm)(am-bl)=(cl-an)^2$ 7. $ab=1$ 8. $35pq=6$.
9. $(b_1c_2-b_2c_1)(a_1b_2-a_2b_1)^2=(c_1a_2-c_2a_1)^3$
10. $(b_1c_2-b_2c_1)^2(a_1b_2-a_2b_1)=(c_1a_2-c_2a_1)^3$
11. $(b_1c_2-b_2c_1)^3(a_1b_2-a_2b_1)=(c_1a_2-c_2a_1)^4$.
12 $(an^2-bn+cm)(am^2-an+b)=(c+amn)^2$
13. $(c^2+3ab)(b^2-2ab-ac)=(3a^2-2ac+bc)^2$.
14. $a^2+b^2=m^2+n^2$. 15. $(ab_1+bc_1)^2+(a_1b+b_1c)^2$
$=(cc_1-aa_1)^2$ 16. $a^2n+b^2l=abm$ 17. $ab+bc+ca+2abc$
$=1$. 18. $a+b+c+abc=0$ 19 $a^2+b^2+c^2-abc=4$
20. $d^2(a+b+c)+abc=0$ 21. $x^2+y^2+z^2+2xyz=1$.

Miscellaneous Exercises (4).

[Pages 374—389.]

I.

1. $1\frac{2}{3}$ 2. 0 3. $5b(a+b)$ 4. $2x^2-4xy+5y^2$.
5. $\frac{b^3}{(a+b)^3}$. 6. $\frac{ab}{b-a}$ 8. $5+\sqrt{6}$

II.

1. 21. 2. $4x^2-6x-1$ 3. 12. 4. $\frac{3\sqrt{2}}{5}$.

5. x^2-3x+2 6. $\frac{1}{5}$. 7. 11.

III.

1. -30 2. $\frac{4x^2}{1-x^4}$. 3. $(a+b-3c)(a-b+3c)$.

4. $x-a$ 5. $(x-1)(x-2)(x-3)$. 6. $\frac{x(x+2)}{x^2-2x+4}$.

7. $\frac{1}{3}$

IV.

1. $\frac{y^2}{x^2}$. 2. $x^6+x^5y-x^3y^3+xy^5+y^6$.

3. (i) $(x-1)(x+1)^2$; (ii) $(a+1)(a-1)(b+1)(b-1)$.

5. $(64x^6-729)(3x+2)$. 6. $-\frac{5}{7}$ 7. $x=a^2b, y=ab^2$.

V.

1. 1. 2. $a^2(b-c)+b^2(c-a)+c^2(a-b)=-(b-c)(c-a)(a-b)$.

3. $\frac{9(a^2+3)}{a(a^2+27)}$. 5. $\frac{4\sqrt{y}}{x}-4+\frac{x}{\sqrt{y}}$. 6. $\frac{34\sqrt{5}-18}{11}$.

7 $x=3, y=1$.

VI.

1. 1. 2. $b^2-a^2+\frac{b^4}{a^2}-\frac{a^4}{b^2}$. 3. $x^2+(a-b)x-ab$.

5. $\frac{5x^2-4x-8}{3x^2+4x+24}$. 6. $x=\frac{1}{4}, y=\frac{1}{3}$. 7. $x=3, y=5, z=7$.

VII.

1. $2x^3x^{-3}-3x^4y$. 2. ae^x+e^x+a+1. 4. $\frac{ax+by}{ax-by}$.

5. 7 7. 80, 128

VIII.

2 (i) $(b+c-a)(b+c-5a)$; (ii) $(x-a)(x+2y+a)$.

3. $\frac{a+b}{(a-b+c)(b+c-a)}$ 5. -6 6. $2x^2-3x^{-1}+4x^{-4}$.

7. $x=7\frac{1}{2}, y=3\frac{1}{3}, z=1\frac{1}{5}$

IX.

1. -20. 2. $\frac{1}{x^2-1}$ 3 $(a-b+1)(a^2+b^2+1+ab-a+b)$.

4. 6. 6. $x=16, y=4$. 7. $27\frac{3}{11}$ minutes past 8.

X.

1 $9a^2+4b^2+9c^2-6bc+9ca+6ab$. 2 x^2+2x+3.

3 $\{(a+b)x+(a-b)y\}\{(a-b)x+(a+b)y\}$ 4. $\dfrac{a^2-b^2}{a^2+b^2}$.

6 8. 7. 20 days

XI.

2 (i) $(a+b-c-d)(a-b+c-d)$, (ii) $(x+y-z)(x-y+z+1)$.

3 $\dfrac{3x}{a}-1+\dfrac{a}{3x}$ 4 $x=y=z=1$ 5 $x^2-5xy+7y^2$.

6 480 at 16 a shilling, 90 at 18 7 1.

XII.

2. 0. 3. (i) $(x-b)(x+b-2a)$, (ii) $(x+a)(x+b+c)$.

4. $3x-1$. 5. 20. 6. 10 7. $13\sqrt{3}$.

XIII.

3. 0. 4. 0 5. 30 6. 1. 7. $46\sqrt{2}$.

XIV.

3 47 4. $a+b$ 5. $x=\frac{1}{2}(2a+b+c)$,
$y=\frac{1}{2}(a+2b+c)$ $z=\frac{1}{2}(a+b+2c)$. 6. 5 days.
7. $(x^2+5ax+5a^2)^2-a^4$.

XV.

1. 4. 3. $\dfrac{2x+3}{x^2+x+1}$. 4. $2x-3b$ 5. $\dfrac{1}{n}$. 6. 4.

7. 54 gallons.

XVI.

1. x^2+2x+3. 2 1. 4. $x=2\frac{1}{2}, y=1\frac{1}{2}$.
5. $(xy+ab)(ay^2+b^2x)$. 6 $-a^2-b^2-c^2+2ab+2ac+2bc$.
8 In the 1st the wine is $\frac{1}{3}$ of the whole, in the second $\frac{2}{3}$.

XVII.

1. x^2+x+1. 2. $x=16, y=25$. 3. $n(n-1)$.

6. 72. 7. $\dfrac{x^2-2x+3}{2x^2+5x-3}$.

XVIII.

2. $\frac{5}{4}a$. 3. (i) $(7x-1)(2x-5)$; (ii) $2(a-c)(1-ac)$, (iii) $(m+n).n.2m^2$. 4. 1920 5. $a+b+c$

7. $x=2,\ y=4,\ z=6$ 8. $\frac{c^2}{(b-d)^2}-\frac{a^2}{(b+d)^2}=1$.

XIX.

1. $\frac{4}{3}a$. 3. $ac-bc-b^2+a^2$. 4. $mq:np$.

7. $\sqrt{6}-2$. 8. $a^3+b^3+c^3-3abc=0$.

XX.

3. $\frac{1}{abc}$. 4. $-(a^2+b^2+c^2+ab+ac+bc)$. 6. $x=c$, or, $=c-\frac{a+b}{2}$. 7. $x=\frac{b+c}{2a},\ y=\frac{a+c}{2b},\ z=\frac{a+b}{2c}$.

8. 4 and 3 miles an hour, $3\frac{1}{2}$ miles

XXI.

5. $(a-b)(a+3b-2c)$ 6. x^2-x+3.

7. $(ac'-a'c)^2=(ba'-b'a)^2(b'c-bc')$.

XXII.

4. 1020 yards. 7. $x=b+c,\ y=c+a,\ z=a+b$.

8. $a^3+b^3+c^2-3abc=0$.

XXIII

2. $(7x-2)(4x-1)(3x-1)$ 9. $\frac{a^2+b^2+c^2}{bc+ca+ab}$.

8 $\frac{a^2}{p^2}+\frac{b^2}{q^2}=1$.

XXIV.

1. $x=a,\ y=b,\ z=c$. 2. 54, 81, 108.

6. $\frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{(a-b)^2}$.

XXV.

7. (i) $abc+2fgh-af^2-bg^2-ch^2=0$.
   (ii) $bc+ca+ab+2abc=1$

8. $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\left(\frac{a'^2}{a}+\frac{b'^2}{b}+\frac{c'^2}{c}\right)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)$.

99 [Pages 393, 394.]

1. $\pm 2\frac{1}{3}$ 2. ± 5 3 ± 2. 4 ± 9 5. ± 2 6. $\pm \frac{\sqrt{3}}{2}$.

7 $\pm \frac{a}{2}\left(\frac{a^2-4}{a^4-1}\right)^{\frac{1}{2}}$. 8. $\pm \sqrt{mab}$ 9. $\pm \frac{2}{b\left(4a-b^2\right)^{\frac{1}{2}}}$.

10 $\pm \frac{2a}{\sqrt{5}}$ 11. $\pm \sqrt{\frac{3}{5}}$ 12. $\pm\sqrt{\frac{13}{2}}$. 13. $\pm n\left(a-\frac{n^2}{4}\right)^{\frac{1}{2}}$

14. $\pm\frac{1}{2}$.

100 [Pages 395–397.]

3. $\frac{3}{4}, 7\frac{1}{4}$ 4. $3\frac{3}{5}, 2\frac{2}{5}$. 5. $2\frac{1}{3}, 2\frac{1}{4}$ 6. $6\frac{1}{2}, -1\frac{2}{3}$
7. $1\frac{1}{3}, -2\frac{3}{5}$ 8 4, $\cdot 05$. 9. $2\pm\frac{1}{5}\sqrt{3}$ 11. $9, 8$
12. $\frac{3}{5}, \frac{4}{7}$. 13. $\frac{2}{3}, \frac{3}{10}$. 14. $29, -10$ 15. $10, -29$.
16. $2, -3$ 19. $\frac{4}{3}, 0$ 20. $10, -\frac{2}{5}$. 21. $24, \frac{42}{5}$.
22. $\frac{1}{2}, \frac{57}{14}$ 23. 3. 24. $6, \frac{40}{13}$ 25. $1, -\frac{40}{47}$.
26. $\frac{-11\pm\sqrt{13}}{6}x$.

101 [Page 399.]

1. $3, 2\frac{2}{3}$ 2. $-4, -5$ 3 $\frac{4}{3}, -\frac{5}{2}$. 4. $\frac{5}{3}, -\frac{7}{3}$. 5. $2\frac{1}{2}, -\frac{3}{4}$
6. $5, \frac{5}{3}$ 7. $-1, \frac{5}{4}$.

102 [Page 400.]

1. $\frac{3}{2}, -6$. 2. $1\frac{2}{5}, -1\frac{1}{3}$. 3. $\frac{5}{13}, -8$. 4. $\frac{3}{2}, -34$.

5. $9\frac{15}{17}, -11$ 6. $\frac{a}{2}, \frac{3}{c}$ 7. $ab, -\frac{a}{3}$

103 [Page 403.]

1 $2, 4$. 2. $5, -4$. 3. $1, -4$ 4. $4, -13$.
5. $3, -\frac{4}{3}$. 6. $\frac{1}{2}, -\frac{4}{3}$. 7. $7, -\frac{1}{3}$. 8. $2, \frac{1}{2}$.
9. $2, -\frac{3}{13}$. 10. a, b 11. $1, \frac{2b}{a-b}$. 12. $a, -b$.

13. $\frac{a}{2}, \frac{3a}{4}$. 14. $4, 8$. 15. $0, \frac{2ab-ac-bc}{a+b-2c}$.

16. $a, \frac{a}{5}$.

UNIVERSITY PAPERS.

CALCUTTA UNIVERSITY ENTRANCE PAPERS.

1903.

1. (*a*) Prove that $(ac+bd)^2-(ad+bc)^2=(a^2-b^2)(c^2-d^2)$

(*b*) Divide $8a^3-b^3-27c^3-18abc$ by $2a-b-3c$

2 Simplify $\dfrac{(x+1)^2}{(x-y)(x-z)}+\dfrac{(y+1)^2}{(y-z)(y-x)}+\dfrac{(z+1)^2}{(z-x)(z-y)}$.

3 (*a*) Prove that $a^m\times a^n=a^{m+n}$, where m and n are positive integers

(*b*) Simplify $\left(\dfrac{x^l}{x^m}\right)^l\times\left(\dfrac{x^m}{x^l}\right)^m-\{(x^l)^l\times(x^m)^m\}\times\{(x^m)^l\times(x^l)^m\}$

4 Solve —(1) $\dfrac{x}{x+a-b}+\dfrac{x}{x+b-c}=2$,

(2) $\frac{x}{2}+\frac{y}{3}+\frac{z}{4}=23$, $\frac{x}{3}+\frac{y}{4}+\frac{z}{2}=28$,

and $\frac{x}{4}+\frac{y}{2}+\frac{z}{3}=27$.

5 A general wishing to draw up his regiment in the form of a hollow square found that he had 50 men over when it was 4 deep, but that he wanted 50 men to complete it when it was 5 deep, the number of men in the front being the same in both cases Find the number of men in the regiment

6 If $x\,(b+c)=y\,(c+a)=z\,(a+b)$, prove that
$a\,(y+z-x)=b\,(z+x-y)=c\,(x+y-z)$

1904.

1 Simplify —

(1) $(2x+3y)^3-3(2x+3y)^2(2x-3y)$
$+3(2x+3y)(2x-3y)^2-(2x-3y)^3$,

(2) $\left(\dfrac{x^l}{x^m}\right)^{l^2+lm+m^2}\times\left(\dfrac{x^m}{x^n}\right)^{m^2+mn+n^2}\times\left(\dfrac{x^n}{x^l}\right)^{n^2+nl+l^2}$

2 Reduce $\dfrac{3x^6-5x^3+2}{2x^5-5x^2+3}$ to its lowest terms

3 Show that $(x-1)(x-2)(x-3)(x-4)+1$ is a perfect square

4 (*a*) Distinguish between an *equation* and an *identity*, and give an example of each

(*b*) Solve —(1) $\dfrac{x(a+b)+c}{x+d}+\dfrac{x(a-b)+d}{x+c}=2a$;

(2) $x+y+z=0$, $ax+by+cz=0$,

and $a^2x+b^2y+c^2z+(b-c)(c-a)(a-b)=0$

5. At 7-40 A.M. an ordinary train starts from P, and reaches Q at 11-40 A.M.; an express train which starts from Q at 9 A.M. arrives at P at 11-40 A.M.; if both trains travel at uniform speed without stopping, find the time when they meet.

6. If $a : b :: x : y$, then $a^2+b^2 : \frac{a^3}{a+b} :: x^2+y^2 : \frac{x^3}{x+y}$.

(See page 350, Ex. 7.)

1905.

1. (1) Given $x+y=5$ and $xy=7$, find the value of $x^3+y^3+4(x-y)^2$.

(2) If $x^2+y^2=1$ prove that $(3x-4x^3)^2+(4y^3-3y)^2=1$.

2. Divide $a^3(b-c)+b^3(c-a)+c^3(a-b)$ by $a+b+c$.

(See page 127, Ex. 1.)

3. Simplify $\frac{b-c}{a^2-(b-c)^2}+\frac{c-a}{b^2-(c-a)^2}+\frac{a-b}{c^2-(a-b)^2}$.

(See page 191, Ex. 34.)

4. Solve :—(1) $\frac{x-bc}{b+c}+\frac{x-ca}{c+a}+\frac{x-ab}{a+b}=a+b+c$,

(2) $x-y+z=2$, $4x+6y+5z=31$,
and $5x-11y+13z=22$.

5. A company of men is formed into a hollow square 10 deep. If the company be increased by 1600 men, the whole number may also be formed into a hollow square 10 deep, so that the front in the latter formation shall contain twice the number of men contained in the front of the former. Find the original number of men.

6. (1) If $a : b :: c : d$, prove that $\frac{a+b}{a-b}=\frac{c+d}{c-d}$.

(2) If $(a+b+c)x=(b+c-a)y=(c+a-b)z=(a+b-c)w$, show that $\frac{1}{y}+\frac{1}{z}+\frac{1}{w}=\frac{1}{x}$.

1906.

1. Factorize :— x^4+4a^4 and $(a+b+c)(bc+ca+ab)-abc$.

(See page 112, Ex. 2 and page 125, Ex. 8.)

2. (1) Prove that $(a^m)^n=a^{mn}$, when m and n are positive integers.

(2) If $a^x=b$, $b^y=c$, $c^z=a$, prove that $xyz=1$.

3. (1) If $a^2+b^2+c^2=1$, and $x^2+y^2+z^2=1$, prove that
$(bz-cy)^2+(cx-az)^2+(ay-bx)^2+(ax+by+cz)^2=1$.

(2) If $bc+ca+ab=0$, prove that $\frac{1}{a^2-bc}+\frac{1}{b^2-ca}+\frac{1}{c^2-ab}=0$.

4 Distinguish between an *equation* and an *identity*. Give an example of each.

Solve —(1) $\frac{x-a^2}{b^2-bc+c^2}+\frac{x-b^2}{c^2-ca+a^2}+\frac{x-c^2}{a^2-ab+b^2}=2(a+b+c)$,

(2) $x+y+z=0$, $(b+c)x+(c+a)y+(a+b)z=0$,

and $bcx+cay+abz=1$. (See page 256, Ex. 4.)

5 A man walks from A to B and back in a certain time at the rate of $3\frac{1}{2}$ miles an hour. But if he had walked from A to B at the rate of 3 miles an hour and back from B to A at the rate of 4 miles an hour he would have taken 5 minutes longer. Find the distance between A and B.

6 (1) If $a:b=c:d=e:f$, prove that each ratio $=\frac{a+c+e}{b+d+f}$.

(2) If $\frac{py+qz}{b+c}=\frac{pz+qx}{c+a}=\frac{px+qy}{a+b}$, prove that

$$(x+y+z)\{(b+c)x+(c+a)y+(a+b)z\}=2(a+b+c)(yz+zx+xy)$$

1907.

1. Prove that $a^m\times a^n=a^{m+n}$, where m and n are positive integers.

Simplify $\frac{(x^2yz)^n\,(y^2xz)^n\,(z^2xy)^n}{\left(\frac{yz}{x}\right)^n\left(\frac{zx}{y}\right)^n\left(\frac{xy}{z}\right)^n}$

2 (1) Factorize —(i) $x^2+2x-323$,

(ii) $a^2(b+c)+b^2(c+a)+c^2(a+b)+2abc$.

(See page 127, Ex. 40.)

(2) If $2s=a+b+c$, show that

$$\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c}-\frac{1}{s}=\frac{abc}{s(s-a)(s-b)(s-c)}.$$

(See page 185, Ex. 7.)

3 (1) Find the G. C. M. of $x^4-115x+24$ and $24x^4-115x^3+1$.

(2) Extract the square root of $\left(x+\frac{1}{x}\right)^2-4\left(x-\frac{1}{x}\right)$.

4 Solve —(1) $\frac{2}{2x-5}+\frac{1}{x-3}=\frac{6}{3x-1}$;

(2) $x+y+z=a+b+c$, $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=3$,

and $ax+by+cz=a^2+b^2+c^2$. (See page 368, Ex. 14.)

5 A man met several beggars and wished to give 25 pence to each, but on counting his money he found that he had 10 pence too little for that, and then made up his mind to give 20 pence each. After doing this he had 20 pence over. What had he at first, and how many beggars were there?

6 (1) If $a : b = c : d = e : f$, show that each of these ratios $= \dfrac{ma+nc+pe}{mb+nd+pf}$ (See page 346, Art 14)

(2) Prove that if $\dfrac{ay-bx}{c} = \dfrac{cx-az}{b} = \dfrac{bz-cy}{a}$,

then $\dfrac{x}{a} = \dfrac{y}{b} = \dfrac{z}{c}$ (See page 349, Ex. 4)

1908.

1 (1) Divide —

$a(a+b)(a+c)+b(b+c)(b+a)+c(c+a)(c+b)+3(b+c)(c+a)(a+b)$ by $(b+c)(c+a)+(c+a)(a+b)+(a+b)(b+c)$

(2) Resolve $(ay-bx)^2+(bz-cy)^2+(cx-az)^2+(ax+by+cz)^2$ into two factors

2 (1) Prove that $(a^m)^n = a^{mn}$, m and n being positive integers.

(2) $x = a(b-c)$, $y = b(c-a)$, $z = c(a-b)$, prove that

$$\left(\frac{x}{a}\right)^3+\left(\frac{y}{b}\right)^3+\left(\frac{z}{c}\right)^3 = \frac{3xyz}{abc}.$$

3 If x^2+7x+c is exactly divisible by $x-5$, what is the value of c?

4 (1) Distinguish between an *equation* and an *identity*, and give an example of each

(2) Solve —(i) $\dfrac{x-a^2}{b+c}+\dfrac{x-b^2}{c+a}+\dfrac{x-c^2}{a+b} = 4(a+b+c)$,

(ii) $2x+y-2z = 0$, $7x+6y-9z = 0$,

and $13x+14y-15z = 40$

5 A purse is full of four-anna pieces After ½ of the contents is taken away, there remain 32 pieces more than the number of rupees in the value of the whole purse How many four-anna pieces did it contain at first?

6 (1) If $\dfrac{a}{b} = \dfrac{b}{c} = \dfrac{c}{d}$, then $\dfrac{a}{d} = \dfrac{a^3}{b^3}$,

(See page 341, Art 8 Note)

(2) If $\dfrac{a}{y+z} = \dfrac{b}{z+x} = \dfrac{c}{x+y}$, prove that

$$\frac{a(b-c)}{y^2-z^2} = \frac{b(c-a)}{z^2-x^2} = \frac{c(a-b)}{x^2-y^2}$$

Alternative Questions

I If $a+b+c = 0$, prove that $\dfrac{a^5+b^5+c^5}{5} = \dfrac{a^3+b^3+c^3}{3} \cdot \dfrac{a^2+b^2+c^2}{2}$.

(See page 387 XXIII, Ex. 7)

II. Simplify

$$\frac{a}{(a-b)(a-c)(x-a)}+\frac{b}{(b-c)(b-a)(x-b)}+\frac{c}{(c-a)(c-b)(x-c)}.$$

1909.

1 Divide, by the *ordinary method*, $a^3+b^3+c^3-3abc$ by $a+b+c$; and express the quotient as half the sum of three squares

(See page 104, Ex 5, & page 87, Note)

Or,

Resolve into factors

(1) $(ax+by)^2+(ay-bx)^2$; (2) $2(ab+cd)-a^2-b^2+c^2+d^2$.

2 Find the H C F of $x^5+11x-12$ and x^5+11x^3+54

Or,

Simplify $\frac{x^2-7x+10}{x^2-10x+24}\times\frac{x^2-7x+12}{x^2-2x-35}\quad\frac{x^2-x-6}{x^2-13x+42}$.

3 (1) If $(a^2+b^2)(x^2+y^2)=(ax+by)^2$, shew that $x : y=a : b$

(2) Find the square root of $\frac{x^2}{y^2}-\frac{2x}{y}+3-\frac{2y}{x}+\frac{y^2}{x^2}$.

4 Solve —(1) $\frac{x}{x+2}+\frac{x}{x+1}=2$,

(2) $\frac{1}{x}+\frac{2}{y}=10$, $\frac{4}{y}+\frac{3}{z}=18$, and $\frac{2}{z}+\frac{3}{x}=16$

5 If $ax+hy+gz=0$, $hx+by+fz=0$, and $gx+fy+cz=0$, prove that $abc+2fgh-af^2-ch^2=0$

Or,

Two boys start together to walk 33 miles One walks 12 miles in the same time that the other takes to walk 11 miles, and arrives at the end of the journey an hour beforehand Find their rates of walking in miles per hour

CALCUTTA UNIVERSITY MATRICULATION PAPERS.

1910.

Compulsory Paper.

1 (1) Find the continued product of

$a+b+c$, $b+c-a$, $c+a-b$, $a+b-c$

Or,

If $x-\frac{1}{x}=p$, find the value of $x^3-\frac{1}{x^3}$ in terms of p

(2) Resolve into factors x^3+1 and x^2+x-20

2 (1) Find the G C M of x^2-9, $(x+3)^2$ and x^2+x-6

Or,

Find the L C M of x^2-4, x^2-x-2 and x^2+x-2

(2) If $\frac{x}{a} = \frac{y}{b}$, prove that $(x^2+y^2)(a^2+b^2) = (ax+by)^2$

(See page 351, Ex 9)

3 (1) Solve $\frac{b}{x} = \frac{a}{x-b+a}$ *Or*, $9x-5y=17$, $13y-2x=20$.

(2) Draw the graph of $y = x+1$

Additional Paper

1 (1) Solve $\frac{x}{x+1}+\frac{x+1}{x}=\frac{25}{12}$ (See page 396, Ex 17)

(2) Draw the graph of $y = 4x^2$

2 (1) Find the square root of $4x^4+20x^2-3-\frac{70}{x^2}+\frac{49}{x^4}$

(2) Prove that $a^m \times a^n = a^{m+n}$, for positive integral values of m and n (See page 296, Art 2)

3 (1) Find, without the assumption of any formula, the sum of the first n natural numbers

(2) If a^2, b^2, c^2 be in A P, prove that

$\frac{1}{b+c}$, $\frac{1}{c+a}$, $\frac{1}{a+b}$ are also in A P (See page 561, Ex 7)

Or,

(1) Find, without the assumption of any formula, the sum of n terms of the series $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+$&c

(1) If a, b, c be in A P, and a, b, d be in G P, show that $a, a-b, d-c$ are in G P

4 If $x = b+c-a$, $y = c+a-b$, and $z = a+b-c$, find the value of $\frac{x^3+y^3+z^3-3xyz}{a^3+b^3+c^3-3abc}$ (See page 164, Ex. 8)

1911.

Compulsory Paper.

1 (1) Find the continued product of

$1+x+x^2$, $1-x+x^2$, and $1-x^2+x^4$.

Or,

If $x+y+z = 13$, and $xy+yz+zx = 50$, find the value of $x^2+y^2+z^2$ (See page 88, Ex. 22)

(2) Resolve into factors $x^2+2x-143$ and $a^4+2a^3b-2ab^3-b^4$.

2 (1) Find the G C M of

$$3x^3+17x^2-62x+14 \text{ and } 7x^3-52x^2-46x+8$$

Or,

Find the L C M of a^2-9b^2, $a^2-ab-6b^2$, and $a^2+ab-12b^2$.

(2) If $\frac{x}{a+b-c}=\frac{y}{b+c-a}=\frac{z}{c+a-b}$,

show that each of these fractions $=\frac{x+y+z}{a+b+c}$ (P 375, IV Ex 8)

3 (1) Solve $\frac{2x-3}{3}-\frac{3x-5}{5}+\frac{5x+3}{6}-\frac{7x+5}{10}=4$

Or,

Solve $x+y+z=1$, $2x+3y+z=4$, and $4x+9y-z=16$

(2) Draw the graph of $\frac{x}{2}-\frac{y}{3}=1$

Additional Paper.

1 Solve —(1) $17x^2+19x=1848$, (2) $\frac{x-3}{x+3}-\frac{x+3}{x-3}+6\frac{6}{7}=0$

Or,

(3) Draw the graphs of (a) $x^2+y^2=25$, and, (b) $3x+4y=25$.

Prove that the second graph touches the first, and find the co ordinates of the point of contact

2 (1) Find the square root of

$$(a-b)^4-2(a^2+b^2)(a-b)^2+2(a^4+b^4).$$

Or,

(2) $\frac{x^2}{y^2}+\frac{y^2}{x^2}-6\frac{x}{y}+6\frac{y}{x}+7$

(3) Prove that $(a^m)^n=a^{mn}$, for positive integral values of m and n

3 (1) Find, without assuming any formula, the sum of n terms of the series $1+3+5+7+$,

(2) How many terms of the series $3+5+7+$ must be taken in order that the sum may be equal to 399 ?

Or,

(3) Find, without assuming any formula, the sum of n terms of the series

$$\frac{1}{10}+\frac{6}{100}+\frac{6}{1000}+\frac{6}{10000}+$$

Hence deduce the value of $1\dot{6}$

4 If $x=b+c$, $y=c+a$, $z=a+b$, find the value of

$$\frac{x^3+y^3+z^3-3xyz}{a^3+b^3+c^3-3abc}$$ (Page 385, XIX Ex 5).

1912.

Compulsory Paper

1 (1) Multiply $4x^2+9y^2+z^2+3yz-2zx+6xy$ by $2x-3y+z$

(2) Divide $6x^5-17x^4+42x^3-66x^2+72x-72$ by $2x^2-3x+6$

Or,

(1) Find the co efficient of x^4 in the product of
$1-2x+4x^2-8x^3+16x^4$ by $1+2x+4x^2+8x^3+16x^4$

(2) Find the L C M of $2x^2-x-1$, $2x^2+3x+1$, x^2-1

2 (1) If $a:b::b:c$, show that $(a+b+c)(a-b+c)=a^2+b^2+c^2$.

(2) Solve $\frac{5x-1}{9}+\frac{9x-5}{11}=\frac{9x-7}{5}$.

Or,

(1) Draw the graph of $\frac{x}{4}+\frac{y}{5}=1$

(2) Solve $x+5y=36$ and $\frac{x+y}{x-y}=\frac{5}{3}$.

3 A man performed a journey of 7 miles in 1 hour 15 minutes He walked part of the way at 4 miles an hour and rode the rest of the way at 10 miles an hour How far did he walk?

Additional Paper.

1 (1) Extract the square root of $25x^4-12x+16x^{-8}+4x^{4}-24x^{-3}$

(2) Show that $\left(x^{2^{n-1}}+a^{2^{n-1}}\right)\left(x^{2^{n-1}}-a^{2^{n-1}}\right)=x^{2^n}-a^{2^n}$.

(See page 295, Ex. 3).

Or,

(1) Solve $\frac{x+4}{x-4}+\frac{x-4}{x+4}=\frac{10}{3}$

(2) Simplify — $\frac{bc}{(b-a)(c-a)}+\frac{ca}{(c-b)(a-b)}+\frac{ab}{(a-c)(b-c)}$

2 (1) Find, without assuming any formula, the sum to n terms of $1+\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\frac{1}{3^4}+$&c

(2) Sum to n terms $1.2+2.3+3.4+4.5+$&c (P 556, Ex. 1)

3 Draw the graphs of $x^2+y^2=25$ and $x+y=7$, and measure the co-ordinates of their points of intersection

4 A horse was sold at a loss for Rs 840 but if it had been sold for Rs 1050 the gain would have been three-fourths of the former loss Find its real value

1913.

Compulsory Paper.

1 (1) If $x = b-c$, $y = c-a$, $z = a-b$ find the value of $x^2+y^2-z^2+2xy$

(2) Find the G C M of $2x^3+x^2-5x-3$ and $8x^3+6x^2-21x-18$

Or,

(3) Simplify $\left(\dfrac{x+y}{x-y}-\dfrac{x-y}{x+y}\right) \div \left(\dfrac{1}{\dfrac{x}{y}-\dfrac{y}{x}}-\dfrac{1}{\dfrac{x}{y}+\dfrac{y}{x}}\right)$.

(4) Divide $6x^4-2x^3-23x^2+5x+20$ by $2x^2-5$

2 Solve (1) $\dfrac{2x+1}{5}-\dfrac{3x-2}{6}=\frac{1}{2}$ (2) $2x+y=3y-x=7$

Or,

A man pays Rs 200 more than one-third of his debt and still owes Rs. 210 more than what he has paid What was his original debt ?

3 (1) Draw the graph of $y = x-2$

(2) If $5(x-y)=3(x+y)$, find the ratio of x to y.

Additional Paper

1. (1) Solve $42x^2-41x-20=0$

(2) Find the square root of $x^4-x^3-\dfrac{7x^2}{4}+x+1$

Or,

(1) Divide 50 into two parts such that the sum of their reciprocals may be $\frac{1}{12}$

(2) Simplify $\dfrac{(b-c)^2+(c-a)^2+(a-b)^2}{(a-b)(a-c)+(b-c)(b-a)+(c-a)(c-b)}$

2 (1) The first two terms of an Arithmetical Progression are 3 and 1 Write down the tenth term and the sum of the first ten terms

(2) The first two terms of a Geometrical Progression are 3 and 1 Write down the tenth term and the sum of the first ten terms

3 Draw the graphs of $x^2+y^2=16$ and $x+y=2$, and measure the length of the chord of intersection

1914.

Compulsory Paper

1 (1) Find the value of $a^2+b^2+c^2-bc-ca-ab$ when $a=x+y$, $b=x-y$, $c=x+2y$

Or,

(2) Divide x^6-6x+5 by x^2-2x+1

2 Factorize —(1) $x(x-1)(x-2)-3x+3$,

(2) $a^2(b-c)+b^2(c-a)+c^2(a-b)$

Or,

Simplify $\dfrac{b-c}{a^2-(b-c)^2}+\dfrac{c-a}{b^2-(c-a)^2}+\dfrac{a-b}{c^2-(a-b)^2}$

(See page 191, Example 34)

3 Solve — (1) $\dfrac{x-a}{b}+\dfrac{x-b}{a}+\dfrac{x-3a-3b}{a+b}=0$,

(2) $\dfrac{2x+2y-3}{5}=\dfrac{3x-7y+4}{6}=\dfrac{8y-x+2}{7}$

Or,

(3) Draw the graphs of $3x+4y=25$ and $4x-3y=0$, and measure the co ordinates of their point of intersection

Additional Paper.

1 Solve — (1) $6x^2-91x+323=0$, (2) $x+\dfrac{1}{x}=25\frac{1}{25}$

Or,

(3) Draw the graphs of $4y=x^2$ and $2y=x+4$, between $x=-4$, and $x=+4$ and measure the length of the chord of intersection

2 (1) Extract the square root of $\left(x+\dfrac{1}{x}\right)^2-4\left(x-\dfrac{1}{x}\right)$.

(2) Express $(x-a)(x-b)$ as the difference of two squares.

3 (1) Find seven arithmetical means between 1 and 41.

(2) Find three geometrical means between $\frac{1}{9}$ and 9

1915.

Compulsory Paper

1 (1) Find the product of $(b+c-a)$, $(c+a-b)$, $(-a+b+c)$.

(2) Divide $x^4-y^4+a^4+2a^2x^2$ by $x^2-y^2+a^2$

Or,

(1) Find the H C F of x^2-2x-3 and x^3-2x^2-2x-3.

(2) Find the L C M of a^2-b^2, a^3-b^3, and a^4-b^4.

2 Solve the equations,—

(1) $\frac{5x+6}{12}+\frac{3x-4}{5}=2(x-9)$, (2) $3x+4y=27$ $5x-3y=16$.

3 Simplify $\frac{1}{a(a-b)(a-c)}+\frac{1}{b(b-a)(b-c)}+\frac{1}{c(c-a)(c-b)}$

(See page 192, Example 47)

Or,

Draw the graph of $x-y=4$

Additional Paper.

1 (1) Solve without assuming the formula, $63x^2-62x=221$.

(2) What number must be added to $x^4-6x^3+13x^2-12x+1$ to make a perfect square?

Or,

(1) Divide unity into two parts such that the sum of their cubes is $\frac{7}{16}$

(2) Simplify $\frac{\sqrt{2}(2+\sqrt{3})}{\sqrt{3}(\sqrt{3}+1)}-\frac{\sqrt{2}(2-\sqrt{3})}{\sqrt{3}(\sqrt{3}-1)}$

2 (1) Find the sum of the squares of the first n natural numbers.

(2) In a geometrical progression, show that the product of any two terms equidistant from a given term is equal to the square of the given term

3 Trace the graphs of (i) $y=2x+1$, (ii) $y=3x^2$, and determine the points where they intersect

1916

Compulsory Paper

1 (1) Simplify $2a-2[2a-\{2(a-b)-b\}]$

(2) Divide $6x+x^2-19x+6x^3$ by $2+x$

Or

(3) Find the H C F of $3x^2-11x-4$ and $6x^3-25x^2+3$.

(4) Simplify $\frac{x}{x-y}+\frac{y}{x+y}+\frac{2xy}{y^2-x^2}$

2 (1) Solve the equations (i) $\frac{3x+2}{x-1}+\frac{2(x-2)}{x+2}=5$

(ii) $\left.\begin{array}{l}11x+12y=58\\12x+11y=57\end{array}\right\}$.

Or,

(2) Add 1 to the numerator and denominator of a certain fraction and it reduces to $\frac{1}{2}$, subtract 5 from each and it reduces to $\frac{1}{4}$, required the fraction

3 Draw the graph of $3y-2x=4$, and plot the points on the graph for which $x=-2$, 1, and 4 respectively

Additional Paper

1 (1) Solve $4x^2-65x+126=0$

(2) Find the square root of $x^2-6x+5+\frac{12}{x}+\frac{4}{x^2}$

Or,

(3) Simplify $\left(\frac{x^l}{x^m}\right)^{l+m}\times\left(\frac{x^m}{x^n}\right)^{m+n}\times\left(\frac{x^n}{x^l}\right)^{n+l}$

(4) Find two consecutive numbers, the sum of whose squares is 145

2 (1) Find without the assumption of the formula, the sum of 30 terms of the series 1 3, 5, 7,

(2) Insert two numbers between 5 and 135 so that the four may form a geometrical progression

3 Trace the graphs of $y=x$ and $y=\frac{x^2}{4}$, and determine the points where they intersect

1917

Compulsory Paper

1 (1) Multiply $a^2-ab+a+1$ by $a+b-1$

Or,

(2) Divide a^4-6a-4 by $a-2$

(3) Find the H C F of x^3-7x+6 and x^3-3x^2+4

Or,

(4) Find the L C M of x^2+x-6, x^2+2x-3 and x^2-3x+2

2 (1) Draw the graph of $2x+3y=1$

(2) If $\frac{a}{b}=\frac{b}{c}$, shew that $\frac{a}{c}=\frac{a^2+b^2}{b^2+c^2}$.

3 (1) Solve $\frac{x+3}{4}-\frac{x+4}{5}=\frac{x+5}{6}-\frac{x+6}{7}$

Or,

(2) Solve $x+2y=3=4x-y$

(3) The half of a certain integer exceeds the third of the next greater integer by two Find the integer

Additional Paper.

1 (1) Solve, without assuming formula, $x^2-x=1806$.

(2) Find the square root of $1+2a+2a^2+a^3+\frac{a^4}{4}$.

2 (1) Shew how to find the sum of n terms of an A P, being given the first term and the common difference

Or,

(2) Shew how to find the sum of n terms of a G P, being given the first term and the common ratio

(3) Sum to n terms $1\times2+2\times3+3\times4+4\times5+\ldots$

Or,

(4) Sum to n terms $1+2x+3x^2+4x^3+$

3 (1) Trace the graph of $y=x^2-x$ from $x=-1$ to $x=2$ and therefrom obtain an approximate solution of the equation $1=x^2-x$

MADRAS UNIVERSITY MATRICULATION PAPERS.

1904.

1 (*a*) Find the value of $12x^4-2x^3-3x^2+4x-5$ when $x=-\frac{4}{5}$.

(*b*) Divide $x^8-2a^4x^4+a^8$ by $x^3+ax^2+a^2x+a^3$

(*c*) Resolve $9x^6y^2-576y^2-4x^8+256x^2$ into 6 factors

2 (1) Prove the rule for finding the G C M of two algebraic expressions

(2) Find the L C M of

$$x^6-2x^4y^2+y^4x^2-y^6 \text{ and } x^6-x^2y^4-2xy^5-y^6.$$

3 Reduce to their simplest forms —

(i) $(b+c-a)^3+(c+a-b)^3+(a+b-c)^3-(a+b+c)^3$,

(ii) $$\frac{a+p}{(a-b)(a-c)(a+x)}+\frac{b+p}{(b-c)(b-a)(b+x)}+\frac{c+p}{(c-a)(c-b)(c+x)}.$$

4 Extract (1) the square root of

$$(9a^2+12ab)^2-2(9a^2+12ab)(2b^2+6ab)+4(2b^2+6ab)^2;$$

and (2) the cube root of $\left(a^6+\frac{1}{a^6}\right)+6\left(a^4+\frac{1}{a^4}\right)+15\left(a^2+\frac{1}{a^2}\right)+20$

5 Solve the equations —

(i) $(x-1)(x-5)(x-7)(x-9)=(x-2)(x-4)(x-6)(x-10)$,

(ii) $ay-bx=a^2-2ab-b^2$, $by+ax=a^2+2ab-b^2$,

(iii) $3x-2y=9\frac{1}{2}$, $3y-2z=-13\frac{1}{2}$, $3z-2x=6$

6 A starts to walk from P to Q at 11 A.M., B starts from Q at 11-55 A.M. They meet $5\frac{1}{4}$ miles from Q B stays 20 minutes at P and

A stays 2 hrs $17\frac{1}{2}$ mins at Q, and returning they meet when B has walked $\frac{2}{3}$ of the distance back at 6-30 P M Find the distance from P to Q

1905.

1 If $4x=5$, $3y=-4$, $2z=9$, find the value of each of the following expressions

(1) $\{(4x^2+yz)-(3y^2-zx)\}+1$, (2) $\sqrt{7x+2y+\frac{1}{3}}+\sqrt[3]{5y+7z+\frac{1}{6}}$

2 Factorize (1) x^6-1, (2) $(x-1)^3+(x-2)^3+(3-2x)^3$.

3 Determine the common factors of the expressions

$12x^3-8x^2-3x+2$ and $16x^3+12x^2-4x-3$

Hence find three values of x which when substituted in the expression $(12x^3-8x^2-3x+2)+(16x^3+12x^2-4x-3)$ will give zero for the result

4 Extract the square root of $(3x^2-11x+6)^2+12x(3x-2)^2$

5 Simplify (1) $(x^2+1)(x^2+x\sqrt{2}+1)(x^2-1)(x^2-x\sqrt{2}+1)$,

(2) $\frac{1}{x^2}-\frac{3}{4}\left(\frac{1}{x-1}-\frac{1}{x+1}\right)+\frac{1}{4}\left\{\frac{1}{(x-1)^2}+\frac{1}{(x+1)^2}\right\}$;

(3) $\frac{(x+y)^5-x^5-y^5}{(x+y)^3-x^3-y^3}$

6 State the difference between a conditional and an identical equation Solve the equation $(x-3a-2b)^3-(x-a-b)^3=(3b-2a)^3$.

7 Find an expression of the second degree in x which shall have the values of 9, 24 and 21 when $x=-2$, 1 and 4 respectively

1906.

1 (1) Subtract

$$\left(x-\frac{2}{x^2}\right)\left(x+\frac{x^2}{2}\right)\left(\frac{2}{x^2}-\frac{x^2}{2}\right) \text{ from } \left(x+\frac{x^2}{2}-\frac{2}{x^2}\right)\left(\frac{1}{x}+\frac{2}{x^2}-\frac{x^2}{2}\right);$$

(2) Divide $a(x-b)^4+b(x-a)^4+x(a-b)^4$ by

$x^2-(a+b)x+a^2-ab+b^2$

2 Resolve into factors —

(1) $(a-b)^2x^4-8(a^2-b^2)x^2y^2+16(a+b)^2y^4$,

(2) $a(b-c)^3+b(c-a)^3+c(a-b)^3$

3 (1) Reduce $\frac{55x^4+24x^3+1}{125x^4+24x+11}$ to its lowest terms,

(2) Find the L C M of

$4x^4-12x^3+8x$, $6x^6+18x^4-24x^2$ and $2x^5+3x^4-2x^3$.

4 Simplify $(a+b+c)^5+(a-b-c)^5+(b-c-a)^5+(c-a-b)^5$.

5 Prove that

$$a^3+b^3+c^3-3abc=\tfrac{1}{2}(a+b+c)\{(b-c)^2+(c-a)^2+(a-b)^2\}.$$

Hence, find the square root of

$$(x^2-yz)^3+(y^2-zx)^3+(z^2-xy)^3-3(x^2-yz)(y^2-zx)(z^2-xy)$$

(See page 371, Example 19)

6 Solve (1) $\frac{6}{x+1}-\frac{1}{x+6}=\frac{2}{x-3}+\frac{3}{x+2}$,

(2) $\frac{x^2+3}{x-1}+\frac{x^2-x+3}{x-2}=\frac{2(x-2)^2}{x-5}$

7 A man and his family consume 10 measures of rice in a week. If his wages were raised 5 p. c. and the price of rice raised 2½ p. c. he would gain 8 as. a week. But if his wages were lowered 7½ p. c. and the price of rice lowered 11¼ p. c. then he would lose 9¾ as. a week. Find the price of a measure of rice. (Supposing his other expenses remain the same.)

1907.

1 (1) $a+b+c=abc=6$ and $ab+bc+ca=11$,

find the value of $\frac{a}{1+a^2}+\frac{b}{1+b^2}+\frac{c}{1+c^2}$.

(2) The surface S of a sphere of radius r is given by the formula $S=4+\frac{22}{7}\times r^2$, find the surface of the sphere whose radius is 21 inches and the radius of the sphere whose surface is 57 75 square inches.

2 (1) Prove that $a-(b-c)=a-b+c$

(2) Find the co-efficient of x^3 and x^5 in the expansion of

$$\left(1+\frac{x}{2}-\frac{x^2}{3}-\frac{x^3}{4}\right)^3$$

(3) Divide $(9a^2-4b^2)^3+1728a^2b^4$ by $(3a-2b)^2+16b^2$

3 Factorize

(1) $a(b+c)^2-b(c+a)^2$, (2) x^3-3x+2,

(3) $(b-c)(a-d)(a-x)+(c-a)(b-d)(b-x)+(a-b)(c-d)(c-x)$.

4 Simplify

(1) $\frac{x+27}{x+6x^2-15}-\frac{2x+3}{4x^2+3-8x}+\frac{x+19}{5-7x-6x^2}$;

(2) $\left(\frac{x}{y}+\frac{y}{x}\right)\left(\frac{y}{z}+\frac{z}{y}\right)-\left(\frac{x}{yz}+\frac{y}{zx}+\frac{z}{xy}\right)\left(\frac{yz}{x}+\frac{zx}{y}+\frac{xy}{z}\right)$.

5 Solve the equations —

(1) $\frac{2}{3}(2x-1)-\frac{3}{4}(3x-2)+\frac{5}{6}(5x-4)+4=0$;

(2) $7x+\frac{5y+9x}{11}=17$, $6y+\frac{11y+6x}{17}=21$;

(3) $7x^2-22x=16$

6 A man bought some cloth for Rs 147. He gave away 12 yds and sold the rest at a profit of 4 as. per yd for Rs 120 4 as. How many yards did he buy?

1908.

1 The curved surface of a cone is given by the formula

$$S = \frac{22}{7} r \sqrt{r^2 + h^2}$$

where r is the radius of the base and h the height

Find (a) the surface of a cone the radius of whose base is 6 3 ins and height 8 4 ins , (b) the height of a cone whose curved surface is 550 square ins the radius of the base being 7 ins

2 (1) Divide $x^6 + \frac{p^6}{27}$ by $x^2 + px + \frac{p^2}{3}$

(2) Resolve into factors $2(x^4+a^4+b^4)-(x^2+a^2+b^2)^2$, and state for what values of x the expression becomes zero

(3) If $x = \frac{1}{2}(b+c-a)$, $y = \frac{1}{2}(c+a-b)$ and $z = \frac{1}{2}(a+b-c)$, prove that $x^3+y^3+z^3-3xyz = \frac{1}{2}(a^3+b^3+c^3-3abc)$.

3 Show that $(x-1)^3$ is a common factor of

$$x^3(x^6-1)-3x^4(x^4-1)+3x^5(x^2-1)$$

$$\text{and } x^4(x^5+1)-3x^3(x^3+1)+2x^2(x+1) ,$$

and hence find the L C M of these two expressions

4. Simplify (a) $\frac{1}{5}\frac{(x+2)(x+3)}{(x-2)(x-3)} - \frac{1}{2}\frac{(x-3)(x+1)}{(x-3)(x-1)} + \frac{1}{3}\frac{(x+1)(x+2)}{(x-1)(x-2)}$,

(b) $\dfrac{(b-c)(b+c)^3+(c-a)(c+a)^3+(a-b)(a+b)^3}{(b+c)(b-c)^3+(c+a)(c-a)^3+(a+b)(a-b)^3}$.

5 Find the square root of $(2x+1)(2x+3)(2x+5)(2x+7)+16$.

6 Solve —(1) $\frac{1}{4}(3x+20)-\frac{1}{15}(7x+47) = \frac{1}{5}\left(6+\frac{x+7}{3}\right)$,

(2) $x+\frac{1}{2}(y+z)=17$, $y+\frac{1}{3}(z+x) = z+\frac{1}{4}(x+y) = 15$;

(3) $12x^2-85x-175 = 0$

7 A man buys two horses for Rs 765 By selling one for $\frac{6}{5}$ of its cost price and the other at $\frac{4}{5}$ of its cost price, he makes a profit of Rs. 76 8 as on the whole transaction Find the cost price of each horse

December, 1909.

1 (1) Find the co-fficients of x^2 and x^4 in the expansion of

$$\left(2+\frac{x^2}{4}-\frac{x^3}{3}\right)^8.$$

(2) Show that

$(a-b)^2(a-c)^2+(b-c)^2(b-a)^2+(c-a)^2(c-b)^2$ is a perfect square

(See page 383, XVII, Example 5.)

2. If $x = a^3+a^2$, $y = a^2+a$, and $z = a+1$, prove that

$$(x+y)(x+z)(x^2-z) = (x+y+z)(x-z)(x^2+y^2)$$

3 Solve (1) $\frac{2}{3}(5x-13)-\frac{3}{5}(4x-9) = \frac{1}{2}(x-2)-(10-x)$,

(2) $x+y-\frac{1}{3}(x-y) = 6$, and $x-y+\frac{1}{2}(x+y) = 9$.

Verify the latter by plotting the graphs

4 The distances from P to Q by two different routes are 102 and 121 miles. A motor car taking the longer road travels 4 miles an hour faster than one taking the shorter road and does the journey in 10 minutes less Find the rate of each car

March, 1911.

1. Show that $(a+3b+5c)^3+(2a+6b-5c)^3$ is divisible by $3(a+3b)$ and simplify the quotient

2. (i) Resolve into factors (a) $5x^2-29x+42$;

(b) $5x^2-29x-42$, and (c) $x^3+4x^2-17x-60$

(ii) Shew that

$$(b+c)^3-(c+a)^3+(a-b)^3+3(b+c)(c+a)(a-b) = 0$$

(See Page 277, II, Example 2)

3 Solve (a) $\frac{x-1}{x-2}+\frac{x-3}{x-4} = 3\frac{1}{3}$;

(b) $4x-5y+7 = 0$ and $3x+2y = 12$;

Verify the latter by plotting the graphs.

4. A certain train is 6 mins late when it performs its usual journey at $29\frac{1}{4}$ miles per hour, and 24 mins late when it travels at $27\frac{1}{2}$ miles per hour Find the length of the journey

1912

1 Plot the graph of $y = \frac{1}{2}x^2-3$ between the values $x = -3$ and $x = +3$ (See App. Chap II. Art. 6)

2 (a) Find the co-efficients of x^3 and x^5 in the product of $x-\frac{3}{4}x^2+\frac{4}{5}x^3$ and $\frac{2}{3}-5x^2+\frac{3}{7}x^3$.

(b) If $u = x+\frac{1}{x}$, find the values of

$x^3+\frac{1}{x^3}$ and $x^4+\frac{1}{x^4}$ in terms of u (See Page 73, Ex. 4)

3. Simplify $a(b-c)(b+c-a)^2+b(c-a)(c+a-b)^2+c(a-b)(a+b-c)^2$. (See Example 4, Page 89)

4 (*a*) Solve: $\frac{3}{8}(x-3)+\frac{4}{7}(\frac{2}{3}x+4)=\frac{8}{9}(x-\frac{5}{2})$

(*b*) If the straight line whose equation is

$$\frac{x}{h}+\frac{y}{k}=1$$

passes through the points $(7, \frac{3}{2})$ and $(\frac{3}{5}, -\frac{1}{2})$, find the values of h and k. Check your results graphically.

5 An ordinary train, the average speed of which is 15 miles an hour less than that of an express train, takes 3 hrs 45 mins longer than the express to travel 175 miles What is the average speed of each train?

1913.

1 Plot the graph $xy=80$ between the values $x=-20$ and $x=+20$, and use it to find approximate values for the roots of the equations $xy=80$, $x-y=5$. (App Chap II, Art 8)

2 (i) If the co-efficients of x^5 and x in the product of $3x^4+4x^3+2x^2+ax-5$ and $5x^3-ax^2-3x-6$ are equal to one another, find the value of a

(ii) Write down the remainder when $x^{10}-3x^9+5x^6-15x^5+10x^3-1$ is divided by $x-3$

3 Resolve into elementary factors

(i) $x^3-22x-15$, (ii) $4b^2c^2-(a^2-b^2-c^2)^2$,

(iii) x^6-729y^6

4 Solve $x+y+z=6$, $3x-2y+5z=14$, and $4x+3y-2z=4$.

5 Find two numbers whose difference is 27, such that the larger divided by the smaller gives a quotient 7 and a remainder 3

BOMBAY UNIVERSITY MATRICULATION PAPERS.

1903.

1. Define *quotient* and *power*, giving an example of each

(*a*) Simplify and resolve into factors

$$a(b+c)(b^2+c^2-a^2)+b(c+a)(c^2+a^2-b^2)+c(a+b)(a^2+b^2-c^2);$$

(See Page 185, Ex 8)

(*b*) Divide $8x^9-39x^7+66x^5-43x^3+8$ by $(x-1)^3$.

2 Find the H C F and L C M of $4y^4-5y^2+1$ and $4y^4+4y^3+y^2-1$

3 (*a*) Simplify $\left(y+\frac{m-xy}{x-y}\right)\left(x-\frac{m-xy}{x-y}\right)+\left(\frac{m-xy}{x-y}\right)^2$;

(b) Prove that

$$\left(\frac{a}{b-c}+\frac{b}{c-a}+\frac{c}{a-b}\right)^2=\frac{a^2}{(b-c)^2}+\frac{b^2}{(c-a)^2}+\frac{c^2}{(a-b)^2}-2$$

4 Solve. (i) $1\,2x-\frac{18x-\cdot025}{5}=4x+4\,45$,

(ii) $\frac{1}{x}+\frac{1}{y}=\frac{1}{2},\ \frac{21a}{x}-\frac{12b}{y}=\frac{56a-10b}{7}$.

5 A bag contains £8 12s in half-crowns and shillings, if six half-crowns are added, the number of half-crowns is thrice the number of shillings How many are there of each ?

1904.

1 If $a^2=b^2+c^2$, prove that $(a+b+c)(b+c-a)(c+a-b)(a+b-c)=4b^2c^2$. (See page 90, Ex. 1)

2 Divide $x^3-b(4a+b)x+a^3+2a^2b+3ab^2+6b^3$ by $x+a+2b$.

3 (i) If F be a common factor of the algebraical expressions A and B, prove that F will divide $A\pm B$ exactly

(ii) Find the H C F of

$$x^4-4x^3-11x^2-50x+16 \text{ and } x^4-12x^3+29x^2+46x-16$$

4 (i) Simplify

$$\frac{(a+b)^3-c^3}{(a+b)^2-c^2}+\frac{(b+c)^3-a^3}{(b+c)^2-a^2}+\frac{(c+a)^3-b^3}{(c+a)^2-b^2}-2(a+b+c)$$

(ii) Prove the following equality —

$$\frac{a^2}{a^2x-x^3}+\frac{b^2}{b^2x-x^3}+\frac{c^2}{c^2x-x^3}=\frac{x}{a^2-x^2}+\frac{x}{b^2-x^2}+\frac{x}{c^2-x^2}+\frac{3}{x}.$$

5 Find the square root of $(2n-1)(2n-3)(2n-5)(2n-7)+16$

(See page 130, Ex 13)

6 Find the value of x from the equation $\frac{(x+12)^3}{x^3}=\frac{x+24}{x-12}$.

7 A boatman rows 42 miles up a river and back again in 14 hours, he finds that he can row 7 miles with the stream in the same time as 3 miles against it Find the rate at which the river flows

1905.

1 Simplify $$\frac{\frac{a-x}{b+x}\quad\frac{b-x}{a+x}\quad\frac{a+x}{b+x}-\frac{b-x}{a-x}}{\frac{a+x}{b+x}\quad\frac{b-x}{a-x}\quad\frac{a+x}{b-x}\quad\frac{b+x}{a-x}}$$

2 If $xyz(x+y+z)=2$, show that

$$\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^2=(x+y+z)^2+\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}.$$

3 Find the L C M of

x^3-x^2-4x+4, x^3-2x^2-x+2, and x^3+2x^2-x-2

4 Find the H C F of

$x^4+10x^3+25x^2-36$ and x^4-37x^2+36

5 What expression divides $(2x^2+6x+3)(2x^2+10x+11)$ in order that the quotient may be the same as the divisor and the remainder be equal to -3 ?

6 Solve —(i) $\frac{1}{x+1}-\frac{1}{x+2}-\frac{1}{x+3}+\frac{1}{x+4}=0$,

(ii) $\frac{1}{4}\left(\frac{2}{x}-\frac{3}{y}\right)=3\frac{1}{4}$, $\frac{1}{3}\left(\frac{2}{x}+\frac{3}{y}\right)+1\frac{2}{3}=0$.

7 Find the fraction which is doubled when 2 is added to both the numerator and the denominator and trebled when 8 is added to both the numerator and denominator

1906.

1 Show that

$$\left(a-\frac{1}{b}\right)\left(b-\frac{1}{c}\right)\left(c-\frac{1}{a}\right)+(a+b+c)-\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=abc-\frac{1}{abc}.$$

2. (i) Multiply $a^8-a^6+2a^4+a^2+1$ by a^4+a^2-1

(ii) Divide

$7x^6-26x^5+53x^4-84x^3+53x^2-26x+7$ by $7-5x+3x^2-x^3$.

3. (i) Factorise $(a+b+x)(b+x-a)-(a+x+y)(x+y-a)$

(ii) Simplify $\frac{(b+c)(x^2-a^2)}{(a-b)(a-c)}+\frac{(c+a)(x^2-b^2)}{(b-a)(b-c)}+\frac{(a+b)(x^2-c^2)}{(c-a)(c-b)}$.

4 (i) Find the H C F of $11x^4+24x^3+125$ and $x^4+24x+55$

(ii) Find the L C M of

x^3-3x-2, $x^3-6x^2+5x+12$ and x^3-4x^2+x+6

5 (i) Solve $\frac{(x+1)^3-(x-1)^3}{(x+1)^2-(x-1)^2}=2$

(ii) Solve graphically the simultaneous equations $4x+y=6$, and $2y-x=3$, and explain how you get the result

6 What is the value of silver per tola, when a fall of 2 annas in the price of a tola enables a man to buy for Rs 30 eight tolas more than he bought before for the same amount ?

1907.

1 (i) Divide $6x^5+5x^4-39x^3+39x^2+129x-140$ by $2x^2+3x-5$

(ii) What is meant by the degree of an algebraical expression ? And state what is the dividend, divisor and quotient in the first part of this question

2 (i) Resolve into factors $x^4+y^4+1+2x^2y^2-2x^2-2y^2$.

(ii) Find the L C M of

$$x^2-2x-3,\ x^3+x^2-4x-4, \text{ and } x^3-7x-6$$

3 Simplify $\dfrac{x+x^2}{1+x+x^2}-\dfrac{x-x^2}{1-x+x^2}+\dfrac{1-x^2-x^4}{1+x^2+x^4}$.

4 (i) Plot the following points :—

$$(-5, -2),\ (-1, 1);\ (1, 2.5),\ (3, 4),$$

and verify that all the points lie on a straight line A line is drawn through (0, 3) parallel to the x-axis, find the co ordinates of the point where it meets the above line (Unit ½")

(ii) Solve graphically the following equations, taking for unit ½"

$$\left.\begin{aligned} y-x&=5 \\ \text{and } x+2y&=4 \end{aligned}\right\}$$

5 (i) In solving an equation the following result is obtained *viz*. $10(3x-2)(x-3)=0$ What can be inferred and why?

(ii) Solve (i) $\dfrac{1}{x-1}+\dfrac{4}{x-4}=\dfrac{2}{x-2}+\dfrac{3}{x-3}$, (ii) $\dfrac{x+1}{x-1}-\dfrac{x-1}{x+1}=\dfrac{3}{2}$.

(See Page 397, Ex 20)

6 The difference of two numbers is 5 and the sum of their squares 493 What are the numbers?

7 Three men A, B and C start from the same point at the same time, but at different rates, to walk round a course 16½ miles long A in one direction and B and C in the other A meets C twenty minutes after he met B and a mile further on If B walks at the rate of 4½ miles per hour, find the rate at which C walks

1908.

1 Multiply $a^3+3a^2x+ax^2-x^3$ by $3a^2-2ax-x^2$, and divide the product by a^2-x^2.

2 (i) If $a=x+\dfrac{1}{x}$, find the value of $x^2+\dfrac{1}{x^2}$ and $x^3+\dfrac{1}{x^3}$ in terms of a.

(See Page 73, Ex 4)

(ii) Simplify $\dfrac{x-a}{x^2+a^2}+\dfrac{ax(x^2+a^2)}{x^6-a^6}$

3 Factorise (i) $125x^2-40x-48$, (ii) $x^4-5x^2+4-y^2+2xy$; (iii) $x^3-\dfrac{8}{y^3}$, and (iv) $8a^3+12a^2b+6ab^2+b^3$

4 Plot the points (−5, −4) and (6, −1), and find the co ordinates of the point where the straight line joining them intersects the line $2x-3y=6$ [unit ½"]

5 Solve (i) $11y-x=10$, and $11x-101y=110$;

(ii) $\frac{3}{2x-3}+\frac{5}{3x-2}=2$

6 A man distributes Rs 100 equally among his friends; if there had been 5 more friends, each would have received one rupee less, how many friends had he?

1909.

1 Show graphically that the following equations have a common solution, and find it.

(i) $2x-3y+12=0$, (ii) $2x+3y=0$, and (iii) $4x+3y+6=0$

2 Divide $x^4-2bx^3-(a^2+b^2)x^2+2a^2bx-a^2b^2$ by $x^2-(a+b)x+ab$.

3 Find the H C F of $(x-2)(2x^2-7x+6)+(x-2)(2x-3)$
and $(x-1)(2x^2-7x+6)-(x-1)(2x-3)$

4 (i) Find the square root of $4x^4+32x^2+96+\frac{64}{x^4}+\frac{128}{x^2}$.

(ii) Simplify $\frac{x^2-64}{x^2+24x+128}\times\frac{x^2+12x-64}{x^3-64}\div\frac{x^2-16x+64}{x^2+4x+16}$.

5 Solve the following equations

(i) $\left.\begin{array}{l}\frac{x+2}{7}+\frac{7-x}{4}=2(x-4),\\ \frac{2y-3x}{3}+2y=3x+4,\end{array}\right\}$ (ii) $\frac{x}{2}+\frac{2}{x}=\frac{x}{8}+\frac{8}{x}$

6 One cyclist rides 2 miles an hour faster than another and takes half an hour less for a journey of 30 miles Find the rates at which they travel

1910.

1 If a and b are multiples of c, then $a-b$ is also a multiple of c

The numbers 812 and 957 when divided by a number of two digits leave no remainder find the divisor

2 Interpret and construct the graphs of the equations —

(i) $4x+6y=24$ and (ii) $2x+3y=6$ [unit $=\frac{1}{4}''$]

3 (i) Multiply $a^2+25b^2+4c^2+5ab-2ac+10bc$ by $a-5b+2c$

(ii) Divide $(a+b+c)(ab+bc+ca)-abc$ by $a+b$

(See page 125)

4 (i) If $x=\frac{a+1}{ab+1}$ and $y=\frac{a(b+1)}{ab+1}$, find the value of $\frac{x+y-1}{x-y+1}$.

(ii) If $b^2 = ac$, prove that $\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3} = \frac{a^3+b^3+c^3}{a^2b^2c^2}$.

(See Page 354, Ex. 33).

5 Simplify $\frac{x^3-8y^3}{x^2-xy}\times\frac{(x-y)^2+xy}{(x+2y)^2-2xy}\div\frac{x^3+y^3}{x^3-xy^2}$.

6 Solve (i) $\frac{5}{x}-\frac{4}{x+2}=\frac{3}{x+8}$; (ii) $\left.\begin{array}{l}\frac{x-3y}{2}-\frac{y-3x}{2}+8=0,\\ x-2y=\frac{1}{2}\left(x-\frac{11}{3}y\right)\end{array}\right\}$

7. A man buys oranges at 4 annas a dozen He finds 50 of them spoiled and selling the remainder at 9 for 4 annas made a profit of Rs 6 4 as How many oranges did he buy ?

1911.

1 (a) Multiply together —

$x^2-2x+2,\ x^2+2x+2,\ x^4-2x^3-4x+4$ and $x^4-2x^3+2x^2-4x+4$

(b) Find the L C M of

$8a^3-27b^3,\ 3a^2-ab-2b^2$ and $6a^2-5ab-6b^2$

2 (a) Find the square root of $x^6+\frac{1}{x^6}-4x^4+4\left(x^2-\frac{1}{x^2}\right)+2$

(b) Simplify $\frac{(a+b)^2-ab}{(b-c)(c-a)}+\frac{(b+c)^2-bc}{(c-a)(a-b)}+\frac{(c+a)^2-ca}{(a-b)(a-c)}$

3 Solve the equations

(a) $\frac{x+3}{x+2}-\frac{x+4}{x+3}=\frac{x+5}{x+4}-\frac{x+6}{x+5}$. (See page 208, Ex 3)

(b) $\frac{x+y}{8}+\frac{2x+3y}{19}=2$, and $3x-4y=\frac{9x-4y-3}{10}$.

(c) $8x^2-6x=35$

4 Two boys run a mile race, one of them runs 88 ft more per minute than the other and in consequence reaches the goal 3 minutes earlier At what rates do they run in miles per hour ?

5 Draw the graphs represented by $x-3y=2$, $2x-5y=5$ and $x+1=6y$ Show that the straight lines represented by them meet at a common point and find its co-ordinates [Unit = ½ inch]

1912.

1 (a) How can you determine by inspection whether two straight lines, the equations of which are given, are parallel or not ?

(b) Plot the points A (5, 7), B (7, 10), C (2, 2) and D (−1, −4), and find the equations of the straight lines AB and CD Find the point of their intersection [Unit = ½ inch]

2 Factorise —(a) $x^{12}+x^6-2$ (b) x^8-16y^8

(c) $a(a+1)x^2+x-a(a-1)$ (d) $(a^2-9ab)^2-324b^4$.

3 (a) If $x+y=a$ and $xy=b$, express x^4+y^4 in terms of a and b

(See page 131, Ex 14)

(b) If $x+y+z=6$ and $xy+yz+zx=9$,

prove that $\frac{1}{1-x}+\frac{1}{1-y}+\frac{1}{1-z}=0$

4 (a) Simplify $\dfrac{\frac{a^3}{b^3}-\frac{b^3}{a^3}}{\left(\frac{a}{b}-\frac{b}{a}\right)\left(\frac{a}{b}+\frac{b}{a}-2\right)}\times\dfrac{\frac{1}{b}-\frac{1}{a}}{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{ab}}$

(See page 181, Ex. 21)

(b) Find the square root of $\frac{(a^2+b^2)^2}{a^4+b^4-2a^2b^2}+4\frac{a}{a+b}\times\frac{b}{a-b}$

(See page 328, Ex 3)

5 Solve the following equations —

(a) $\frac{24}{x-12}-\frac{15}{x-3}=\frac{9}{x-7}$, (b) $\frac{10}{x}-\frac{19}{y}+65=0$, and $\frac{3}{5y}+\frac{4}{3x}=7$,

(c) $\frac{2x}{x-1}+\frac{3x-1}{x+2}-\frac{5x-11}{x-2}=0$

6 The difference in the speed of two trains is 15 miles per hour The slower of the two takes two hours more time to travel a distance of 150 miles than the faster takes to travel 160 miles Find their speeds in miles per hour

1913.

1 (a) Resolve into elementary factors —

(i) $\frac{c^2}{4}-a^2-\frac{bc}{2}+ab$, (ii) $10x^2-11xy-35y^2$

(b) Find the H C F and L C M of

$3ab-6b^2$ $12a^3-48ab^2$, and $6a^3-6a^2b-12ab^2$

2 Simplify —

(a) $\frac{3x^2-6x}{x^2-4}\times\frac{6x+12}{27x^3}$, (b) $\frac{x+y+z}{(x+z)^2-y^2}\times\frac{z^2-(x-y)}{xy-y^2-yz}$

3 Solve the following equations —

(i) $(4x-1)(2x+3)-(10x+3)(5-x)=2(3x-7)^2+25$,

(ii) $4y-11=2x=6y-18$, and (iii) $\frac{x}{5}-\frac{2}{x}=\frac{9}{5}$

4 A says to B, 'I am twice old as you were when I was old as you are The sum of their present ages is 63 Find their ages

5 Draw the graphs of $2x+y=5$, $x+2y=4$, from $x=-1$ to $x=4$ and measure the co-ordinates of their point of intersection [unit $=\frac{1}{2}$ inch]

6 When are four quantities a, b, c, d said to be in proportion? What number must be subtracted from each of the numbers 6, 8, 7, 11 so that the remainders may be in proportion?

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1904.

1 Solve the following equations:

(i) $\frac{x-4}{2}+\frac{x-9}{3}+\frac{x-16}{4}+\frac{x-25}{5}=63$,

(ii) $\left.\begin{array}{l} x+y+z=6 \\ x+1=y+2=z+3 \end{array}\right\}$

2 Simplify $\frac{x+3}{x^2-3x+2}+\frac{x+2}{x^2-4x+3}+\frac{x+1}{x^2-5x+6}$

3 If $x=b+c$, $y=c+a$, and $z=a+b$, express $x^3+y^3+z^3-3xyz$ in terms of a, b and c (See page 385, Ex 5)

4 (i) Show that the product of the H C F and the L C M of two algebraical expressions is equal to the product of two expressions.

(ii) Prove that the product of any five consecutive integers is divisible by 120

5 What number must be added to the numerator and denominator of $\frac{3}{7}$ to make it equal to [illegible]?

1905.

1 (i) Multiply $x^{-\frac{1}{3}}y^{\frac{1}{3}}+2-3x^{\frac{1}{3}}y^{-\frac{1}{3}}$ by $y^{-\frac{1}{3}}x+yx^{-1}$;

(ii) Divide $x^{\frac{4}{3}}y+2x^{\frac{2}{3}}y^{\frac{1}{2}}+9$ by $x^{\frac{2}{3}}+2x^{\frac{1}{3}}y^{-\frac{1}{4}}+3y^{-\frac{1}{2}}$

2 Simplify (i) $\frac{a\left(\frac{1}{b}-\frac{1}{c}\right)+\text{two similar terms}}{a\left(\frac{1}{b}-\frac{1}{c}\right)+\text{two similar terms}}$,

(See page 193, Ex. 52)

(ii) $\frac{3x-8}{x^3-1}-\frac{5x+7}{x^2+x+1}+\frac{2}{x+1}$

3 Solve—(i) $24x+20\{3(3x+1)-2x\}-7\{6(6x+1)-16x\}=6$,

(ii) $\frac{66x}{13}+\frac{30y}{91}=1$ and $\frac{6y}{7}-330x+13=0$

4 (i) If $\frac{a}{b}=\frac{c}{d}$, prove that $\frac{a+b}{a-b}=\frac{c+d}{c-d}$

(ii) If $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}$, prove that $a(a+d)=(a+b)(a-b+c)$.

5 (1) Factorise (i) $16x^4+4x^2+25$,

(ii) $(y-z)^3+(z-x)^3+(x-y)^3$

(2) Prove that

$$\left(\frac{y}{z}+\frac{z}{y}\right)^2+\left(\frac{z}{x}+\frac{x}{z}\right)^2+\left(\frac{x}{y}+\frac{y}{x}\right)^2=4+\left(\frac{y}{z}+\frac{z}{y}\right)\left(\frac{z}{x}+\frac{x}{z}\right)\left(\frac{x}{y}+\frac{y}{x}\right).$$

(See page 184, Ex 6)

1906.

1 (1) Find the value of $3x^5+5x^4+5x^3+7x^2+7x+9$,

(i) when $x=10$, (ii) when $x=\frac{1}{10}$

(2) Simplify $\dfrac{2}{x^2+3-\dfrac{1}{\dfrac{x^2+2}{x^2+1}}}-\dfrac{1}{x^2+2+\dfrac{1}{x^2+2}}$

2 Factorise (i) $x^2-xy-2y^2+x+y$,

(ii) $(x-1)^3+(x-2)^3+(3-2x)^3$,

(iii) $(x^2-yz)^2-(y^2-zx)(z^2-xy)$

3 Prove, algebraically, that if three consecutive numbers be taken, the difference of the squares of the first and the last of them is equal to four times the middle number

4 Solve (i) $\sqrt{(x-a^2)}+\sqrt{(x-b^2)}=b-a$,

(ii) $y=m_1x+\frac{a}{m_1}$, $y=m_2x+\frac{a}{m_2}$.

5 The sum of *Rs* 116 14¼ *as* was collected from 40 persons some of whom gave *Rs* 2 7½ *as* each, and the rest *Rs* 3 2¼ *as* each; how many gave *Rs* 2 7½ *as* each?

1907.

1 Solve $\frac{33}{x}+6y=75$, and $y-\frac{11}{x}-1=0$

2 Divide x^6-y^6 by $x-y$, and put the quotient in factors

3 A man who went out between 5 and 6 and returned between 6 and 7 found that the hands of his watch had exactly changed places When did he go out?

4 Extract the square root of $(2x+1)(2x+3)(2x+5)(2x+7)+16$.

5 Simplify $\frac{yz}{(x-y)(x-z)}+\frac{zx}{(y-z)(y-x)}+\frac{xy}{(z-x)(z-y)}$.

(See Page 188)

6 Show that a ratio of greater inequality is diminished, and a ratio of less inequality is increased, by adding the same quantity to both its terms (Art 2, Page 335)

1908.

1 (a) Prove that the L C M of two given expressions may be found by dividing their product by their H C F

(b) Find the H C F and L C M. of $x^3+6x^2+11x+6$ and x^3+2x^2-x-2

2 Extract the square root of

(i) $\frac{7}{2}-\frac{3}{2}\sqrt{5}$, and (ii) $x^2+\frac{1}{x^2}-4\left(x+\frac{1}{x}\right)+6$

3 Factorise (i) $12x^2+13x-14$,

(ii) $4(ab+cd)^2-(a^2+b^2-c^2-d^2)^2$.

(See Page 94, Ex. 23)

4 Simplify

$$\frac{(a-b)^3+(b-c)^3}{a-c}+\frac{(b-c)^3+(c-a)^3}{b-a}+\frac{(c-a)^3+(a-b)^3}{c-b}.$$

5. (i) If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, show that each ratio $=\frac{al+cm+en}{bl+dm+fn}$.

(ii) Divide £1230 among three persons so that if their shares be diminished by £5, £10, £15 respectively, the remainders shall be in the ratio 3 4 5

6 A man walks a certain distance at a certain rate Had he gone two miles an hour faster, he would have walked it in $\frac{4}{5}$ths of the time, if he had gone $\frac{1}{2}$ a mile slower, he would have taken $2\frac{1}{2}$ hours longer Find the distance

1909

1 A room is a ft long, b ft broad, and c ft high How many yards of paper d ft wide will be required to cover the walls ? Find also the cost in rupees of this paper at the rate of n annas per piece of 12 yards

2 (i) If $X=3a(x-1)^2-a(x-1)-4$, $Y=16+b(x-1)-3b(x-1)^2$, find the value of $bX+aY$ in its simplest form.

(ii) Factorise (a) $125x^5y^2-27x^2y^5$;

(b) $(y-z)^3+(z-x)^3+(x-y)^3$.

(See page 87, Ex 3.)

3 (i) Add together $\frac{1}{x-2y}$, $\frac{1}{x+2y}$, $\frac{2x}{x^2+4y^2}$ and $\frac{4x^3}{x^4+16y^4}$.

(ii) Simplify $\frac{ax}{(a-b)(a-c)}+\frac{bx}{(b-c)(b-a)}+\frac{cx}{(c-a)(c-b)}$.

4 Solve (i) $\frac{3}{2}\left(x-\frac{1}{2}\right)-\frac{2x}{5}=\frac{1}{3}\left(\frac{4}{5}+\frac{x}{4}\right)$,

(ii) $\frac{a}{x}-\frac{b}{y}=3a-2b$, and $\frac{a+b}{x}+\frac{a-b}{y}=5a+b$

5 A man has Rs 15,000 invested partly at $3\frac{1}{2}$ per cent and partly at 5 per cent His total income is Rs 672 15a Find the amount of his investments.

6 (i) If $\frac{x}{b-c}=\frac{y}{c-a}=\frac{z}{a-b}$, show that $x+y+z=0$

(See page 348 Ex. 3).

(2) Prove that a ratio is made more nearly equal to unity by adding the same positive quantity to each of its terms

1910.

1 Solve the following equations

(i) $(x-a)^3+(x-b)^3+(x-c)^3=3(x-a)(x-b)(x-c)$,

(See Page 198, Ex 40)

(ii) $\frac{1}{y}-\frac{1}{x}=\frac{1}{z}-\frac{1}{y}=\frac{2}{x}-\frac{1}{y}=1$

2 Resolve into elementary factors

(i) $81x^4+64y^4$,

(ii) $(y+z)^3+(z+x)^3+(x+y)^3-3(y+z)(z+x)(x+y)$

3 Simplify $\frac{bc(x-a)^2}{(a-b)(a-c)}+\frac{ca(x-b)^2}{(b-a)(b-c)}+\frac{ab(x-c)^2}{(c-a)(c-b)}$

(See page 193, Ex. 56)

4 There is a number of two digits which is three times the sum of its digits, if 45 be added to the number, the digits change their places Find the number

5 (i) If $a : b :: c : d$, show that $a+b : a-b :: c+d : c-d$

(ii) If $(a+b+c+d)(a-b-c+d)=(a-b+c-d)(a+b-c-d)$, prove that a, b, c, d are proportionals

(See page 344, Ex 4.)

1911.

1 (i) Reduce to their simplest terms $\frac{x+y}{1-xy}$ and $\frac{x-y}{1+xy}$,

where $x=\frac{a+b}{1-ab}$ and $y=\frac{a-b}{1+ab}$.

(ii) Simplify $\frac{1}{x^2-5x+6}+\frac{1}{x^2-4x+3}-\frac{1}{x^2-3x+2}$.

2 (1) Factorise (i) $p^2q^2-15pq+44$, (ii) $729x^6-8y^6$.

(2) Express in factors the square root of

$$(x^2+8x+7)(2x^2-x-3)(2x^2+11x-21)$$

3 Solve the following equations:

(i) $(x+\frac{1}{3})^2-(x-\frac{2}{3})^2=2x+3$, and (ii) $\frac{x+3}{5}=\frac{8-y}{4}=\frac{3(x+y)}{6}$.

4 Simplify $\frac{a^2}{(a-b)(a-c)(x-a)}+$ two similar expressions

(See page 192, Ex. 50.)

5 A school is divided into a certain number of divisions. If there were 36 more boys in the school, the average number of boys in a division would be increased by 3, and if the school were divided into 3 less divisions the number of boys in a divison would be twice the number of divisions How many boys are there in the school?

1912

1 (i) If $a=371$, $b=064$, and $c=435$, find the value of

$$a^3+b^3+3abc-c^3.$$

(ii) Reduce to its lowest terms,

$$\frac{7x^3-2x^2y-63xy^2+18y^3}{5x^4-3x^3y-43x^2y^2+27xy^3-18y^4}$$

2 (1) Factorise (i) a^6-64, and

(ii) $2xyz+x^2(y+z)+y^2(z+x)+z^2(x+y)$.

(See page 94, Ex. 32.)

(2) Extract the square root of $9x^4-18x^3+9x+\frac{9}{4}$

3 Solve (i) $\frac{x}{3x-\frac{1}{3}}=\frac{15x+\frac{1}{4}}{45x-5}$,

(ii) $\frac{1}{x}+\frac{3}{y}=2\,15$ and $\frac{5}{x}-\frac{4}{y}=3\,31+\frac{2}{x}$.

4 (1) Prove the following identity:—

$$\frac{(a^2-b^2)^3+(b^2-c^2)^3+(c^2-a^2)^3}{(a-b)^3+(b-c)^3+(c-a)^3}=(a+b)(b+c)(c+a)$$

(See page 87, Ex. 3.)

Or,

(2) Eliminate x from the equations —

$$ax^2+b=0 \text{ and } lx^2+mx+n=0$$

5 A party of traveller coming to a hotel find that there are too few rooms for each to have one If they sleep two in a room, there are 2 empty rooms How many rooms are left empty if they sleep four in a room ?

1913.

1 (i) Add the product of $(a+b+1)$ and $(a-b+2)$ to the product of $(a+b-1)$ and $(a-b+2)$, and verify the result when $a=4$, $b=1$

(ii) Divide $x^5+x^4-2x^3-2x^2+x+1$ by $(x+1)^3$

2 Solve *graphically* $2x+3y=6$ and $3x-\frac{5y}{2}=2$

3 Factorise —(i) $(a^2+b^2+c^2)^2-4a^2b^2$, (See P 89, Ex 1)

(ii) $x^3(y-z)+y^3(z-x)+z^3(x-y)$

(See page 125, Ex 9)

4 A number of three digits is such that if the order of the digits is reversed the number is diminished by 99, also the sum of the digits is 14, and the middle digit is equal to the sum of the other two Find the number

5 (i) If $a:b=c:d=e:f$, show that each of these ratios $=la+mc+ne:lb+md+nf$

(ii) If $2a+3c:3a-4c=2b+3d:3b-4d$, prove that a, b, c, d, are proportionals

1914.

1 (i) Divide $4x^4+1$ by $2x^2+2x+1$, and by choosing suitable values of x find two factors of 40001

(ii) If $x+y+1=0$ and $p^2x+q^2y+r(p+q)=pq$, find the value of x^2-y^2 in terms of p, q, and r

2 (i) Resolve $4x^3-3x+1$ into elementary factors

(ii) Prove that the product of any four consecutive odd numbers increased by 16 is a perfect square

3 (i) Find x from the equation —

$$\frac{3\frac{1}{2}}{x}+\frac{5}{x}=\frac{2{\cdot}2-\frac{1}{x}}{25}.$$

(ii) Solve *graphically* the equations $3x+y-14=x-2y=0$.

4 Divide the number 77 into three parts such that the sum of the first and second multiplied by 3, the sum of the second and third diminished by 3, and the sum of the first and third increased by 3, may be all equal (*Verify your answer.*)

5 (i) If $a : b = c : d$, show that $ab : cd = a^2-b^2 : c^2-d^2$.

(ii) Eliminate t from the equations — $x = at^2$, $y = 2at$.

1915.

1. (i) Find the co efficient of x^4 in the expansion of

$$(x+3)^3(x+4)^2(x+5)$$

(ii) Prove that a number is divisible by 9, when the sum of its digits is divisible by 9.

2. (i) Resolve $18x^3-27x^2-35x$ into elementary factors.

(ii) Show that $(x+y+z)^4-(x+y)^4-(y+z)^4-(z+x)^4+x^4+y^4+z^4$ contains each of the factors x, y, z.

3 (i) Find x from the equation

$$\frac{3}{11}\left(4x-8\right)-\frac{2}{3}\left(5x-14\right)=\frac{4}{11}\left(2x-4\right)-\frac{6}{7}\left(3x-4\right).$$

(ii) Solve *graphically* the equations $3x+5y=13$, $5x-3y+1=0$.

4 (i) If a, b, c, d be in continued proportion, prove that $\sqrt{(a+b+c)(b+c+d)}=\sqrt{ab}+\sqrt{bc}+\sqrt{cd}$ (See page 352, Ex 10.)

(ii) Eliminate x from the equations $x+y=a$, $x^2+y^2=b^2$.

1916.

1 (i) Divide $\frac{1}{2}x^4+\frac{2}{3}-\frac{54}{35}x-\frac{7}{3}x^3+\frac{3}{10}x^2$ by $\frac{x}{2}-\frac{7}{3}$.

(ii) Show that if a two-digited number is 8 times the sum of its digits, the number formed by reversing the digits is thrice their sum.

2 (i) Find the H. C. F. of $3x^3+10x^2+7x-2$ and $3x^3+13x^2+17x+6$.

(ii) Show that $a^n(b-c)+b^n(c-a)+c^n(a-b)$ contains each of the factors $a-b$, $b-c$, $c-a$.

3 (i) Find x, y from the equations $\frac{5}{x}+3y=8$, $\frac{4}{x}-10y=56$.

(ii) Solve *graphically* the equations $2x-3y=5$, $3x-4y=6$.

4 A number consists of three digits, that to the left being 4. When the 4 is changed into 9 and 3 subtracted from the number, the result is twice the original number. Find the original number, if the sum of its digits be 20. (Verify your answer.)

5 (i) If a, b, c are in continued proportion, and if $a(b-c)=2b$, prove that $a-c=\dfrac{2(a+b)}{a}$

(ii) Eliminate x from the equations $ax+b=0$, $px^2+q=0$

1917.

✓ 1 (i) Simplify $\dfrac{6^n\times 2^{2n}\times 3^{3n}}{30^n\times 3^{2n}\times 2^{3n}}$ and find its numerical value when $n=\frac{1}{2}$ correctly to two places of decimals

(ii) Find, without unnecessary calculations, the co-efficient of x^3 in the product of $5x^3+2x^2-7x-8$ and $2x^3-3x^2-10x+4$

2 (i) Simplify $\dfrac{a(b^2-c^2)}{bc}+\dfrac{2b(c^2-a^2)}{ac}-\dfrac{c(2b^2-a^2)}{ab}$.

(ii) Resolve into factors

(a) $a^2-b^2+c^2+2ac$, (b) $(a+b+c)(bc+ca+ab)-abc$

✓ 3 (i) Find p and q from the equations

$$p+3q=4p+q=2\tfrac{1}{2}$$

✓ (ii) Draw the graphs of the three equations

$3x-4y=12$, $4x+3y=41$, and $x+y=11$,

and find the value of x and y which satisfy all of them.

✓ 4 Some pages of a book have 30 lines each and the remaining pages have 25 lines each The total number of pages in the book is 36, and the total number of lines is 1055 Find how many pages have 25 lines each

5 If $a : b = b : c$ then show that

$$\frac{a^2+b^2}{a+b}-\frac{b^2+c^2}{b+c}=\frac{(a-b)^2+(b-c)^2}{a-c}.$$

ALLAHABAD UNIVERSITY MATRICULATION PAPERS

1905.

1 (a) Find the H C F of x^4-5x^2+4 and $x^5-11x+10$

(b) Extract the square root of $a^2+(1+a^2)(1+a)^2$,

2 Simplify $\dfrac{(b+c)(x^2+a^2)}{(a-b)(a-c)}+\dfrac{(c+a)(x^2+b^2)}{(b-a)(b-c)}+\dfrac{(a+b)(x^2+c^2)}{(c-a)(c-b)}$.

3. Solve (i) $\dfrac{1}{x+2}+\dfrac{1}{x+10}=\dfrac{1}{x+1}+\dfrac{1}{x+8}$;

(ii) $\dfrac{x+3}{5}=\dfrac{8-y}{4}=\dfrac{3(x+y)}{8}$.

4 If a, b, c, d are in continued proportion, prove that

(i) $a : d = a^3+b^3+c^3 : b^3+c^3+d^3$; (See Page 354, Ex 38)

(ii) $a^2 d = b^3$.

5 Coffee is bought at 1*s* and chicory at 3*d.* per lb in what proportion must they be mixed, that 10 per cent may be gained by selling the mixture at 11*d* per lb ?

1906.

1 Divide $2x^3y - 7xy^3 + x^4 - 4y^4$ by $x^2 - xy - y^2$.

2 If the product of three successive odd or even integers be increased by four times the middle number, prove that the sum will be a perfect cube

3 Factorise (i) $x^4 - 64$ and (ii) $x^3(y-z)+y^3(z-x)+z^3(x-y)$

(See Page 125, Ex. 9)

4 Simplify $\dfrac{1}{(1-a)(1-b)}+\dfrac{a^2}{(1-a)(b-a)}-\dfrac{b^2}{(b-1)(a-b)}$.

5 Solve (i) $\dfrac{x-2a}{b+c-a}+\dfrac{x-2b}{c+a-b}+\dfrac{x}{a+b+c}=3$,

(ii) $(x-1)^2+(x-1)^2+(x-8)^2 = 3(x-2)^2$.

6 If 11 be taken from a certain integer, we get the square of a whole number, and if 24 be added to the same integer, we get the square of the next greater number Find the integer

1907.

1 Show that $(a^2+b^2)(c^2+d^2) = (ac+bd)^2+(ad-bc)^2$.

(See Page 93, Ex 9)

2. Simplify $\dfrac{a^2}{(a-b)(a-c)}+\dfrac{b^2}{(b-c)(b-a)}+\dfrac{c^2}{(c-a)(c-b)}$.

(See Page 192, Ex 43)

3 Resolve into factors x^6-y^6.

4 Solve $\dfrac{x-1}{2}-\dfrac{x-3}{4}+\dfrac{x-5}{6}=4$

5 Solve $\dfrac{x+y}{8}+\dfrac{x-y}{6}=5$ and $\dfrac{x+y}{4}-\dfrac{x-y}{3}=10$

6 A can do half as much work as B , B can do half as much as C, and together they can complete a piece of work in 42 days in what time could each alone complete the work ?

7 If $a : b = c : d$, prove that $ab+cd$ is a mean proportional between a^2+c^2 and b^2+d^2. (See Page 351, Ex. 9).

1908.

1 Simplify $(a+b+c)^2-2(b+c)(a+b+c)+(b+c)^2$

(See Page 68, Ex. 4).

2 Find the G C M of

$8x^3+38x^2+59x+30$ and $6x^3-13x^2-13x+30$

3 Solve the equation $13x^2-90x-7=0$

4 Solve $x+y=5$ and $x+3y=13$

5 If $\frac{a}{b}$ be a proper fraction and x a positive number, prove that

$\frac{a+x}{b+x}$ is greater than $\frac{a}{b}$ (See Page 335, Art, 2)

6 Break into factors $x^2-2x-15$

7 Trace the graph of $y=2x+1$

1909.

1 Find the G C M of

$21x^3-28x^2-46x-7$ and $21x^2-58x+21$

2 The sum of two numbers is 4225 and their G C M is 845. Show that there are two pairs of numbers satisfying these conditions and find them

3 Simplify

$$\left(1+\frac{2a}{\frac{z^2}{a}-a}\right)\left(1+\frac{2a}{\frac{z^2}{a}+a}\right)\left(1-\frac{a}{z-2a}\right)\left(1-\frac{a}{z+2a}\right)$$

4 Solve the equations

(i) $\frac{7-3x}{5}=\frac{14-5x}{3}+\frac{25-4x}{9}$.

(ii) $x^3=7xa^2-6a^3$

5 An ordinary train, the average speed of which is 20 miles an hour less than that of the express, takes two hours longer than the express to go 150 miles What is the average speed of each train ?

6 To what number must 1, 5 and 13 be severally added so that they may be a continued proportion ?

7 Obtain by means of a graph the square root of 5

1910.

1 (a) Simplify $\frac{bc(x-a)^2}{(a-b)(a-c)}+\frac{ca(x-b)^2}{(b-c)(b-a)}+\frac{ab(x-c)^2}{(c-a)(c-b)}$.

(See Page 193, Ex. 56

(b) Prove that $x(y+z)^2+y(z+x)^2+z(x+y)^2-4xyz$

$=(y+z)(z+x)(x+y)$

2 Split into factors —

(i) x^6-y^6, (ii) $x^4+x^2y^2+y^4$, (iii) $x^3-6x^2+11x-6$

3 Solve —(i) $\frac{x-1}{x-2}-\frac{x-2}{x-3}=\frac{x-5}{x-6}+\frac{x-6}{x-7}$,

(ii) $\frac{x+y}{2}-\frac{x-y}{3}=8$, and $\frac{x+y}{3}+\frac{x-y}{4}=11$;

(iii) $(3x-2)(x-1)=14$

4 The product of two numbers is 750, and the quotient when one is divided by the other is $3\frac{1}{3}$, find the numbers

5 If $\frac{a}{b}=\frac{c}{d}$, prove that

(i) $\frac{pa+qb}{pa-qb}=\frac{qc+qd}{pc-qd}$,

(ii) $a^2+b^2 \quad \frac{a^2}{a+b}=c^2+d^2 \quad \frac{c^2}{c+d}$. (See Page 350, Ex 7)

6 Draw the graphs of $5x+7y=35$, and $xy=120$

1911.

1 (a) Simplify $\frac{2(x-3)}{(x-4)(x-5)}-\frac{x-1}{(x-3)(x-4)}-\frac{x-2}{(x-5)(x-3)}$.

(b) If $x+\frac{1}{y}=1$, $y-\frac{1}{z}=1$, prove that $xyz=1$

2 Split into factors

(i) $x^2-\left(a-\frac{1}{a}\right)x-1$, (ii) x^3-3x^4+3x-1, (iii) $x^4-11x^2y^2+y^4$.

3 Solve —(i) $\frac{1}{7}(x-\frac{1}{2})+\frac{1}{6}(x+\frac{1}{3})=\frac{43}{40}$;

(ii) $\frac{5x+6y-7}{2}=\frac{2x+5y+3}{3}=\frac{8-4x+3y}{2}$;

(iii) $x=\cfrac{3}{4-\cfrac{3}{4-x}}$.

4 A, B and C do a piece of work in a certain time A could have done it alone in 6 hours more, B in 15 hours more, and C in twice the time How long did they take working together ?

5 If $x=2^{\frac{1}{3}}-2^{-\frac{1}{3}}$, prove that $2x^3+6x=3$

6 What is a graph ? Explain this clearly

Find four points on each of the graphs of $5x+4y=10$, $3x+2y=6$, and *thence* solve the equations (Unit = 1 inch)

1912.

1 Simplify —(i) $\frac{1}{(x+1)^2(x+2)^2}-\frac{1}{(x+1)^2}+\frac{2}{x+1}-\frac{2}{x+2}$;

(ii) $\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{x}}}}$

2 (a) Factorise —(i) $x^2+2ax+a^2-b^2$, (ii) x^3+4x^2+x-6;
(iii) x^4+4y^4

(b) Simplify $\frac{x^4-1}{x^4+x^2+1}\times\frac{x^3-x}{(x^4-1)^2}\div\frac{x^2+1}{x^6-1}$.

3 Solve —(i) $\frac{lx}{l-x}-\frac{mx}{m-x}=m-l$, (ii) $\frac{x}{x-10}=\frac{2x}{x-5}$,
(iii) $x-y=1$, and $x^2-y^2=5$

4 (i) Multiply $x^{\frac{2}{3}}-x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}}$ by $x^{\frac{1}{3}}+y^{\frac{1}{3}}$ and (ii) find the value of x^3-6x when $x=2^{\frac{2}{3}}+2^{\frac{1}{3}}$

5 (i) If a, b, c are in continued proportion, prove that $a^2+ab \quad b^2 = b^2+bc \quad c^2$

(ii) Find what number must be subtracted from 11, 14 and 20 in order that the remainders may be in continued proportion

6 Find the square root of $x^4+9y^4+4x^3y-12xy^3-2x^2y^2$

7 What is a graph? Are all graphs straight lines? Solve by means of graphs the simultaneous equations $3x = 12$ and $x+3y=11$.

1913.

1 If ax^3+bx^2+cx+d be divided by $x-p$ until the remainder is independent of x, find the remainder without actually performing the division

Use this theorem to prove that $x^6+ax^5+cx^3+dx^2-1$ is divisible by $x+1$ if $a+c = d$

2 (a) Factorise —$(a^2+b^2)c+(a+b)(ab+c^2)+3abc$,

(b) Simplify $\frac{1}{a^2+2bc-b^2-c^2}+\frac{1}{b^2+2ca-c^2-a^2}+\frac{1}{c^2+2ab-a^2-b^2}$.

3 Solve (1) $\frac{1}{x-1}+\frac{1}{y-2}=3$ and $\frac{2}{x-1}+\frac{3}{y-2}=5$,

(ii) $x+2y=5$ and $x^2+4y^2=2x(1-y)+y(1-2x)$;

(iii) $\frac{3x^2+x+1}{3x-1}=\frac{2x^2-x+1}{2x-1}$.

4 The area of the floor of a rectangular room is 40 square yards, if the length and breadth of the room were each increased by 3 yards, the area of the floor would be 88 square yards Find the length and breadth of the room

5 A cyclist starts travelling at the rate of 12 miles an hour, and after one hour's riding stops for half an hour A motorist starts from the same place two hours after the cyclist and travels at the rate of 28 miles an hour. Find by graphical methods when and where the motorist passes the cyclist

1914.

1 Factorise —(i) $(x-y)^3+(y-z)^3+(z-x)^3$

(See page 87, Ex. 3)

$$\text{(ii)}\quad 1-\left\{\frac{a^2+b^2-c^2-d^2}{2(ab+cd)}\right\}^2.$$

(See page 94, Ex. 23)

2 Simplify — $\dfrac{a^3}{(a-b)(a-c)}+\dfrac{b^3}{(b-a)(b-c)}+\dfrac{c^3}{(c-a)(c-b)}.$

(See page 187, Ex. 12)

3 Solve —(i) $\dfrac{x^2-4x+5}{x^2+6x+10}-\left(\dfrac{x-2}{x+3}\right)^2=0$,

(ii) $\dfrac{3}{x}-y=12$, and $\dfrac{1}{x}-3y=21$;

(iii) $12x^2+160x-27=0$

4 Two men, A and B run a race A runs at the uniform rate of 22 ft. per second, and B at the uniform rate of 20 ft per second If B has 20 yards' start, and A wins by 3 seconds, find the length of the race

5 Find *graphically* at what time between 2 and 3 o'clock the two hands of a watch are together

1915.

1 Find the factors of

(i) $6x^2-13x+6$, and (ii) $a^2+2ab-2ac-3b^2+2bc$

2 Simplify $\dfrac{x-y}{(a+x)(a+y)}+\dfrac{y-z}{(a+y)(a+z)}+\dfrac{z-x}{(a+z)(a+x)}.$

3 If $a+b+c=0$, show that $a^3+b^3+c^3-3abc=0$

(See page 86, Art 13)

4 Solve —(i) $\dfrac{4x-7}{2x-5}+\dfrac{9x+7}{3x+1}=5$;

(ii) $\dfrac{1}{x}+\dfrac{1}{y}=3$, $y+z=5yz$, $z+x=4xz$

(See page 257, Ex. 2).

5 Give the graphical solution of the equation $2x^2-x-2=0$.

6 A father is 30 years older than his son, and one year ago he was four times as old as his son Find their present ages

1916.

1 Split into the simplest factors —

(i) $10a^4b^2-46a^2b^3-20b^4$,

(ii) $a^2x^2-b^2x^2+4b^2+4a^2+4a^2x$,

(iii) $a^3+b^3+c^3-3abc$

2 Solve —(i) $\frac{x+10}{5}-4\frac{3}{4}-\frac{x}{4}=\frac{x-2}{3}-(x-1)$,

(ii) $\frac{4x-2}{3}-\frac{4y-5x}{2}=15(x+y)$, $3x+1+\frac{2x-y}{2}=\frac{x-2y}{8}$;

(iii) $(3a^2+b^2)(x^2-x+1)=(3b^2+a^2)(x^2+x+1)$

3 Five persons A, B, C, D, E, play at cards After A has won half of B's money, B one-third of C's, C one-fourth of D's, D one sixth of E's, they have each 30 shillings Find how much each had to begin with.

4 If x and y be unequal, and x have to y the duplicate ratio of $x+z$ to $y+z$, prove that z is a mean proportional between x and y

5 If $a^b=b^a$, show that $\left(\frac{a}{b}\right)^{\frac{a}{b}}=a^{\frac{a}{b}-1}$

and if $a=2b$, prove that $b=2$

6 Find a number such that its square added to its cube is 16 times the next number

7 Taking suitable units, draw the graphs of the equations $xy=12$ and $y=x-4$, and from your figure find the values of x and y which satisfy both equations

Verify your result by solving the equations algebraically.

1917.

1 Factorize —

(i) $(x^2-y^2)(a^2-b^2)+4abxy$.

(ii) $x^3+y^3-3xy+1$

(iii) $x^2+12yz-4y^2-9z^2$

2 Simplify —

$$\frac{a}{bc(a-b)(a-c)}+\frac{b}{ac(b-a)(b-c)}+\frac{c}{ab(c-a)(c-b)}$$

3 Solve —

(i) $$\left.\begin{aligned} xy &= \tfrac{2}{7}(2y+3x) \\ yz &= \tfrac{3}{11}(4z+5y) \\ xz &= \tfrac{1}{9}(6x+7z) \end{aligned}\right\}$$

(ii) $$\frac{10}{x-5}-\frac{9}{x+5}=4\frac{1}{4}.$$

4 A man finds that if he walked half a mile per hour faster than his ordinary rate, he would take an hour less than usual in walking 28 miles Find his ordinary rate in miles per hour

5 The expenses of a school are partly constant and partly proportional to the number of boys The expenses were £650 for 105 boys, and £742 for 128 boys Draw a graph to represent the expenses for any number of boys, and find (1) the expenses for 115 boys, and (2) the number of boys that can be maintained at a cost of £710

ANSWERS TO UNIVERSITY PAPERS.

Cal. 1903.

1 (*b*) $4a^2+b^2+9c^2+2ab-3bc+6ca$ 2. 1 3 (*b*) 1.

4 (1) $x=\dfrac{2(a-b)(b-c)}{c-a}$, (2) $x=12, y=24, z=36$ 5 530.

Cal. 1904.

1 (1) $216y^3$, (2) 1 2 $\dfrac{3x^3+6x^2+4x+2}{2x^3+4x^2+6x+3}$.

3 $(x^2-5x+5)^2$ 4 (*b*) (1) $x=\dfrac{c^2+d^2-2acd}{c(a-b)+d(a+b)-(c+d)}$

(2) $x=b-c, y=c-a, z=a-b$ 5 At 10-4 A.M.

Cal. 1905.

1 (1) ∞ 2 $(a-b)(a-c)(b-c)$ 3. 0 4 (1) $x=ab+bc+ca$

(2) $x=1, y=2, z=3$ 5 1200

Cal. 1906.

1 $(x^2+2ax+2a^2)(x^2-2ax+2a^2)$, $(a+b)(b+c)(c+a)$

2 (2) $[c=b^y=a^{xy}=c^{xyz}$, &c] 3 (2) $[a^2-bc$
$=a(a+b+c)$, &c]

4 (1) $x=a^3+b^3+c^3$ $\left[\dfrac{x-a^3}{b^2-bc+c^2}-(b+c)=\dfrac{x-a^3-b^3-c^3}{b^2-bc+c^2}\right.$, &c $\Big]$

(2) $\dfrac{x}{b-c}=\dfrac{y}{c-a}=\dfrac{z}{a-b}=\dfrac{-1}{(b-c)(c-a)(a-b)}$ 5 7 miles.

6 (2) [Each ratio $=\dfrac{(p+q)(x+y+z)}{(b+c)+(c+a)+(a+b)}$

and also $=\dfrac{(p+q)(xy+yz+zx)}{(b+c)x+(c+a)y+(a+b)z}$; hence &c]

Cal 1907.

1 1 2 (1) (i) $(x-17)(x+19)$, (ii) $(b+c)(c+a)(a+b)$.

3 (1) x^2-5x+1, (2) $x-2-\dfrac{1}{x}$. 4 (1) $x=\dfrac{79}{29}$, (2) a, b, c.

5 11*s* 8*d*, 6 6 (2) [Taking each ratio $=k$, show that $k(a^2+b^2+c^2)=0$]

Cal 1908.

1 (1) $a+b+c$ [Putting $b+c=x$, $c+a=y$, $a+b=z$, the dividend becomes $azy+bxz+cyx+3xyz=yz(a+x)+zx(b+y)+xy(c+z)$]).
(2) $(a^2+b^2+c^2)(x^2+y^2+z^2)$ 2 (2) (See Ex 3, page 87.

3 -60. 4 (2) (i) $x=(a+b+c)^2$.

[We have $\left(\frac{x-a^2}{b+c}-a\right)+\left(\frac{x-b^2}{c+a}-b\right)+\left(\frac{x-c^2}{a+b}-c\right)=3(a+b+c)$;

whence, $\left\{\frac{x-a(a+b+c)}{b+c}-(a+b+c)\right\}+\left\{\frac{x-b(a+b+c)}{c+a}-(a+b+c)\right\}$

$+\left\{\frac{x-c(a+b+c)}{a+b}-(a+b+c)\right\}=0$]

(ii) $x=6, y=8, z=10$ 5 128 6 (2) [Putting each ratio $=k$, show that $a(b-c)=-k^2(y^2-z^2)$, $b(c-a)=-k^2(z^2-x^2)$ &c]

I [$a^2+b^2+c^2=2(c^2-ab)$, $a^3+b^3+c^3=3abc$; $a^5+b^5+c^5$ $=-5ab(a^3+2a^2b+2ab^2+b^3)=-5ab(3abc-c^3-2abc)=5abc(c^2-ab)$]

II See Example 11, Page 187

Cal. 1909.

1 $\frac{1}{2}\{(a-b)^2+(b-c)^2+(c-a)^2\}$, (1) $(a^2+b^2)(x^2+y^2)$;
(2) $(c+d+a-b)(c+d-a-b)$ 2 x^2-2x+3, 1

3 (2) $\frac{x}{y}-1-\frac{y}{x}$. 4 (1) $x=4$, (2) $x=\frac{1}{4}, y=\frac{1}{3}, z=\frac{1}{2}$

5 3 and $2\frac{2}{3}$ miles per hour.

Cal. 1910. Compulsory.

1 (1) $2a^2b^2+2b^2c^2+2c^2a^2-a^4-b^4-c^4$, p^3+3p (2) $(x+5)(x-4)$

2 (1) $x+3$, $(x+2)(x-2)(x+1)(x-1)$ 3 (1) $x=b$, $x=3, y=2$

Additional 1 (1) $x=3$ or -4 (2) the graph is a Parabola

2 (1) $2x^2+5-\frac{7}{x^2}$ 3 (1) [Let S denote the required sum, then

$$\left.\begin{aligned} S&=1+2+3+\ldots+(n-2)+(n-1)+n \\ \text{also}\quad S&=n+(n-1)+(n-2)+\ldots+3+2+1\end{aligned}\right\}$$

$\therefore 2S=(n+1)+(n+1)+(n+1)+\ldots$ to n terms $=n(n+1)$, &c]

(3) [If S denote the sum of n terms of the series $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+$&c

$$\left.\begin{aligned} \text{we have } S&=1+\tfrac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots+\frac{1}{2^{n-1}} \\ \therefore \tfrac{1}{2}S&=\tfrac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots+\frac{1}{2^{n-1}}+\frac{1}{2^n}\end{aligned}\right\}$$

Hence, $\frac{1}{2}S=1-\frac{1}{2^n}$, &c]

(4) [$a+c=2b$, and $ad=b^2$ Hence, $a(a+c-2b)=ad-b^2$, or, $a^2-2ab+b^2=ad-ac$, &c] 4 4

Cal. 1911. Compulsory.

1 (1) $1+x^4+x^8$, 69 (2) $(x+13)(x-11)$, $(a-b)(a+b)^2$
2 (1) x^2+8x-2, $(a+3b)(a-3b)(a+2b)(a+4b)$
3 (1) $x=20$, $x=-3, y=3, z=1$

Additional 1 (1) $x=-11$ or $9\frac{1}{7}$, (2) $x=4$ or $-2\frac{1}{4}$

2 (1) a^2+b^2, $\frac{x}{y}+3-\frac{y}{x}$ 3 (1) n^2 [The nth term $=1+(n-1)2=2n-1$, let S denote the required sum, then

$$\left.\begin{aligned} S &= 1+3+5+\ \ +(2n-3)+(2n-1) \\ \text{also}\quad S &= (2n-1)+(2n-3)+(2n-5)+\ \ +3+1 \end{aligned}\right\}.$$

$2S=2n+2n+2n+$ to n terms, $=2n^2$]

(2) 19 (3) $\frac{1}{6}$ [If S denote the required sum we have

$$S=\frac{1}{10}+\frac{6}{10^2}+\frac{6}{10^3}+\frac{6}{10^4}+\ \ +\frac{6}{10^n}$$

$$\therefore\ \frac{1}{10}S=\qquad \frac{1}{10^2}+\frac{6}{10^3}+\frac{6}{10^4}+\ \ +\frac{6}{10^n}+\frac{6}{10^{n+1}}.$$

Hence, $\frac{9}{10}S=\frac{1}{10}+\frac{5}{10^2}-\frac{6}{10^{n+1}}$, &c Clearly therefore, if S' be the sum of an infinite number of terms of this series, we must have $\frac{9}{10}S'=\frac{1}{10}+\frac{5}{10^2}=\frac{15}{10^2}$, and $S'=\frac{15}{90}=\frac{1}{6}$,

Hence, $1\dot{6}=16666=1+06+006+0006$

$$=\frac{1}{10}+\frac{6}{10^2}+\frac{6}{10^3}+\frac{6}{10^4}+\qquad=\frac{15}{90}=\frac{1}{6}\ \Big]$$ 4 2.

Cal 1912. Compulsory.

1 (1) $8x^3-27y^3+z^3+18xyz$ (2) $3x^3-4x^2+6x-12$ (3) 16
(4) $(x+1)(x-6)(2x+1)$ 2 (2) $x=3$ (4) 16, 4 3 $3\frac{1}{3}$ miles.

Additional. 1 (1) $2x^2-3x^{-1}+4x^{-4}$ (3) ± 8, (4) 1

2 (1) $\frac{3}{2}\left(1-\frac{1}{3^1}\right)$, (2) $\frac{n(n+1)(n+2)}{3}$ 3 (See Ex. 1, page 557).

4 Rs 960

Cal. 1913. Compulsory.

1 (1) 0, (2) $2x+3$ (3) $\frac{2(x^2+y^2)}{y^2}$ (4) $3x^2-x-4$

2 (1) $\frac{1}{3}$, (2) 2, 3 (3) Rs 1830 3 (2) 4 1

Additional 1 (1) $\frac{4}{3}$, $-\frac{5}{14}$, (2) $x^2-\frac{x}{2}-1$ (3) 20, 30, (4) 2

2. (1) -15, -60 (2) $\frac{1}{3^3}$, $4\frac{3280}{6561}$ 3 75

Cal. 1914. Compulsory.

1. (1) $7y^2$ [the given expression$=\frac{1}{2}\{(a-b)^2+(b-c)^2+(c-a)^2\}$&c.]
(2) $x^4+2x^3+3x^2+4x+5$ 2 (1) $(x+1)(x-1)(x-3)$
(2) $(a-b)(a-c)(b-c)$ (3) 0 3 (1) $a+b$ (2) 3, 1 (3) 3, 4

Additional 1 (1) $9\frac{1}{2}, 5\frac{2}{3}$ (2) 25, $\frac{1}{25}$ $\left[\begin{array}{l} x^2-(25+\frac{1}{25})x+1=0 \\ \text{or } (x-25)(x-\frac{1}{25})=0 \end{array}\right]$

(3) 6 7 2 (1) $x-2-\frac{1}{x}$ (2) $\left(x-\frac{a+b}{2}\right)^2-\left(\frac{a-b}{2}\right)^2$.

3 (1) 6, 11, 16, 21, 26, 31, 36 (2) $\frac{1}{3}$, 1, 3

Cal. 1915. Compulsory.

1 (1) $a^3-b^3+c^3-3a^2b+3ab^2-a^2c-ac^2-b^2c+bc^2+2abc$;
(2) $x^2+y^2+a^2$, (3) $x-3$, (4) $(a^4-b^4)(a^2+ab+b^2)$

2 (1) 18, (2) 5, 3 3 $\frac{1}{abc}$

Additional 1 (1) $2\frac{3}{7}$ or $-1\frac{4}{5}$, (2) $7x^2+6x$, (3) $\frac{1}{4}, \frac{3}{4}$,
(4) $\sqrt{\frac{2}{3}}$ i e 816 2 (1) $\frac{n(n+1)(2n+1)}{2}$. 3 $(-\frac{1}{3}, \frac{1}{3})$, (1, 3)

Cal. 1916. Compulsory.

1 (i) $2(a-3b)$ (ii) $3-11x+6x^2$ (iii) $3x+1$ (iv) 1
2 (i) $x=6$ (ii) $x=2, y=3$ (iii) $\frac{2}{3}$

Additional 1 (i) $x=14$ or $2\frac{1}{4}$ (ii) $x-3-\frac{2}{x}$
(iii) $=1$ (iv) 8, 9

2 (i) 900 (ii) 5, 15, 45, 135

Cal. 1917. Compulsory.

1 (a) $a^3-ab^2+2ab+b-1$, a^3+2a^2+4a+2
(b) $x-2$, $(x-1)(x-2)(x+3)$
3 (a) $x=1$, $x=1, y=1$ (b) 14

Additional 1 (a) $x=43$ or -42 (b) $\pm\left(1+a+\frac{a^2}{2}\right)$.

2 (b) $\frac{n(n+1)(n+2)}{3}$, $\frac{1-(n+1)x^n+nx^{n+1}}{(1-x)^2}$ 3 1·65 or $-$ 65.

Mad. 1904.

1. (a) 27 (b) $x^5-ax^4-a^4x+a^5$
(c) $(3y-2x)(3y+2x)(x-2)(x+2)(x^2+2x+4)(x^2-2x+4)$
2. (2) $x^9-x^7y^2+x^6y^3-x^5y^4-2x^4y^5+x^2y^7-xy^8-y^9$.

3. (i) $-24abc$. (ii) $\frac{x-p}{(a+x)(b+x)(c+x)}$ 4 (1) $9a^2+6ab+4b^2$.

(2) $a^2+2+\frac{1}{a^2}$ 5 (i) $5\frac{1}{2}$ (ii) $a+b, a-b$ (iii) $1\frac{1}{2}, -2\frac{1}{2}, 3$

6 $12\frac{1}{2}$ miles

Mad. 1905.

1. (1) $\frac{1}{7}$ (2) $5\frac{1}{3}$ 2. (1) $(x-1)(x+1)(x^2-x+1)(x^2+x+1)$

(2) $-3(x-1)(x-2)(2x-3)$ 3 $4x^2-1$, $\frac{1}{2}, -\frac{1}{2}, -\frac{3}{4}$ 4 $3x^2+7x-6$.

5 (1) x^3-1 (2) $\frac{1}{x^2(x-1)^2}$ (3) $\frac{4}{3}(x^2+xy+y^2)$ 6 $a+b$ or $3a-2b$

3 $-x^2+4x+21$

Mad. 1906.

1 (1) 1 (2) $(a+b)x^2+(a^2-6ab+b^2)x+ab(a+b)$

2 (1) $\{(a-b)x^2-4(a+b)y^2\}^2$. (2) $(b-c)(c-a)(a-b)(a+b+c)$

3 (1) $\frac{11x^2-4x+1}{25x^2-20x+11}$ (2) $12x^3(x-1)^2(x+2)^2(2x-1)$

4 $80abc(a^2+b^2+c^2)$ 5 $x^3+y^3+z^3-3xyz$

6. (1) -3 (2) $\frac{1}{6}(\pm\sqrt{256}-2)$ 7 3 annas

Mad. 1907.

1 (1) $\frac{5}{6}$ (2) 55 44 sq in, 2 143 in 2 (2) $-\frac{13}{8}, \frac{23}{4}$

(3) $(3a+2b)^2-16b^2$ 3 (1) $(a-b)(c^2-ab)$ (2) $(x-1)^2(x+2)$.

(3) $-(b-c)(c-a)(a-b)$ 4 (1) $-\frac{1}{2x-1}$ (2) -1

5 (1) $-\frac{6}{13}$ (2) 2, 3 (3) 3 75, $-$ 61. 6 Rs 49

Mad. 1908.

1 (a) 207 9 sq in (b) 24 in

2 (1) $x^4-px^3+\frac{2p^2}{3}x^2-\frac{p^3}{3}x+\frac{p^4}{9}$.

(2) $(x+a+b)(x+a-b)(x-a+b)(x-a-b)$

3 $x^4(x-1)^4(x^3+1)$ 4 (a) $\frac{1}{30}$ (b) -1 5 $4x^2+16x+11$.

6 (1) $-1\frac{2}{3}$ (2) 7, 9, 11 (3) $8\frac{1}{4}, -1\frac{2}{3}$ 7. Rs 255, Rs 510.

Mad. 1909.

1 (1) 3, $\frac{3}{8}$ (2) $\{(a-b)(a-c)-(b-c)(b-a)+(c-a)(c-b)\}^2$.

3 (1) 11 (2) $5\frac{1}{2}, 2\frac{1}{2}$. 4 22 and 18 miles per hour.

Mad. 1911.

1 $3(a+3b)^2-15c(a+3b)+75c^2$ 2 (i) (i) $(x-3)(5x-14)$
(ii) $(x-7)(5x+6)$ (iii) $(x+3)(x-4)(x+5)$
3 (a) $4\frac{1}{2}, 5$ (b) $2, 3$ 4 132 miles

Mad. 1912.

2 (a) $-4\frac{7}{15}, 4\frac{8}{15}$ (b) u^3-3u, u^4-4u^2+2. 3 0
4 (a) $4\frac{1}{2}$ (b) $3\frac{1}{2}, -1\frac{1}{2}$ 5 20 and 35 miles per hour.

Mad. 1913.

1 $x=11\cdot8$ or $-6\cdot8$, $y=6\cdot8$ or $-11\cdot8$ 2 (i) 7 (ii) 269
3 (i) $(x-5)(x^2-5x+3)$ (ii) $(a+b+c)(b+c-a)(c+a-b)(a+b-c)$.
(iii) $(x+3y)(x-3y)(x^2+3xy+9y^2)(x^2-3xy+9y^2)$
4 $x=1, y=2, z=3$ 5 31, 4

Bom. 1903.

1 (a) $2abc(a+b+c)$ (b) $8x^6+24x^5+9x^4-37x^3-48x^2-24x-8$.
2 $(2y-1)(y+1)$, $(2y^2+y+1)(4y^4-5y^2+1)$ 3 (a) m.
4 (i) 10 (ii) $2\frac{1}{8}, 4\frac{2}{5}$ 5 60, 22

Bom. 1904.

1 $x^2-(a+2b)x+a^2+3b^2$. 3 x^2-7x+2 4 (1) $\dfrac{a^2+b^2+c^2}{a+b+c}$
5 $4n^2-16n+11$ 6 -6 7 $2\frac{6}{7}$ miles per hour

Bom. 1905.

1 $\dfrac{2x(a-b)}{(a+x)(b+x)}$ 3 x^4-5x^2+4 4. x^2+5x-6
5 $2(x^2+4x+3)$ 6 (i) $-2\frac{1}{2}$ (ii) $\frac{1}{2}, -\frac{1}{2}$ 7 $\frac{1}{4}$

Bom. 1906.

2 (1) $a^{12}+4a^6-1$ (2) $1-3x+5x^2-7x^3$ 3. (1) $(2x+y+b)(b-y)$.
(2) 0 4 (1) x^2+4x+5 (2) $(x+1)^2(x-2)(x-3)(x-4)$.
5 (1) 1 or $\frac{1}{3}$ (2) 1, 2 6 12 annas per tola.

Bom. 1907.

1 $3x^3-2x^2-9x+28$
2 (1) $(x^2+y^2-1)^2$ (2) $(x-3)(x-2)(x+1)(x+2)$
3 $\dfrac{1-x^2+x^4}{1+x^2+x^4}$ 4 (1) $\left(\frac{5}{3}, 3\right)$ (2) $-2, 3$. 5. (1) 0 $\frac{5}{2}$.
(2) $3, -\frac{1}{2}$. 6 17, 12 or $-12, -17$. 7. $3\frac{1}{2}$ miles per hour.

Bom. 1908.

1 $3a^3+7a^2x-ax^2-x^3$ 2 (1) a^2-2, a^3-3a, (2) $\dfrac{x^4+a^4}{x^6-a^6}$

2 (i) $(5x-4)(25x+12)$ (ii) $(x^2-x+y-2)(x^2+x-y-2)$

(iii) $\left(x-\dfrac{2}{y}\right)\left(x^2+\dfrac{2x}{y}+\dfrac{4}{y^2}\right)$ (iv) $(2a+b)^3$

4 $(-1\frac{8}{13}, -3\frac{1}{13})$ 5 (i) 111, 11 (ii) $1, 2\frac{3}{4}$ 6 20

Bom 1909.

1 $-3, 2$ 2 $x^2+(a-b)x-ab$ 3 $(x-1)(2x-3)$,

$(x-1)(x-2)(x-3)(2x-3)$ 4 (1) $2x^2+8+\dfrac{8}{x^2}$ (2) $\dfrac{1}{x-8}$.

5 (1) 5, 9 (2) ± 4 6 12 and 10 miles per hour

Bom. 1910.

1 29 3. (1) $a^3-125b^3+8c^3+30abc$ (2) $(a+c)(b+c)$ 4 (1) a

5 $x-2y$ 6 (1) 10, -4 (2) 2, 6 7 1100

Bom. 1911.

1 (a) $x^{12}+64$, (b) $(8a^3-27b^3)(3a^2-ab-2b^2)$

2 (a) $x^3-2x+\dfrac{1}{x^3}$ (b) 0 3 (a) $-3\frac{1}{2}$, (b) $5, \bar{3}$, (c) $2\frac{1}{2}, -1\frac{3}{4}$

4 4 and 5 miles per hour 5 (5, 1)

Bom. 1912.

1 (b) $3x-2y=1$, $2x-y=2$, (3, 4)

2 (a) $(x-1)(x+1)(x^2-x+1)(x^2+x+1)(x^6+2)$

(b) $(x^2-2y^2)(x^2+2y^2)(x^2+2xy+2y^2)(x^2-2xy+2y^2)$

(c) $\{(a+1)x+a\}\{ax-(a-1)\}$

(d) $(a-3b)(a-6b)(a^2-9ab-18b^2)$

3 (a) $a^4-4a^2b+2b^2$ 4. (a) $a-b$, (b) $1+\dfrac{2ab}{a^2-b^2}$

5 (a) 6, (b) $\frac{1}{4}, \frac{1}{5}$, (c) $4, 1\frac{1}{2}$ 6 40 and 25 miles per hour

Bom. 1913.

1 (a) (i) $\left(\dfrac{c}{2}+a-b\right)\left(\dfrac{c}{2}-a\right)$, (ii) $(5x+7y)(2x-5y)$

(b) (i) $3(a-2b)$, (ii) $12ab(a-2b)(a+2b)(a+b)$

2 (a) $\dfrac{2}{3x^2}$, (b) $-\dfrac{1}{y}$ 3 (i) 1, (ii) $1\frac{1}{2}$, $3\frac{1}{2}$, (iii) 10 or -1

4. 36 and 27 years 5 (2, 1) 6 5

Pun. 1904.

1 (1) 60, (2) 3, 2, 1   2 $\frac{-3x^2-13}{(x-1)(x-2)(x-3)}$

3 $2(a^3+b^3+c^3-3abc)$ 5 -11

Pun. 1905.

1 (1) $f^{\frac{4}{3}}k^{-\frac{4}{3}}+2fk^{-1}-3f^{\frac{1}{2}}k^{-\frac{1}{2}}+k^{\frac{2}{3}}+2f^{\frac{1}{3}}k^{2}-3f^{\frac{5}{6}}k^{\frac{5}{2}}$,

(1) $x^{\frac{2}{3}}y-2x^{\frac{1}{3}}y^{\frac{3}{4}}+3y^{\frac{1}{2}}$ 2 (1) $a+b+c$ (2) $\frac{1}{x^3-1}$.

3 (1) $-\frac{1}{2}$, (2) $\frac{1}{12}, \frac{1}{3}$ 5 (i) $(4x+5)(4x^2-4x+5)$;
(ii) $5(y-z)(z-x)(x-y)(x^2+y^2+z^2-yz-zx-xy)$

Pun. 1906.

1 (1) (i) $\cdot 355779$, (ii) $9\cdot 77553$ (2) 2

2 (i) $(x+y)(x-2y+1)$,   (ii) $3(x-1)(x-2)(3-2x)$;
(iii) $a(x+y+z)(x^2+y^2+z^2-yz-zx-xy)$ 4 (i) a^2+b^2,
(ii) $\frac{a}{m_1m_2}$, $a\left(\frac{1}{m_1}+\frac{1}{m_2}\right)$ 5 13.

Pun. 1907.

1 $1\frac{19}{33}$ $8\frac{2}{3}$ 2 $(x+y)(x^2-xy+y^2)(x^2+xy+y^2)$.

3 At $32\frac{4}{13}$ mins past 5   4 $4x^2+16x+11$ 5 1

Pun. 1908.

1 (b) $(x+1)(x+2)$, $(x-1)(x+1)(x+2)(x+3)$.   2 (1) $\frac{1}{2}(3-\sqrt{5})$;
(ii) $x+\frac{1}{x}-2$ 3. (i) $(3x-2)(4x+7)$,
(ii) $(a+b+c-d)(a+b-c+d)(a-b+c+d)(-a+b+c+d)$

4 $3(a^2+b^2+c^2-bc-ca-ab)$ 5 (2) £305, £410, £515

6 $37\frac{1}{2}$ miles

Pun. 1909.

1 $\frac{2c(a+b)}{3d}$ yds, $\frac{nc(a+b)}{288}$ rupees   2 (1) $16a-4b$

(2) (i) $x^2y^2(5x-3y)(25x+15xy+9y^2)$; (ii) $3(y-z)(z-x)(x-y)$.

3 (1) $\frac{8x^{\frac{7}{2}}}{x^8-256y}$, (2) 0   4. (1) 1, (2) $\frac{1}{3}, \frac{1}{2}$

5. Rs 513. 8 as., Rs 9862 8 as

Pun. 1910.

1 (i) $\dfrac{a+b+c}{3}$, (ii) $\frac{1}{3}, \frac{1}{3}, \frac{1}{4}$

2 (i) $(9x^2+12xy+8y^2)(9x^2-12xy+8y^2)$
(ii) $2(x+y+z)(x^2+y^2+z^2-yz-zx-xy)$ 3 x^2 4 27.

Pun. 1911.

1 (i) $\dfrac{2a}{1-a^2}, \dfrac{2b}{1-b^2}$, (ii) $\dfrac{x}{(x-1)(x-2)(x-3)}$.

2 (1) (i) $(pq-4)(pq-11)$,
(ii) $(9x-2y^2)(81x^2+18xy^2+4y^4)$, (2) $(x+1)(x+7)(2x-8)$

3 (i) $2\frac{1}{2}$, (ii) $12, -4$ 4 $\dfrac{x^2}{(x+a)(x+b)(x+c)}$ 5 162

Pun. 1912.

1 (1) 0, (2) $\dfrac{7x-2y}{5x^2-3xy+2y^2}$

2 (i) $(a-2)(a+2)(a^2-2a+4)(a^2+2a+4)$,
(ii) $(y+z)(z+x)(x+y)$, (2) $3(x^2-x-\frac{1}{2})$ 3. (i) $\frac{7}{38}$;
(ii) $\dfrac{1300}{1853}, \dfrac{1300}{314}$ 4 (2) $(lb-na)^2+m^2ab=0$

5 One more than half the number of rooms

Pun. 1913.

1. (i) $2\{a^2-b^2+2(a+b)\}$, (ii) x^4-2x+1 2 $(1\frac{1}{2}, 1)$

3 (i) $\{(a+b)^2+c^2\}(c+a-b)(c-a+b)$,
(ii) $-(a-b)(b-c)(c-a)(a+b+c)$.

4. 473

Pun. 1914.

1 (i) $2x^2-2x+1$, (ii) 181, 221
(iii) $\left\{\dfrac{pq-r(p+q)+q^2}{p^2-q^2}\right\}^2-\left\{\dfrac{pq-r(p+q)+p^2}{n^2-q^2}\right\}^2$.

2. (i) $(x+1)(2x-1)^2$ 3 (i) 5' (ii) (4, 2). 4 8, 14, 55.

5. (ii) $y^2=4ax$

Pun. 1915.

1. (i) 200 2 (i) $x(3x-7)(6x+5)$ 3 (i) 13; (ii) (1, 2).
4 (ii) $2y^2-2ay+a^2-b^2=0$.

Pun. 1916.

1 (i) $x^3+\frac{3}{5}x-\frac{2}{7}$ 2 (i) $x+2$ 3. (i) $x=\frac{1}{4}$, $y=-4$.
4 497. 5 (ii) $a^2n+b^2p=0$.

Pun. 1917.

1 (i) $\frac{3^n}{10^n}$, 17 (ii) 5 2 (i) $-\frac{ab}{c}$.
(ii) (a) $(a+c+b)(a+c-b)$ (b) $(a+b)(b+c)(c+a)$
3 (i) $p=\frac{2}{5}, q=\frac{3}{5}$ (ii) $x=8, y=3$ 4. 5.

All. 1905

1 (a) x^2+x-2, (b) a^2+a+1 2. 0. 3 (i) -6, (ii) 12, -4
6 7, 12

All. 1906.

1 $x^2+3xy+4y^2$ 3 (i) $(x^2+8)(x^2-8)$,
(ii) $-(y-z)(z-x)(x-x)(x+y+z)$ 4 1
5 (i) $a+b+c$, (ii) $4\frac{1}{4}\frac{4}{5}$. 6. 300.

All. 1907.

2 1 3 $(x-y)(x+y)(x^2-xy+y^2)(x^2+xy+y^2)$.
4 11 5 20, 30 6 $29\frac{1}{4}$, 147 and $70\frac{1}{2}$ days.

All. 1908.

1 a^2 2 $2x+3$ 3 $7, -\frac{1}{13}$ 4 1, 4 6. $(x-5)(x+3)$

All. 1909.

1 $3x-7$ 2 845, 3380, 1690, 2535 3 $\frac{z^2+4a^2}{z^2-4a^2}$ 4. (i) 4;
(ii) $a, 2a, -3a$ 5 30 and 50 miles per hour. 6. 3 7. 2·24.

All. 1910.

1 (a) x^2. 2 (i) $(x-y)(x+y)(x^2-xy+y^2)(x^2+xy+x^2)$;
(ii) $(x^2-xy+y^2)(x^2+xy+y^2)$, (iii) $(x-1)(x-2)(x-3)$
3 (i) $4\frac{1}{2}$, (ii) 18, 6, (iii) $3, -\frac{1}{3}$. 4 15 50.

All. 1911.

1 (a) $\frac{5}{(x-3)(x-4)(x-5)}$. 2. (i) $(x-a)\left(x+\frac{1}{a}\right)$; (ii) $(x-1)^3$
(iii) $(x^2+3xy-y^2)(x^2-3xy-y^2)$
3 (i) $2\frac{275}{288}$ (ii) 1, 2; (iii) 1 3 4. 3 hours. 6 2, 0.

All. 1912.

1 (i) $\frac{1}{(x+2)^2}$, (ii) $\frac{2x+1}{3x+2}$ 2 (a) (i) $(x+a+b)(x+a-b)$.
(ii) $(x-1)(x+2)(x+3)$, (iii) $(x^2+2xy+2y^2)(x^2-2xy+2y^2)$.
(b) $x(x^2-1)$ 3. (i) $\frac{lm}{l+2}$, (ii) 0, 15, (iii) 3, 2

4 (i) $x+y$, (ii) 6 5 (ii) 8 6 $x^2+2xy-3y^2$ 7 2, 3

All. 1913.

1 ap^3+bp^2+cp+d 2 (a) $(a+b+c)(ab+ac+bc)$
(b) $\frac{a+b+c}{(a+b-c)(b+c-a)(c+a-b)}$ 3 (i) 2, 3, (ii) 15, −5 (iii) $\frac{3}{4}$

4. 8, 5 5. After $3\frac{1}{8}$ hours and $31\frac{1}{4}$ miles from the starting point.

All. 1914.

1 (i) $3(x-y)(y-z)(z-x)$
(ii) $\frac{(a+b+c+d)(a+b-c-d)(c+d+a-b)(c+d-a+b)}{4(ab+cd)^2}$
2 $a+b+c$ 3. (i) $-\frac{1}{2}$, (ii) $\frac{5}{16}$, $-6\frac{3}{8}$, (iii) $\frac{1}{6}$ or $-13\frac{1}{2}$.
4 440 yds

All. 1915.

1 (i) $(3x-2)(2x-3)$, (ii) $(a-b)(a+3b-2c)$ 2 0
4 (i) 1, (ii) 1, $\frac{1}{3}$, $\frac{1}{3}$ 6 41 and 11 years.

All. 1916.

1 (i) $2b^2(a^2-5b)(5a^2+2b)$, (ii) $(x+2)\{(a^2-b^2)x+2(a^2+b^2)\}$;
(iii) $(a+b+c)(a^2+b^2+c^2-ab-ac-bc)$

2 (i) 5, (ii) $x=-\frac{1}{4}$, $y=\frac{1}{8}$, (iii) $\frac{a+b}{a-b}$ or $\frac{a-b}{a+b}$

3 $A=11s$, $B=38s$, $C=33s$, $D=32s$ $E=36s$ 6 4

All. 1917.

1 (i) $(ax+by-ay+bx)(ax+by+ay-bx)$
(ii) $(x+y+1)(x^2+y^2+1-xy-x-y)$
(iii) $(x+2y-3z)(x-2y+3z)$

2. $\frac{1}{abc}$. 3 (i) $x=1$ $y=2$, $z=3$ (ii) $x=7$ $-6\frac{1}{7}$.

4. $3\frac{1}{2}$ miles 5. (1) £690, (2) 120 hors

APPENDIX.

CHAPTER I.

GRAPHS.

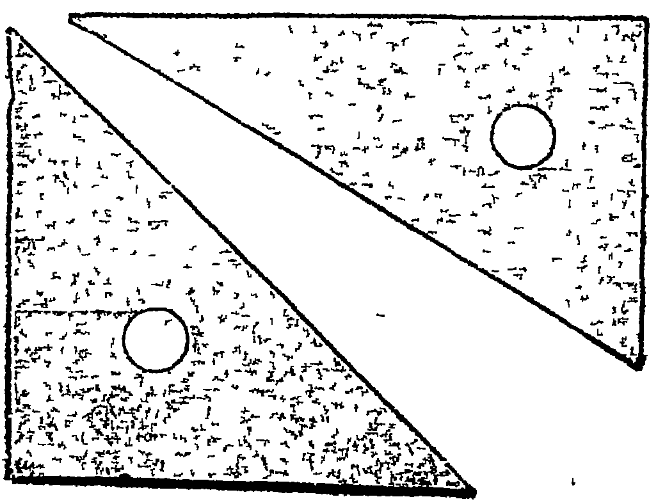
1. Instruments required The student should first of all provide himself with the following instruments and acquire skill in manipulating them with accuracy and neatness

(1) A Hard Pencil.

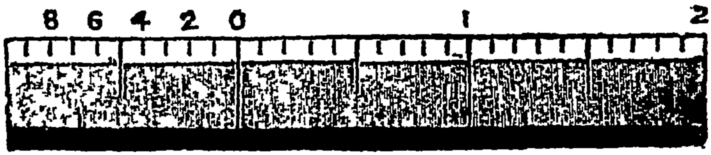
Note It must be well sharpend so that the lines drawn may be very fine

(2) A Pair of Compasses (also called Dividers).

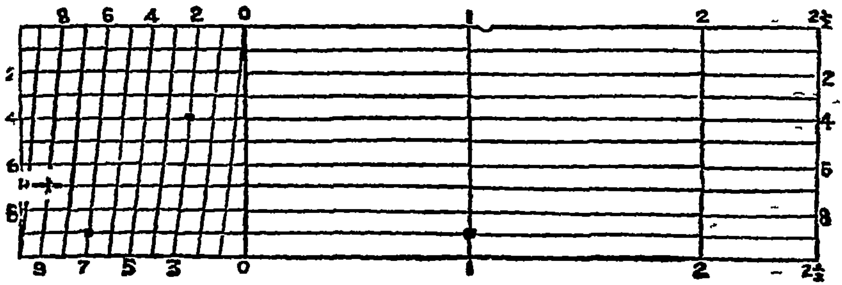
(3) Two Set-squares.



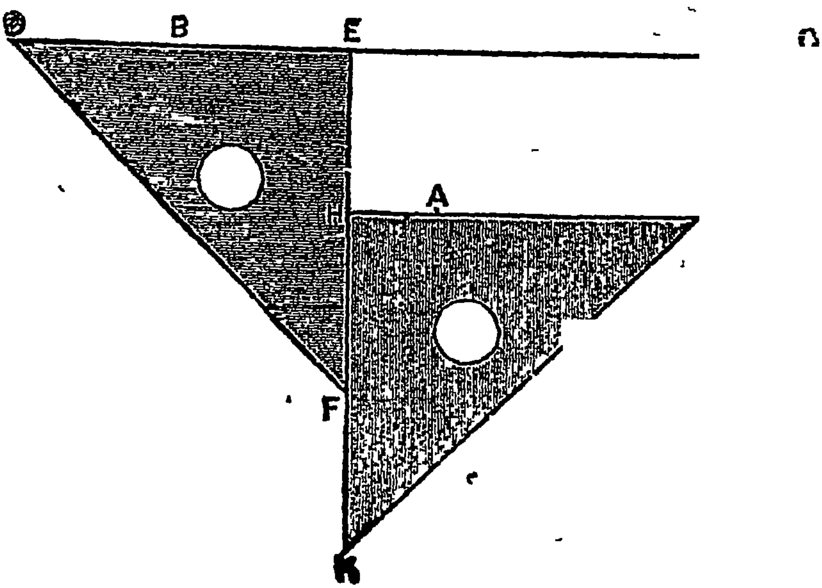
(4) A graduated Flat Ruler (of moderate length) shewing tenths of an inch.



(5) A Diagonal Scale, giving hundredths of an inch.

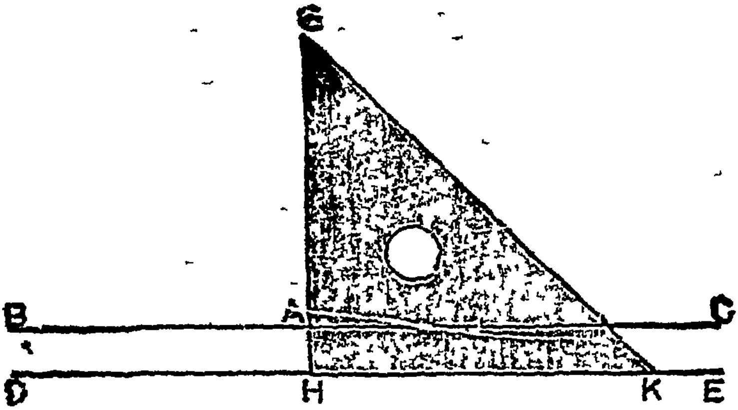


Example 1. Through the point A draw a straight line parallel to BC.



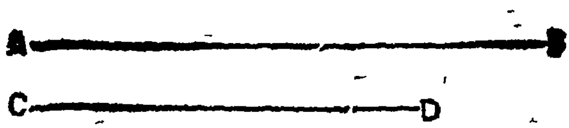
Place the Set-square DEF in such a way that the edge DE may fall along BC. Then slip the other Set-square GHK into the position shewn in the diagram, so that HG may pass by A. Now trace a line along HG, which will evidently be parallel to BC.

Example 2. Through the point A in the straight line BC draw a straight line perpendicular to BC.



First trace a line DE parallel to BC. Then place the Set-square GHK in such a way that HK may fall along DE and GH may pass by A. Now trace a line along HG, which will evidently be perpendicular to BC.

Example 3. Find the lengths of the straight lines AB and CD :—



(1) By means of the Pair of Compasses and the Diagonal Scale we find that the length of AB is equal to the distance between the two points marked on the line 4—4 in the diagram. Hence the required length = 2·24 inches.

(2) The length of CD is found to be equal to the distance between the two points marked on the line 9—9 in the diagram. Hence the required length = 1·69 inches.

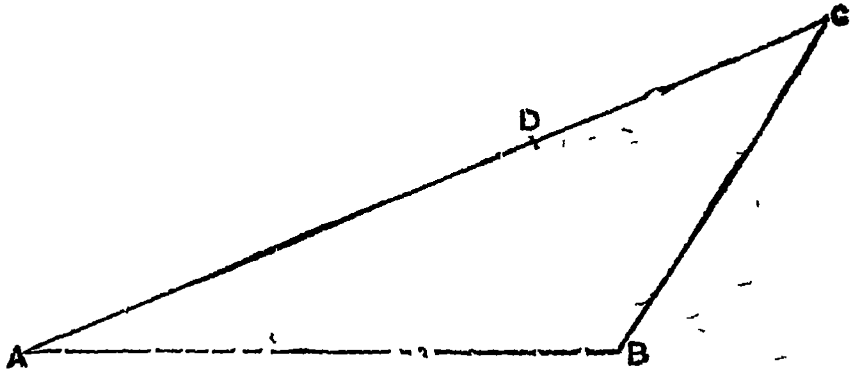
Exercise (1).

1. Produce the straight line AB to double its length

A———————————B

2. On a given straight line AB a point D is taken supposing it to be the middle point By means of a Pair of Compasses however it is found that AD is a trifle shorter than BD How is the mistake to be corrected ?

3. ABC is a triangle and D a point on AC, as in the following diagram Through D draw, towards AB, a straight line parallel to CB



4. In the same diagram, through D draw, away from AB, a straight line parallel to BC.

5. In the diagram of example 3, through B draw a straight line parallel to AC.

6. From the vertices of a given triangle draw ...rpendiculars to its opposite sides

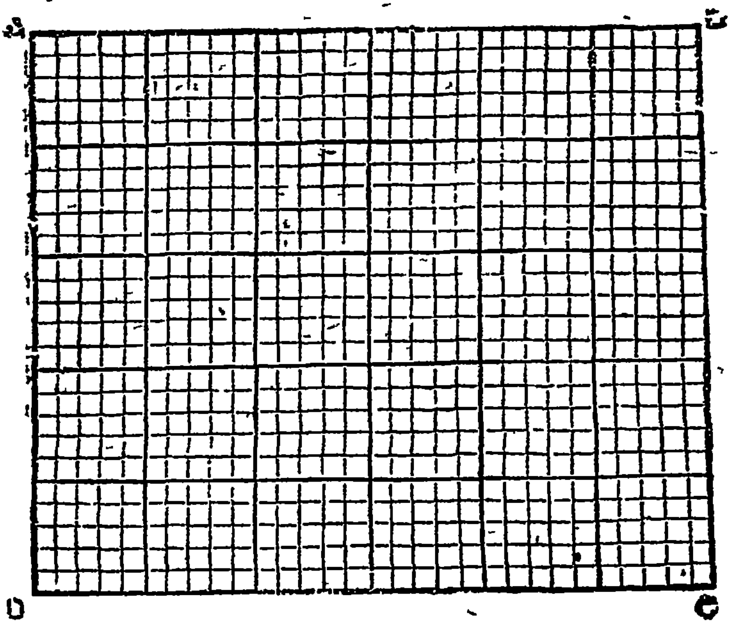
...In example 3, measure the lengths of the sides of ...le, and also measure the lengths of AD and DC

...ave

...red Paper. A specimen of a sheet of ...r is given on the next page

... two sets of parallel straight lines on the ...et being parallel to the length, and the other,

parallel to the breadth, of the paper, it is clear that every line of the first set is perpendicular to every line of the



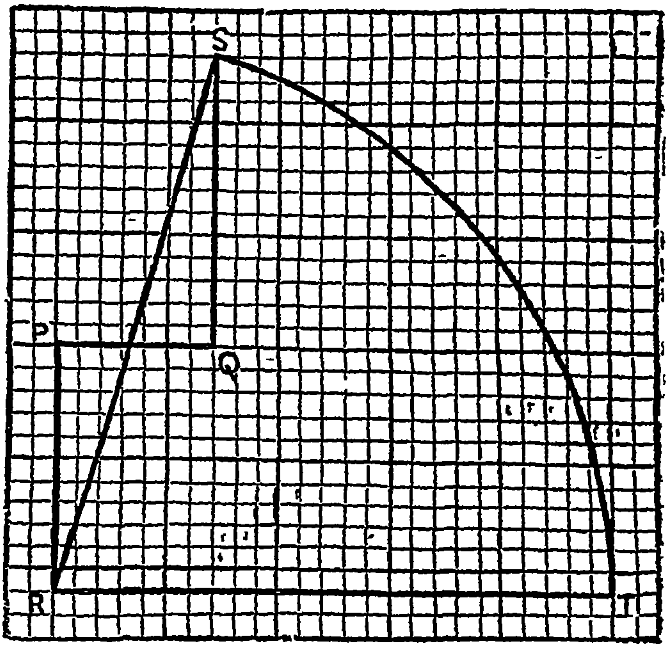
second. The distance between every two consecutive parallels is one-tenth of an inch, whilst every two consecutive *thick* parallels are half an inch apart The whole paper is thus divided into a large number of small squares which are equal to one another, each side of each square being one-tenth of an inch in length The paper is also divided into a number of thick-bordered squares, each side of each such square being half an inch in length. It is clear also that twenty-five of the small squares are contained in each of the thick-bordered squares.

Note 1 Lines parallel to AB may be regarded as *east-and-west* lines, and those parallel to AD, *north-and-south* lines They may also be considered as *horizontal* and *vertical* lines respectively.

Note 2 For the sake of convenience the length of a side of a small square may be denoted by the symbol *a*

Note 3 The paper may also be ruled so that the length of a side of a small square is only one-tenth of a centimetre (*i e* a millimetre) instead of one-tenth of an inch In that case the distance between every two consecutive *thick* parallels is evidently half a centimetre or 5 millimetres (One centimetre is approximately equal to 39 of an inch)

Example 1. P, Q, R, S are four stations such that Q is 7 miles east of P, R is 11 miles south of P, and S is 13 miles north of Q. Find the distance between R and S.



Taking the length of a side of a small square (*i.e.*, a) to represent one mile, we have P, Q, R, S as in the above figure, where $PQ = 7a$, $PR = 11a$ and $QS = 13a$

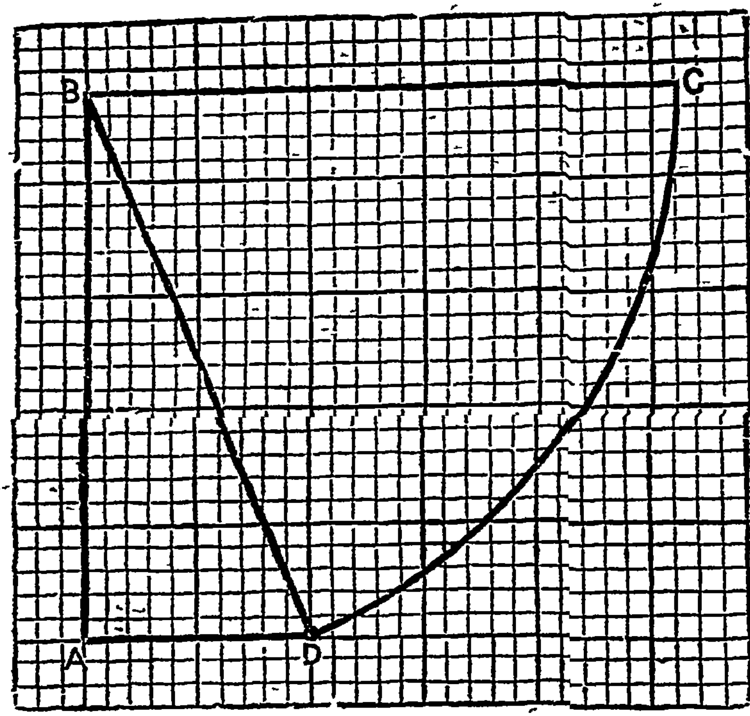
With R as centre and RS as radius describe an arc of a circle cutting the east-and-west line through R at T.

Now as $RT = 25a$, we have RS also $= 25a$. Hence the required distance $=$ **25 miles.**

Example 2. An upright post is 8 feet high. A string of length $8\frac{2}{3}$ feet has one end attached to the top of the post and is held tight with the other end in contact with the ground. How far is this end from the foot of the post?

Let $3a$ (*i.e.*, 3 times the length of a side of a small square) represent one foot. Then 8 feet will be represented by $24a$ and $8\frac{2}{3}$ feet by $26a$.

Let AB represent the post, so that $AB = 24a$. Take a point C on the horizontal line through B such that $BC = 26a$.



With B as centre and BC as radius describe an arc of a circle cutting the horizontal line through A at D. Join BD; then BD represents the string.

Now, AD is equal to $10a$, which is $9a+a$. Hence the required distance $= 3\frac{1}{3}$ feet.

Exercise (2).

1. A is $5\frac{1}{3}$ units of length east of O, and P is 4 units of length north of A. How far is P from O?

2. B is 8 feet west of O, and Q is $7\frac{1}{5}$ feet south of B. How far is Q from O?

3. C is 2 yards north of O, and R is $6\frac{6}{7}$ yards west of C. How far is R from O?

4. D is 2 1 inches south of O, and S is 2 8 inches east of D. How far is S from O?

5. A is 2 7 feet east of O P is north of A and 4 5 feet from O. How far is P from A ?

6. Q is 2 4 feet south of B O is east of B and 2 5 feet from Q. How far is B from O ?

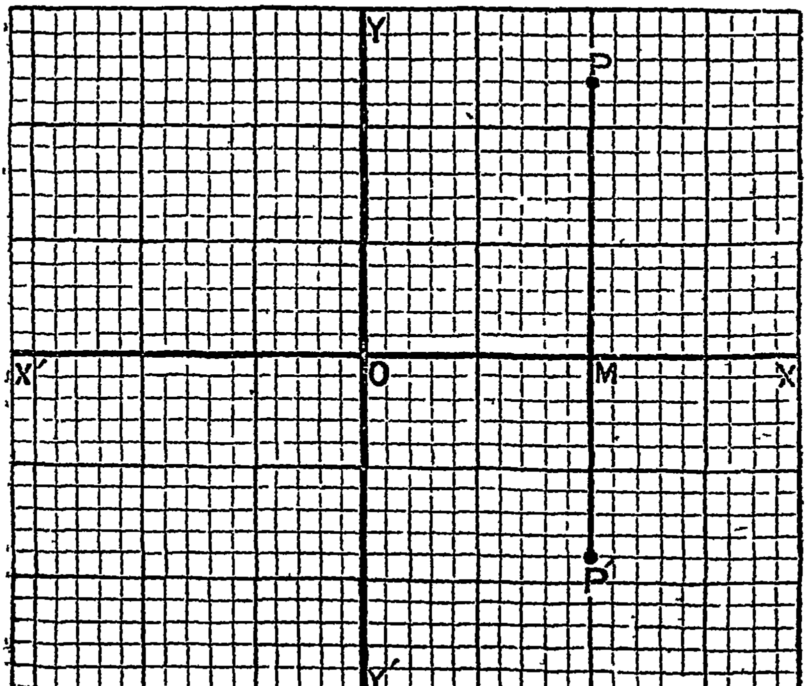
7. B is $4\frac{4}{5}$ yards east of A C is $\frac{3}{5}$ yard north of A, and D is 2 yards north of B. How far is D from C ?

8. B is 25 feet north of A P is 40 feet west of A, and Q is 20 feet east of B How far is Q from P ?

9 Two vertical posts, 14 feet and $3\frac{4}{5}$ feet high, are $13\frac{3}{5}$ feet apart. Find the distance between the tops of the posts

10 A ladder 30 feet long has its foot at a distance of 10 feet from a vertical wall How far up the way does it reach ? (The Diagonal scale may be used if necessary).

3. If in a plane, a point and two straight lines passing through it at right angles to each other be given, the position of any point in the plane can be easily defined.



In the plane of the paper as shewn in the last diagram, let XOX′ and YOY′ be the two given straight lines at right angles to each other. If P be any point in the plane, how to know its position?

We may regard XOX′ as the *east-and-west* line and YOY′ as the *north-and-south* line. Draw PM parallel to YOY′ meeting XOX′ at M. Evidently then M is due east of O, and P, due north of M. Hence if OM and MP be known, we know the position of P at once.

Taking the length of a side of a small square as the unit of length, we have OM = 9 units of length and M = 12 units of length Hence the position of P may be briefly defined as follows :—

9 units east, 12 units north.

Note 1 If Q be a point whose position is defined to be 5 units east, 8 units north, to find Q all that we have to do is to take a point 5 units due east of O and thence proceed 8 units northwards

Note 2 If R be a point whose position is defined to be 7 units west, 4 units south, to find R all that we have to do is to take a point 7 units due west of O and thence proceed 4 units southwards

Exercise (3).

[SQUARED PAPER IS TO BE USED IN EVERY CASE.

1. Find the points whose positions are defined as follows —

(1)	5 units east,	7 units north.
(2)	8 units west,	7 units north
(3)	10 units west,	12 units south
(4)	15 units east,	6 units south
(5)	8 units west,	13 units north
(6)	14 units east,	15 units south

2 It is clear from Chapter II (Positive and Negative Quantities) that "6 units west" is the same as "-6 units east," and "8 units south" is the same as "-8 units north." Hence find the points whose positions are defined as follows —

(1)	7 units east,	-3 units north.
(2)	-10 units east,	6 units north.
(3)	-9 units east,	-13 units north

3. In defining the position of a point the words "east" and "north" may be omitted if it is accepted as a rule that the distance measured towards the east should invariably be mentioned first On this convention, find the points whose positions are defined as follows :—

(1) 8 units, 9 units. (2) 6 units, -11 units.

(3) -12 units, 15 units. (4) -10 units, -14 units

4. We may define the position of a point still more briefly if the word "units" be omitted, Find, then, the points whose positions are defined as follows :—

(1) 6, 4. (2) 13, 8. (3) -7, 6.

(4) 8, -6 (5) -10, -13 (6) -9, -15

5. Definitions The student is referred to the diagram of the last article The given lines XOX′ and YOY′ with reference to which the positions of all points in the plane are defined, are called the **axes of co-ordinates**; and the point O, where these lines intersect, is called the **origin.**

The straight lines XOX′ is called the **axis of** x and the straight line YOY′, the **axis of** y.

The lengths OM and MP which define the position of the point P are called its **co-ordinates**, OM being called the **abscissa** (or x **co-ordinate**) and MP, the **ordinate** (or y **co-ordinate**).

"The point (x, y)" or simply "(x, y)" means "the point whose abscissa $= x$ units of length, and ordinate $= y$ units of length.

Note 1 When we speak of the "x and y" of a point, we mean its "abscissa and ordinate"

Note 2 The abscissa is positive or negative according as M is on the right or on the left of O The ordinate is positive or negative according as P is above or below XOX′.

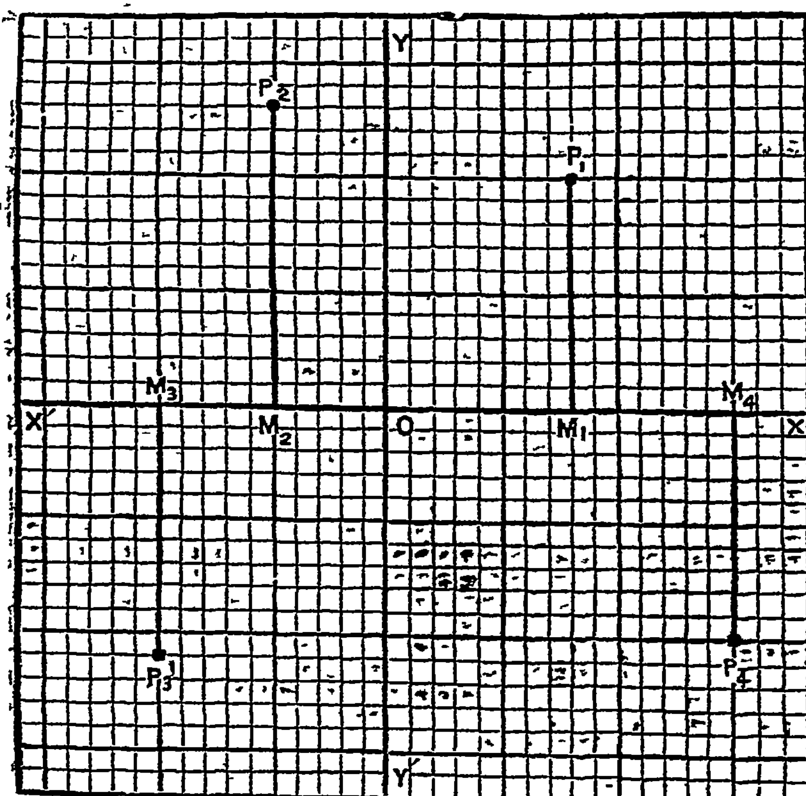
Note 3 "To Plot a point" is to find the position of a point when its co ordinates are given

Example 1. In the diagram given on the next page, write down the co-ordinates of the points P_1, P_2, P_3, P_4

The figure explains itself Take the length of a side of a small square as the unit of length.

(1) $OM_1 = 8$ units and M_1 is on the *right* of O; $M_1P_1 = 10$ units and P_1 is *above* the line XOX'. Hence the co-ordinates of P_1 are 8 and 10.

(2) $OM_2 = 5$ units and M_2 is on the *left* of O; $M_2P_2 = 13$ units and P_2 is *above* the line XOX'. Hence the co-ordinates of P_2 are -5 and 13

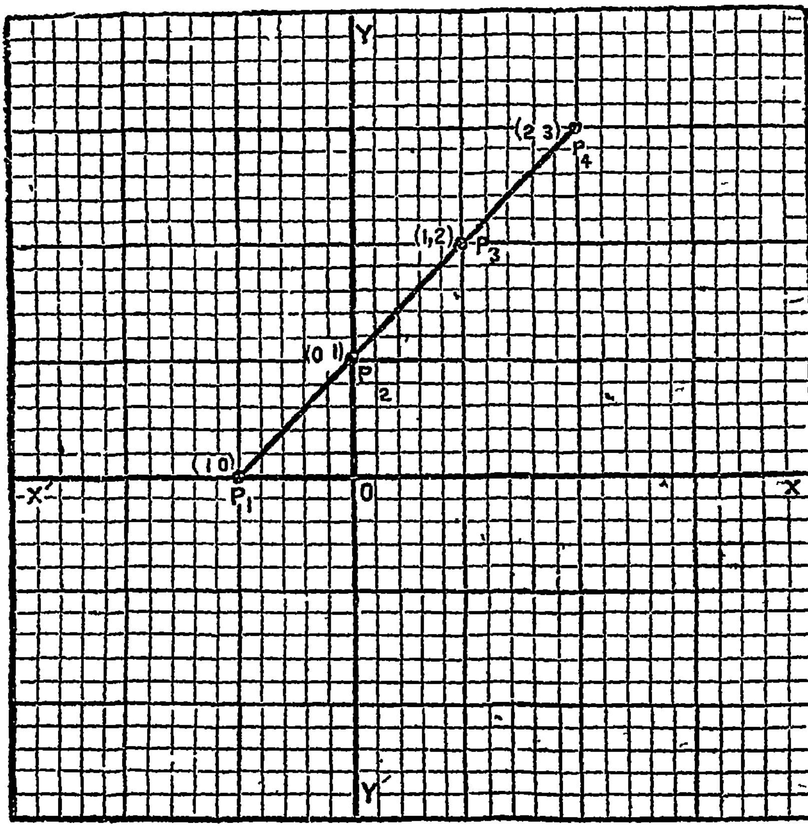


(3) $OM_3 = 10$ units and M_3 is on the *left* of O; $M_3P_3 = 11$ units and P_3 is *below* the line XOX'. Hence the co-ordinates of P_3 are -10 and -11.

(4) $OM_4 = 15$ units and M_4 is on the *right* of O; $M_4P_4 = 10$ units and P_4 is *below* the line XOX'. Hence the co-ordinates P_4 are 15 and -10.

Example 2. Plot the points $(-1, 0)$, $(0, 1)$, $(1, 2)$ and $(2, 3)$, and show that they all lie in a straight line.

Let 5 times the side of a small square represent the unit of length, and let P_1, P_2, P_3, P_4, respectively denote the four given points Then the positions of the points will be as shewn in the figure

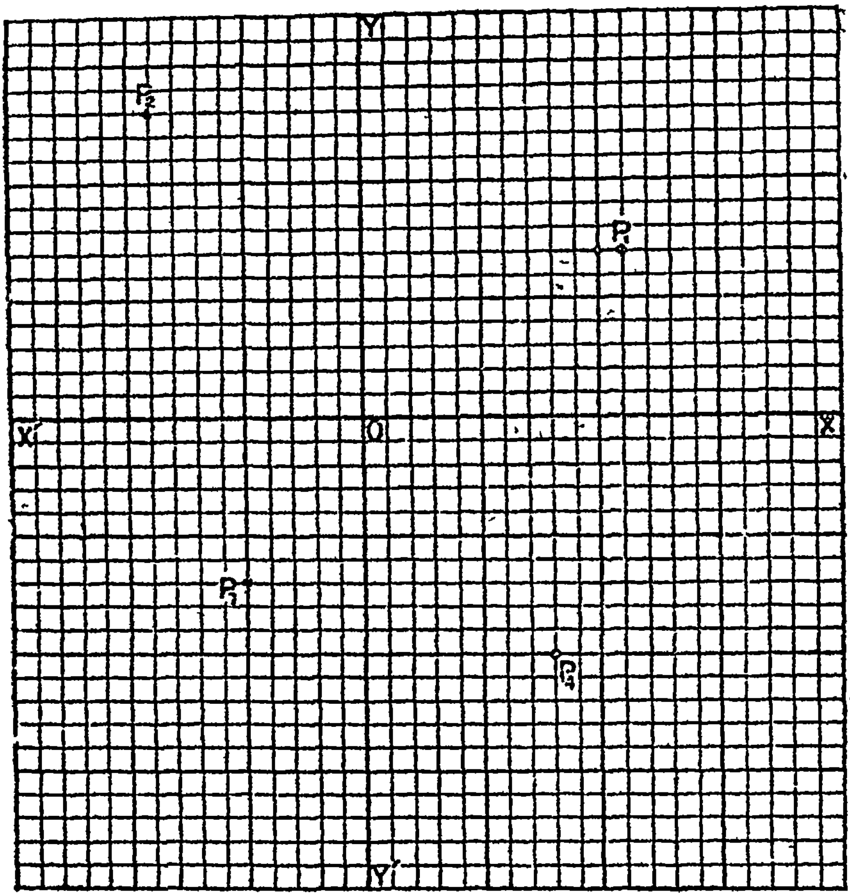


Now we find that a Flat Ruler may be so placed that its edge will pass through all the four points. Hence they all lie in the same straight line

Exercise (4).

1. In the diagram given on the next page, what are the co-ordinates of the points P_1, P_2, P_3, P_4, (i) when the unit of length is represented by a side of a small square, (ii) when the unit of length is represented by 5 times the side of a small square ?

2. In the following diagram what will be the co-ordinates of the points if the unit of length be represented by three times the side of a small square ?



3. Plot the points (−4, −4), (7, 7), (13, 13) and satisfy yourself that they lie in a straight line passing through the origin.

4. Plot the points (−8, 4) and (10 −5), and satisfy yourself that the straight line joining them passes through the origin.

5. Plot the points (8, 5) and (−4, −11), and find the distance between them.

6. Plot the points (−7, 9) and (−12, 21), and find the distance between them.

7. Plot the points $(-11, 13)$ and $(3, -35)$, and find the distance between them

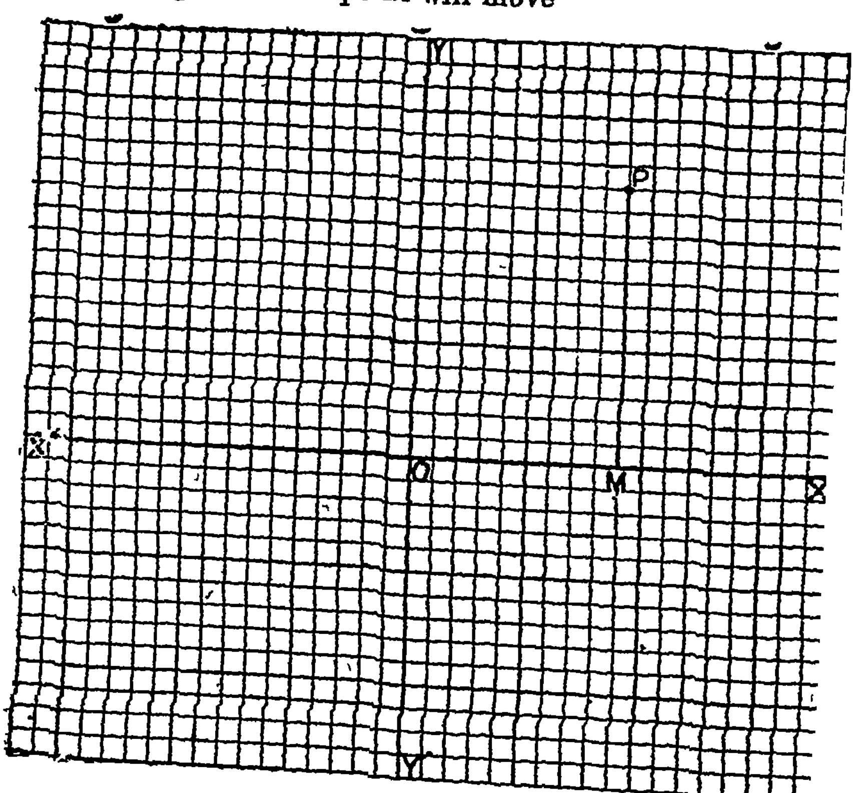
8. Join the points $(0, 0)$ and $(5, 5)$, and produce the straight line both ways Find the ordinate of the point on this straight line whose abscissa is 11, and the abscissa of the point whose ordinate is -13

9. Join the points $(0, 7)$ and $(12, 0)$, and produce the straight line both ways Find the ordinate of the point on the straight line whose abscissa is -18, and the abscissa of the point whose ordinate is -14

10. Join the points $(-4, 0)$ and $(0, -8)$, and produce the straight line both ways Find the ordinate of the point on the straight line whose abscissa is -10, and the abscissa of the point whose ordinate is -24

5 **Graphs of Simple Equations.** The following examples will make the subject clear

Example 1. If a point moves in such a manner that its abscissa is always equal to 5 units of length, find the path along which the point will move



Let twice the side of a small square represent the unit of length (The figure is on page 512)

On OX take the point M such that OM = 5 units of length, through M draw the straight line PMP′ parallel to YOY′

Now, if any point be taken on the straight line PMP′ its x will evidently be equal to 5 units of length, but this will *not* be so if the point be taken on either side of the line PMP.

Hence the moving point will always be on the line PMP′

We see therefore that if a point moves in such a manner that its x is always equal to 5 units of length, the path along which the point will move is the straight line PMP′ This fact is briefly expressed by saying that **the straight line PMP′ is the Graph of the equation $x = 5$.**

Note 1 From the above it is clear that the graph of the equation $y = 5$ is a straight line parallel to XOX′

Note 2 Generally speaking the graph of the equation $x = a$ is a straight line parallel to the axis of y, and passing through a point on the axis of x which is at a distance of a units of length from the origin, and the graph of the equation $y = b$ is a straight line parallel to the axis of x and passing through a point on the axis of y which is at a distance of b units of length from the origin

Note 3 Evidently therefore the graph of the equation $x = 0$ is the axis of y itself, and the graph of the equation $y = 0$ is the axis of x itself

Example 2. If a point moves in such a manner that its x and y are always connected by the relation $y = 3x$, find the path along which the point will move

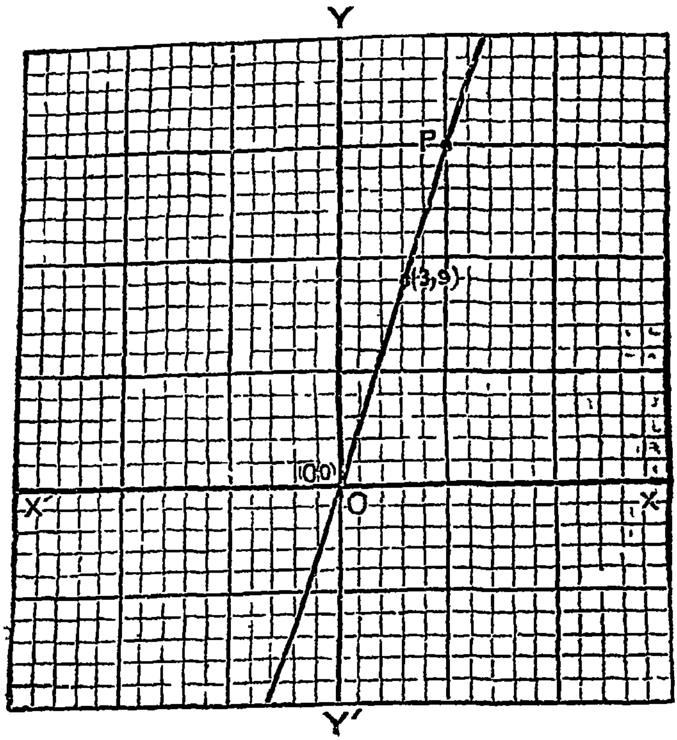
Since $y = 3x$, when $\left.\begin{array}{r} x = 0 \\ \text{we have } y = 0 \end{array}\right\}$, and when $\left.\begin{array}{r} x = 3 \\ \text{we have } y = 9 \end{array}\right\}$.

Evidently therefore (0, 0), and (3, 9) are two positions of the moving point

Take the length of a side of a small square as the unit of length (The figure is on the next page).

Join the points (0, 0), and (3, 9) and produce the straight line both ways Then this straight line will be the required path

Take any point P on this straight line The co-ordinates of P are found to be 5 and 15, which evidently satisfy the given relation Similarly, the co-ordinates of any other point on this straight line may be shewn to satisfy the given relation But the co-ordinates of a point which is outside the line OP will *not* satisfy the given relation, as can be easily verified.



Hence the moving point will always be on the line OP and never stray out of it

Thus it is found that if a point moves in such a way that its x and y are invariably connected by the relation $y = 3x$, the path along which the point will move is the straight line OP In other words **the line OP is the Graph of the equation $y = 3x$.**

Note Generally speaking, the graph of the equation $y = mx$, where m is any given number, is a straight line passing through the origin

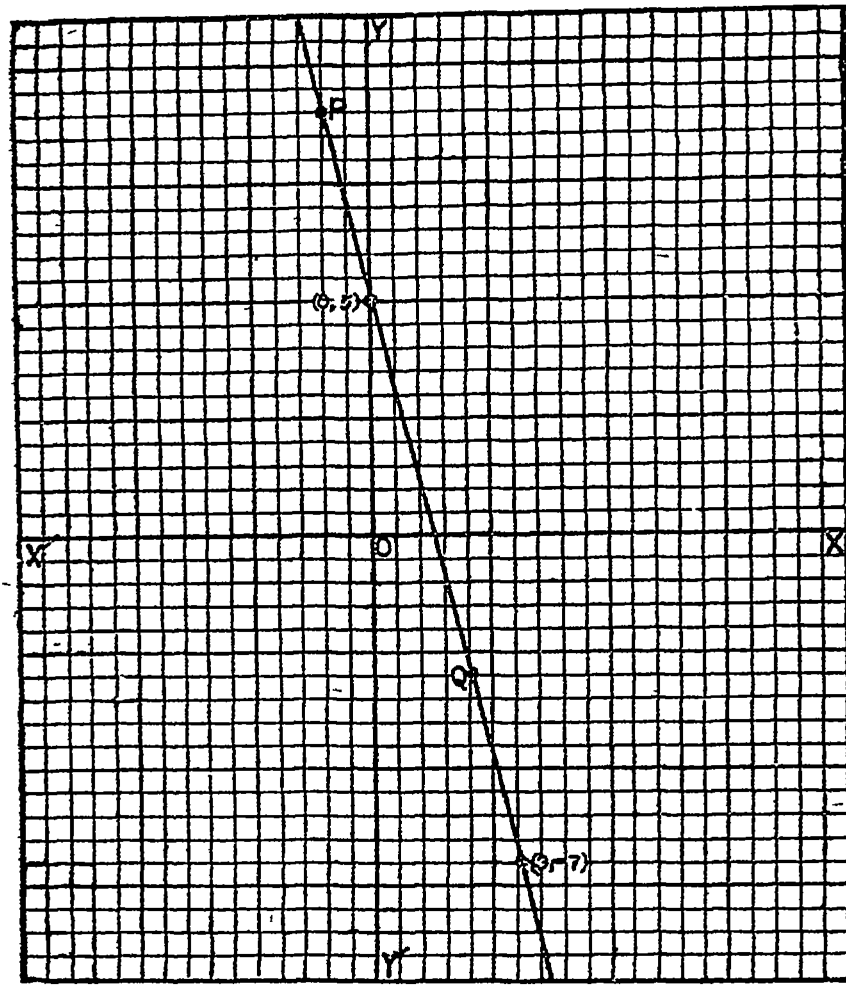
Example 3. If a point moves in such a way that its x and y are invariably connected by the relation $y = -4x + 5$, find the path along which the point will move.

From the given relation,

$$\left.\begin{array}{ll}\text{when} & x = 0\\ \text{we have} & y = 5\end{array}\right\}, \qquad \left.\begin{array}{ll}\text{and when } x = & 3\\ \text{we have} & y = -7\end{array}\right\}.$$

Evidently therefore $(0, 5)$ and $(3, -7)$ are two positions of the moving point

Let twice the side of a small square represent the unit of length. Join the points (0, 5) and (3, −7), and produce the straight line both ways Then this straight line will be the required path



Take a point P on this straight line The co-ordinates of P, which are found to be −1 and 9, satisfy the given relation Take another point Q on the straight line, its co-ordinates also, which are found to be 2 and −3, satisfy the given relation Similarly, the co-ordinates of any other point on this straight line may be shewn to satisfy the

given relation But if a point be taken outside the line PQ, its co-ordinates will *not* satisfy the given relation, as can be easily seen Hence the moving point will always be on the line PQ and never stray out of it

Thus it is found that if a point moves in such a manner that its co-ordinates always satisfy the equation $y=-4x+5$, the path along which the point will move is the line PQ. In other words, **the line PQ is the Graph of the equation $y=-4x+5$**

Note 1 Generally speaking the graph of the equation $y=mx+c$ where m and c are any given numbers is a straight line passing through the point (o, c)

Note 2 As every equation of the first degree in x and y can be reduced to the form $y=mx+c$, it is clear that graphs of all simple equations are straight lines

Note 3 The graph of the *equation* $y=mx+c$ is also said to be the graph of the *expression* $mx+c$

Note 4 The graph of any given equation may be defined to be the path described by a point which moves in such a manner that in every position of the point its co-ordinates satisfy the given equation

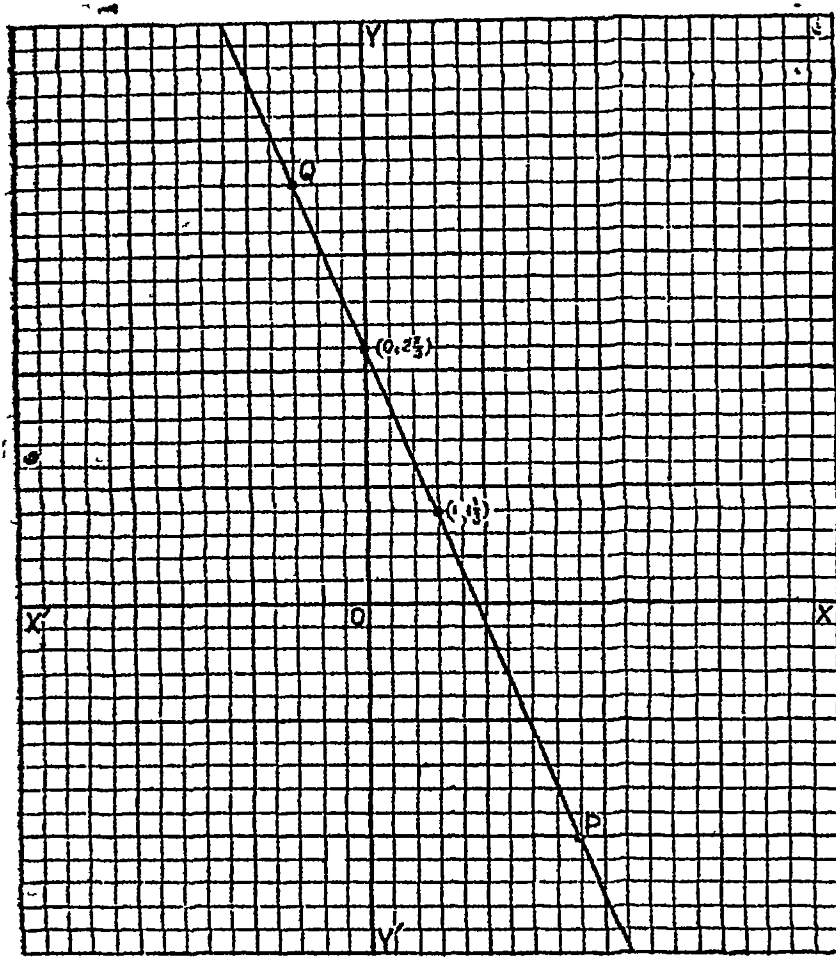
Example 4 Draw the graph of the equation $7x+3y=11$.

When $\left.\begin{array}{l} x=0, \\ y=3\frac{2}{3} \end{array}\right\}$, when $\left.\begin{array}{l} x=1, \\ y=1\frac{1}{3} \end{array}\right\}$.

Evidently therefore $(0, 3\frac{2}{3})$ and $(1, 1\frac{1}{3})$, are two points on the graph

Let 3 times the side of a small square represent the unit of length Join the points $(0, 3\frac{2}{3})$ and $(1, 1\frac{1}{3})$, and produce the straight line both ways Then this straight line will be the required graph (See diagram on page 517).

Take any point P on the line, its co-ordinates, which are found to be 3 and $-3\frac{1}{3}$, satisfy the given relation. Take any other point Q on the line, its co-ordinates also, which are found to be -1 and 6, satisfy the given relation Similarly it may be shewn that the co-ordinates of any point that may be taken on the line PQ will satisfy the given relation, but the co-ordinates of any point which is outside PQ will *not* Hence the line PQ is the required graph.



Note 1. The graph of the *equation* $7x+3y=11$ is also said to be the graph of the *expression* $\dfrac{11-7x}{3}$

Note 2 The straight line PQ being the graph of the equation $7x+3y=11$, this equation is said to be the equation of the straight line PQ

Note 3 Hence the equation of a given straight line means the equation which is satisfied by the co-ordinates of every point on that straight line

Example 5. Find the equation of the straight line which passes through the points $(1, 1)$ and $(3, -\frac{1}{2})$.

Let $y = mx+c$ be the required equation.

This equation being satisfied by (1, 1) and also by $(3, -\frac{1}{2})$, we must have

$$\left.\begin{aligned} 1 &= m+c \\ \text{and} \quad -\tfrac{1}{2} &= 3m+c \end{aligned}\right\}. \quad \begin{aligned} &\text{Hence, } 2m = -\tfrac{3}{2}, \text{ and } \therefore m = -\tfrac{3}{4}, \\ &\text{whence } c = 1+\tfrac{3}{4} = \tfrac{7}{4} \end{aligned}$$

Thus the required equation is, $y = -\frac{3}{4}x + \frac{7}{4}$

or $3x+4y = 7$.

Exercise (5).

1. Draw the graphs of the following equations —

(1) $x = 8$ (2) $x = 13$ (3) $x+11 = 0$.
(4) $y = -7$. (5) $y-9 = 0$. (6) $y+10 = 0$.

2. Draw the graphs of the following equations —

(1) $y = x$. (2) $y = -x$ (3) $y = 2x$.
(4) $y+2x = 0$. (5) $y = -3x$. (6) $3y = 5x$
(7) $7y+8x = 0$ (8) $6y+13x = 0$

3. Draw the graphs of the following equations :—

(1) $y = 3x+4$. (2) $y = 7x-8$ (3) $y = -5x+9$.
(4) $y = -8x-11$. (5) $3y = 7x+4$ (6) $-6y = 7x-10$

4. Draw the graphs of the following equations —

(1) $2x+7y = 10$ (2) $4x-5y-7 = 0$
(3) $5x+6y+8 = 0$. (4) $-3x+7y+8 = 0$
(5) $10y-9x = 13$. (6) $8x-11y+13 = 0$

5. Draw the graphs of the following equations —

(1) $\dfrac{x}{3}+\dfrac{y}{4} = 1$. (2) $\dfrac{x}{7}+\dfrac{y}{-9} = 1$ (3) $\dfrac{x}{-8}+\dfrac{y}{13} = 1$.

(4) $y = \dfrac{5-7x}{6}$. (5) $y = \dfrac{9x-13}{4}$ (6) $\dfrac{3x}{4}-\dfrac{4y}{3} = 1$.

6. Draw the graphs of the following expressions :—

(1) $x-3$ (2) $3x+4$ (3) $-7x-8$

(4) $\dfrac{7-4x}{3}$. (5) $\dfrac{5x-9}{4}$. (6) $\dfrac{8x+11}{5}$.

7. Find the equation of the straight line which passes through each of the following pairs of points —

(1) (0, 0), (5, 6). (2) (0, 5), (7, 0).

(3) (6, −8), (−7, 5) (4) (−4, 8), (−9, −13).

(5) (−11, 0), (7, −10)

6. Graphical Solution of Equations.

Example. Solve graphically—

$$\left.\begin{aligned} 2x-7y+12 &= 0 \\ 3x+2y &= 32 \end{aligned}\right\}$$

Let us draw the graphs of the two equations.

We find that

$\left.\begin{aligned} x &= -6 \\ y &= 0 \end{aligned}\right\}$, $\left.\begin{aligned} x &= 1 \\ y &= 2 \end{aligned}\right\}$ are points on the graph of the 1st. equation;

whilst

$\left.\begin{aligned} x &= 0 \\ y &= 16 \end{aligned}\right\}$, $\left.\begin{aligned} x &= 6 \\ y &= 7 \end{aligned}\right\}$ are points on the graph of the 2nd. equation.

Hence, taking the length of a side of a small square as the unit of length, the two graphs are as shewn on the next page

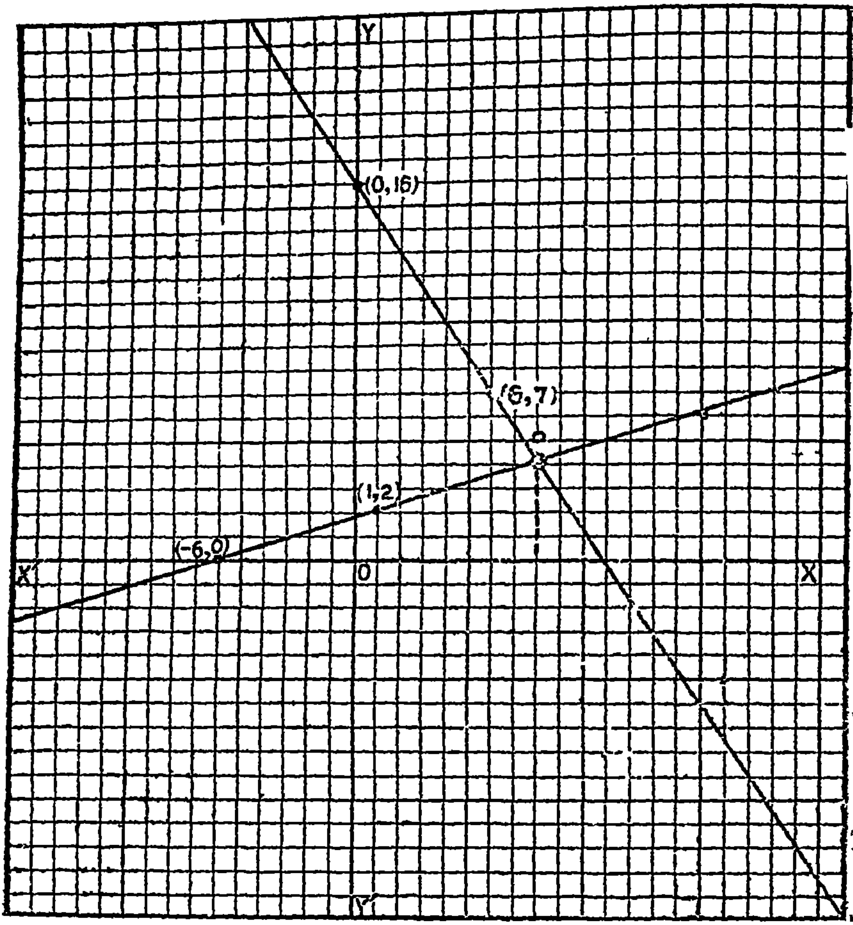
Let P be the point where the two graphs intersect P being common to the graphs, its co-ordinates will satisfy both the given equations

Now the co-ordinates of P are found to be 8 and 4.

Hence, $\left.\begin{aligned} x &= 8 \\ y &= 4 \end{aligned}\right\}$ is the required solution.

Note 1 By actual verification we find that both the equations are satisfied when $x = 8$ and $y = 4$

Note 2 If it is required to "solve graphically the equation $\frac{x-3}{5} = \frac{3x-22}{2}$," all that we have to do is to draw the graphs of the expressions $\frac{x-3}{5}$ and $\frac{3x-22}{2}$ and take the abscissa of the point common to the two graphs



Exercise (6).

Solve the following equations graphically —

1. $x+y=9$, $3x-2y=7$. 2. $4x+3y=13$, $3x+2y=11$.

3. $\frac{x}{4}+\frac{y}{3}=4$, $4x-5y=2$.

4. $y-x=2$, $3x-2y=5$

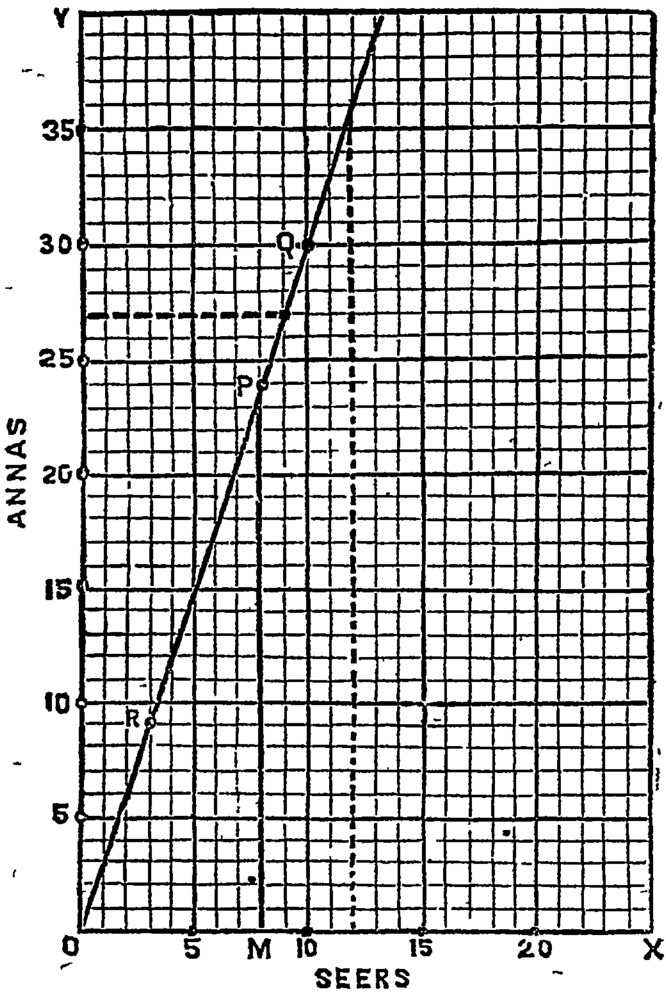
5. $5x-3y=11$, $2y-3x+4=0$.

6. $\frac{x-2}{2}=\frac{-5x+4}{5}$. 7. $\frac{2x+7}{3}=\frac{3x-7}{2}$.

8. $\frac{4x-3}{5}=\frac{6x}{7}-1$.

7. Graphical Problems.

Example 1. Given that the price of a seer of rice is three annas, shew that a straight line can be drawn such that if any point be taken on it, the abscissa of the point will represent the quantity of rice of which the price is represented by the ordinate



In the above figure let the length of a side of a small square measured along OX represent one seer, and let an equal length measured along OY represent one anna Then the meaning of the figures along OX and OY is clear

Since the price of a seer is 3 annas, the price of 8 seers must be 24 annas. Clearly therefore P is a point such that its abscissa OM represents a quantity of rice of which the price is represented by the ordinate PM.

Join OP and produce it Then this is the straight line every point on which will satisfy a condition similar to that satisfied by P

Q is the point (10, 30) , consequently its abscissa represents a quantity of rice of which the price is represented by its ordinate R is the point (3, 9) ; its abscissa therefore represents a quantity of rice of which the price is represented by its ordinate Similarly this is true of every point on the line OP

Hence OP is the required straight line

Note 1 The line OP is called the graph of the price of rice, or more simply the price-graph of rice

Note 2 The graph enables us to determine readily the price of any given number of seers of rice For instance, if the abscissa be taken to be 12, the ordinate is immediately found to be 36 thus we know that the price of 12 seers of rice is 36 annas Similarly for any other abscissa the corresponding ordinate can be immediately found

Note 3 The graph also enables us to determine quickly the number of seers of rice that can be had for any given price For instance, if the ordinate is taken to be 27, the corresponding abscissa is immediately found to be 9, which shews that we can have 9 seers of rice for 27 annas

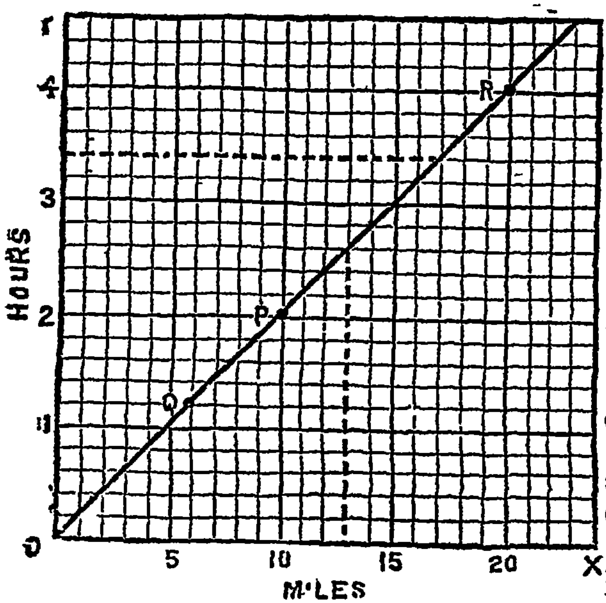
Example 2. A person, named B, starting from a given spice, travels at the rate of 5 miles an hour Shew that a straight line can be drawn such that if any point be taken on it, the abscissa of the point will represent the number of miles that B travels in the time represented by the ordinate

In the figure on the next page let the length of a side of a small square measured along OX represent one mile, and let an equal length measured along OY represent 12 minutes. Then the meaning of the figures along OX and OY is clear.

Since B travels 5 miles in one hour, he travels 10 miles in 2 hours Clearly therefore P is a point such that its abscissa represents the number of miles that the person travels in the time represented by its ordinate

Join OP and produce it. Then this is the straight line every point on which will satisfy a condition similar to that satisfied by P.

Let Q be any point on the line. Its abscissa represents 6 miles and ordinate represents 1 hour 12 minutes, but we *know* that the person travels 6 miles in 1 hour 12 minutes. Hence Q satisfies the condition above mentioned.



Let R be some other point on the line Its abscissa represents 20 miles and ordinate represents 4 hours; but we *know* that the person travels 20 miles in 4 hours Hence R also satisfies the proposed condition.

Similarly for any other point on the line. Hence OP is the required straight line

Note 1 The line OP is called the graph of B's motion, or the motion-graph of B

Note 2 The graph enables us to determine readily the time in which B travels any given number of miles For instance, if the abscissa be taken which represents 13 miles, the corresponding ordinate is immediately found to be that which represents 2 hours 36 minutes, thus it is known that the time taken by the person to travel 13 miles is 2 hours 36 minutes

Note 3 The graph also enables us to determine readily the number of miles that the person travels in any given time For instance, if the ordinate be taken which represents 3 hours 24 minutes, the corresponding abscissa is immediately found to be that which represents 17 miles, thus it is known that in 3 hours and 24 minutes the person travels 17 miles

Example 3. If one inch be equal in length to 2 5 centimetres, shew that a straight line can be drawn such that the abscissa of any point on the line will represent the number of inches that are equivalent to the number of centimetres represented by the ordinate.

In the figure on the next page let the length of a side of a small square measured along OX represent one inch,

and let an equal length measured along OY represent one centimetre Then the meaning of the figures along OX and OY is clear.

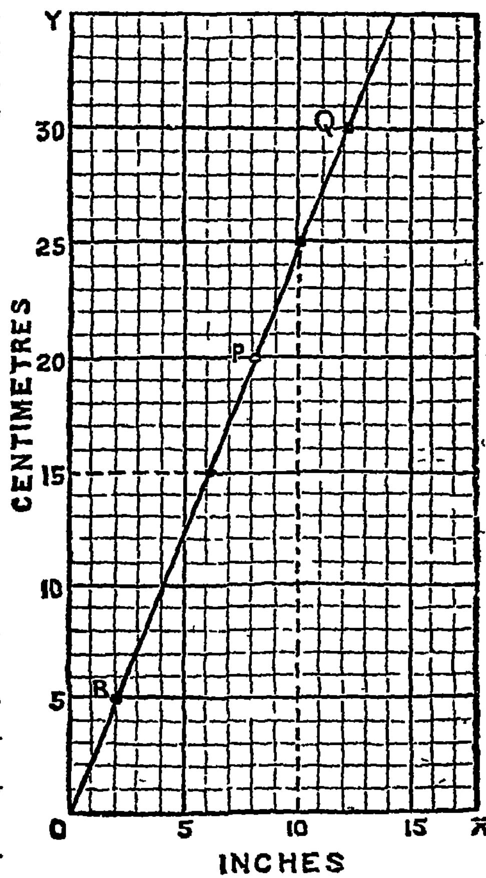
Since 1 inch = 2 5 centimetres, we have 8 inches = 20 centimetres Clearly therefore P is a point such that its abscissa represents the number of inches that are equivalent to the number of centimetres represented by its ordinate

Join OP and produce it. Then this is the straight line every point on which will satisfy a condition similar to that satisfied by P

Let Q be any point on the line Its abscissa represents 12 inches, whilst its ordinate represents 30 centimetres, but we *know* that these two are equivalent Hence Q satisfies the condition above mentioned

Let R be some other point on the line Its abscissa represents 2 inches, whilst its ordinate represents 5 centimetres, but we *know* that these two are equivalent Hence R also satisfies the proposed condition

Similarly for any other point on the line Hence OP is the required straight line.



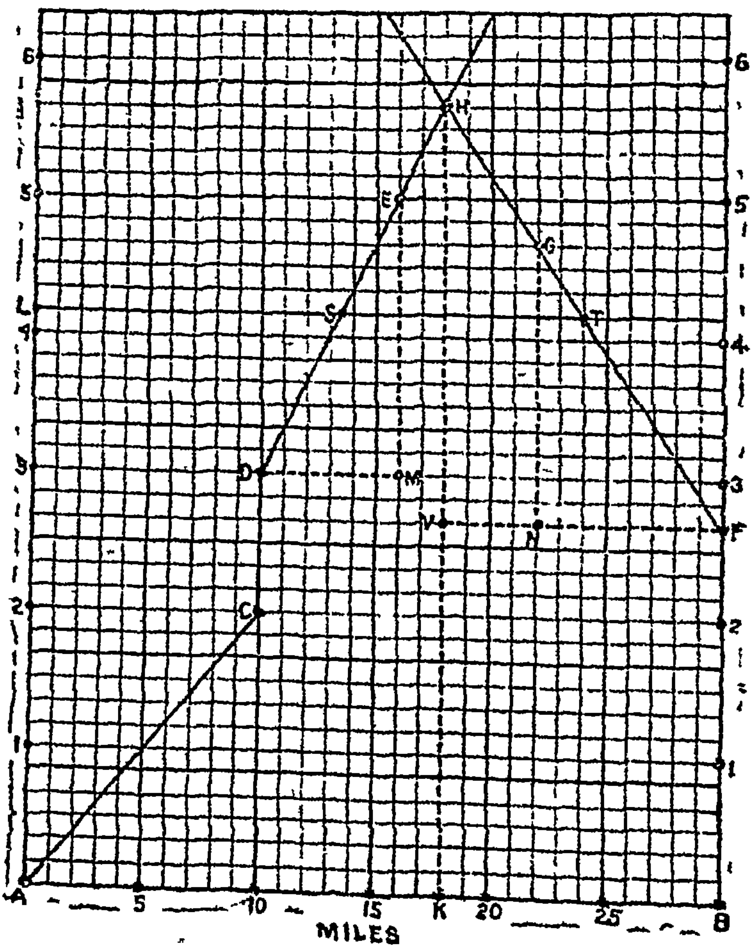
Note 1 The line OP is called the graph for converting inches into centimetres and *vice versa*, or more briefly the conversion-graph for inches and centimetres

Note 2 The graph enables us to determine readily the number of centimetres that are equivalent to any given number of inches. For instance, if the abscissa be taken which represents 10 inches, the corresponding ordinate is immediately found to be that which

represents 25 centimetres; thus it is known that 10 inches are equivalent to 25 centimetres.

Note 3. The graph also enables us to determine readily the number of inches that are equivalent to any given number of centimetres. For instance, if the ordinate be taken which represents 15 centimetres, the corresponding abscissa is immediately found to be that which represents 6 inches; thus it is known that 15 centimetres are equivalent to 6 inches.

Example 4. A and B are two stations 30 miles apart. P starts from A and travels towards B at the rate of 5 miles an hour; at the end of 2 hours he takes rest for one hour, and then resumes his journey at the rate of 3 miles an hour. Q leaves B 2 hours 40 minutes after P leaves A, and travels towards A, without stoppage, at the rate of 4 miles an hour. When and where will the two travellers meet?



Let the length of a side of a small square measured horizontally represent one mile, and let an equal length measured vertically represent 10 minutes Then the meaning of the figures along the lines in the above diagram is clear.

(i) P starts from A, and travelling at the rate of 5 miles an hour, completes 10 miles in 2 hours Hence, if the point C be taken such that its co-ordinates respectively represent 10 miles and 2 hours, AC is the graph of P's motion for the first two hours

The graph for the 3rd hour must be such that the abscissa of any point on it may represent 10 miles, because P is supposed to be at rest throughout this hour Hence CD drawn vertically to represent one hour, as in the diagram, will be the graph of P's rest

After the 3rd hour P travels at the rate of 3 miles an hour Hence if DM be taken to represent 6 miles and ME to represent 2 hours, the straight line DE is the graph of P's motion after the 3rd hour.

Thus the broken line ACDE is the complete graph of P's motion.

(ii) Q starts from B 2 hours 40 minutes after P leaves A Hence if BF be measured vertically to represent 2 hours 40 minutes, BF may be regarded as the graph of Q's rest at B.

When Q leaves B he moves towards A at the rate of 4 miles an hour Hence if FN be taken to represent 8 miles and NG to represent 2 hours, the straight line FG will be the graph of Q's motion.

(iii) Let the two graphs intersect at H, and draw HK perpendicular to AB Produce FN to meet HK at V

Now it is clear that at the end of time HK, P will have gone a distance AK towards B, and Q will have gone a distance BK (*i e*, FV) towards A Hence they will meet at this instant Thus the required time of meeting = that represented by HK = 5 hours 40 minutes after the commencement of P's motion

Also, the distance of the place of meeting from A = that represented by AK = 18 miles

Note 1 As HV represents 3 hours, it is clear that P and Q meet at the end of 3 hours after Q starts from B.

Note 2 The horizontal line through L meets the graphs at the points S and T As AL represents 4 hours 10 minutes and ST represents $10\frac{1}{2}$ miles, it is clear that at the end of 4 hours 10 minutes from the commencement of P's motion, P and Q are at a distance of $10\frac{1}{2}$ miles from each other

Exercise (7).

1. If milk sells for 4 annas per seer, construct the price-graph of milk, giving the price of any quantity of milk up to 5 seers From the graph read off the price of 3 seers and 5 chattacks of milk, and also the quantity of milk that can be had for 10 annas and 9 pies.

2. If *Fazli* mangoes be worth one rupee two annas a dozen, construct a price-graph for mangoes, giving the price of any number up to 30 Read off from the graph the price of 17 mangoes, and also the number of mangoes that can be had for 1 Re 12 as 6 p

3. If a man walks at the rate of 4 miles an hour construct a graph of his motion Read off from the graph the time in which he travels 13 miles, and also the number of miles he travels in $4\frac{3}{4}$ hours.

4. If one cubit be equal to 1 5 feet, construct a conversion-graph for cubits and feet Read off from the graph the number of feet that are equivalent to $5\frac{1}{4}$ cubits, and also the number of cubits that are equivalent to $6\frac{3}{4}$ feet.

5. A starts from a place and walks in a given direction at the rate of 3 miles an hour; B starts from the same place one hour later and moves in the same direction at the rate of 5 miles an hour. Draw the motion-graphs of A and B, and find when and where B overtakes A.

6. A and B are two stations 20 miles apart. P starts from A and travels towards B at the rate of 3 miles an hour, whilst Q starting from B travels towards A at the rate of 2 miles an hour Construct the motion-graphs of P and Q and find when and where they meet.

7. Fifty articles of the same kind cost Rs. 3 2 as Construct a graph from which you can read off the cost of any number of articles up to 50 Hence find the cost of 19 articles, and the number of articles that you would get for Rs 2 7 as.

8. Given that 1 kilogramme = 2 2 lbs, construct a graph which will enable you to read off the number of kilo-

grammes that are equivalent to any given number of lbs. up to 15 lbs Read off the number of kilogrammes in 11 lbs.

9. A man travels for 3 hours at the rate of 2 miles an hour, at the end of which he takes rest for an hour and a half, and then starts to walk at the rate of two and a half miles an hour. Construct the graph of his motion.

10. A man starts from a place B to walk towards C at the rate of 4 miles an hour. After 3 hours he changes his mind and walks back towards B at the rate of 3 miles an hour At the end of 2 hours again he suddenly changes his mind and begins to run towards C at the rate of 7 miles an hour Draw a graph of his motion

11. A, B and C are three stations in order on the same road, the distance between A and B being 6 miles. Q starts from B at noon to walk towards C at the rate of 3 miles an hour, and at 1-30 P M P starts from A to run towards B at the rate of $6\frac{1}{2}$ miles an hour. Draw graphs of their motion, and find when and where P will overtake Q

CHAPTER II

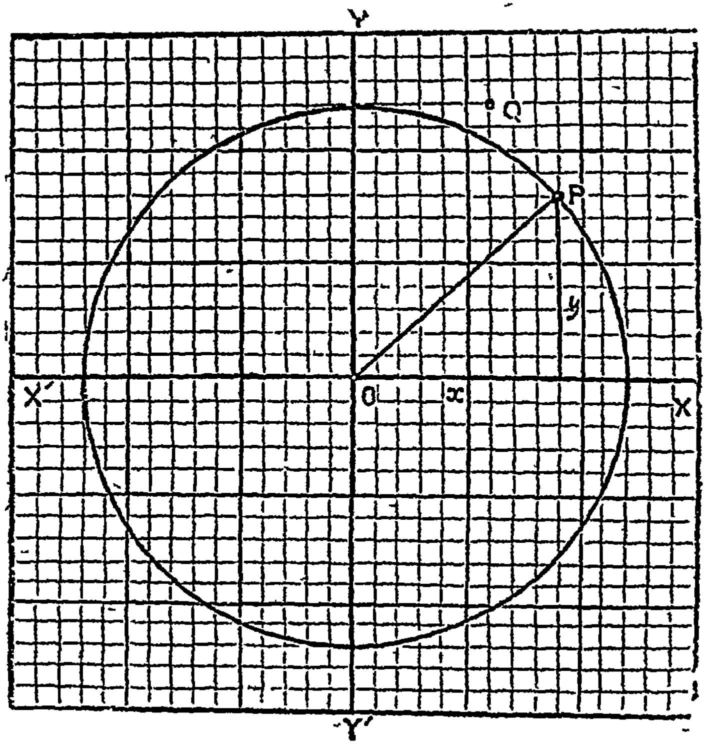
GRAPHS (*Continued*).

1. Draw the graph of the equation $x^2+y^2=36$.

Let twice the length of a side of small square represent the unit of length.

With centre O and a radius equal to 6 units of length describe a circle, as in the diagram on the next page Then this circle will be the required graph.

Take *any* point P on the circle, and let its co-ordinates be denoted by x and y, evidently then $x^2+y^2=OP^2=36$. But if a point, such as Q, be taken anywhere *not on the circle*, it is easy to see that its co-ordinates will *not* satisfy the given equation.



Thus it is shewn that the co-ordinates of every point on the circle, and of no other point, satisfy the given equation. Hence the circle drawn is the required graph.

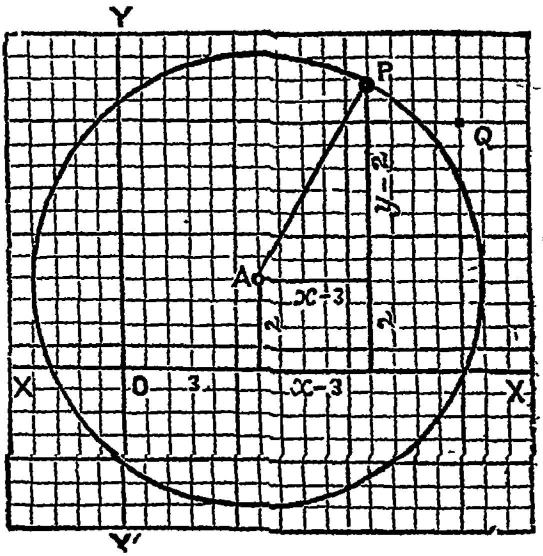
2. Draw the graph of the equation $(x-3)^2+(y-2)^2=25$.

Let twice the length of a side of a small square represent the unit of length.

Let A be the point (3, 2) With centre A and a radius equal to 5 units of length describe a circle as in the diagram on the next page Then this circle will be the required graph.

Take *any* point P on the circle, and let its co-ordinates be denoted by x and y Now from the diagram it is clear that AP is the hypotenuse of a right-angled triangle of which the sides are $(x-3)$ and $(y-2)$ units of length respectively.

Hence, $(x-3)^2+(y-2)^2 = AP^2 = 25$, which shews that the co-ordinates of P satisfy the given equation. But if a point such as Q, be taken anywhere *not on the circle*, it is easy to see that its co-ordinates will *not* satisfy the given equation



Thus it is clear that the co-ordinates of every point on the circle and of no other point, *satisfy the given equation.* Hence the circle described is the required graph.

Note 1 It may be similarly shewn that the graph of the equation $(x+2)^2+(y+5)^2 = 49$ is a circle of which the centre is the point $(-2,-5)$ and the radius is equal to 7 units of length

Note 2 The equation $x^2+y^2-8x+10y+25 = 0$ can be easily reduced to the from $(x-4)^2+(y+5)^2 = 16$ Hence its graph is a circle of which the centre is the point $(4, -5)$ and the radius is equal to 4 units of length.

3. Draw the graph of the equation $y^2 = 4x^2$.

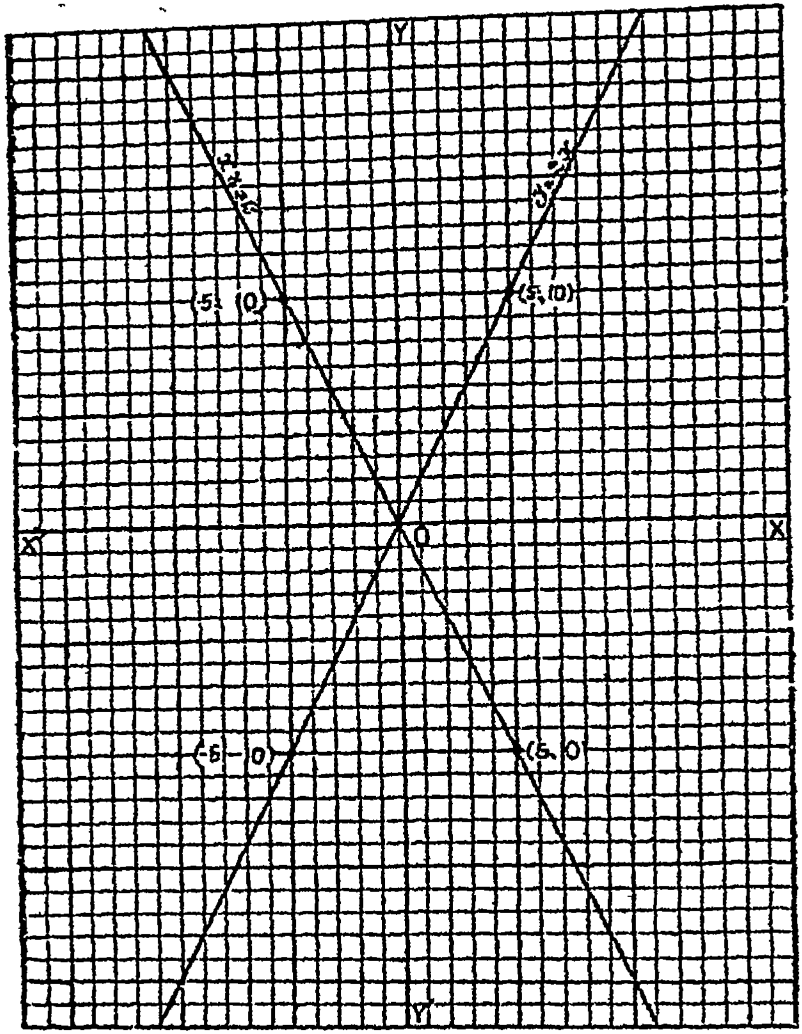
From the given equation we have

$$\left.\begin{aligned} y^2-4x^2 &= 0 \\ \text{or, } (y+2x)(y-2x) &= 0 \end{aligned}\right\}.$$

Hence, it is clear that the given equation is satisfied by (1) all those points which satisfy the equation $y+2x = 0$, and also (2) by all those points which satisfy the equation $y-2x = 0$.

Hence, the required graph consists of *two straight lines*, one being the graph of the equation $y+2x = 0$, and the other being the graph of the equation $y - 2x = 0$

Hence the required graph is as shewn below :—



4. Draw the graph of the equation $4x^2+9y^2 = 36$.

(1) When $x=0$, we have $y^2=4$, ann therefore $y=\pm 2$. Hence the points (0, 2) and (0, -2) are on the required graph

(2) When $y=0$, we have $x^2=9$, and therefore $x=\pm 3$. Hence the points (3, 0) and (-3, 0) are on the required graph.

(3) When $x = \pm 1$, we have $9y^2 = 32$, and therefore

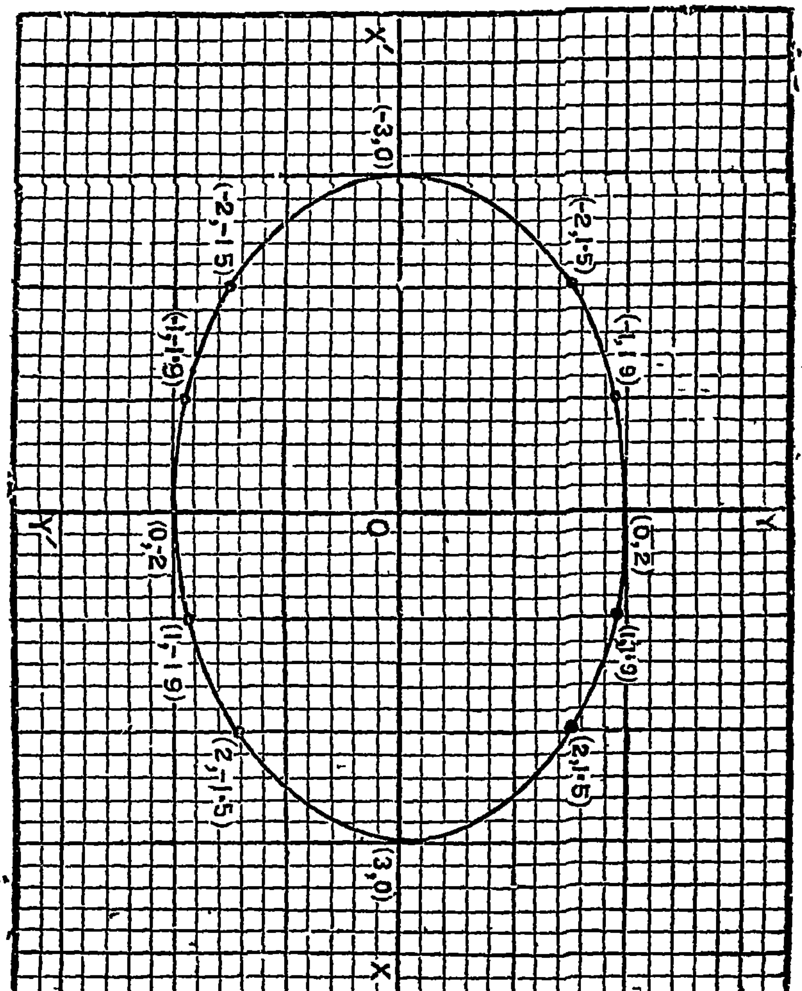
$$5y = \pm \frac{4}{3}\sqrt{2} = \pm \frac{4 \times 1{\cdot}414}{3} = \pm \frac{5{\cdot}656}{3} = \pm 1{\cdot}885 \quad = \pm 1{\cdot}9$$

approximately Hence the four points $(1, 1{\cdot}9)$, $(1, -1{\cdot}9)$, $(-1, 1{\cdot}9)$ and $(-1, -1{\cdot}9)$ are on the required graph

(4) When $x = \pm 2$, we have $9y^2 = 20$, and therefore

$$y = \pm \frac{2}{3}\sqrt{5} = \pm \frac{2 \times 2{\cdot}236}{3} = \pm \frac{4{\cdot}472}{3} = \pm 1{\cdot}490 \quad = \pm 1{\cdot}5$$

nearly. Hence the four points $(2, 1{\cdot}5)$, $(2, -1{\cdot}5)$, $(-2, 1{\cdot}5)$ and $(-2, -1{\cdot}5)$ are on the required graph



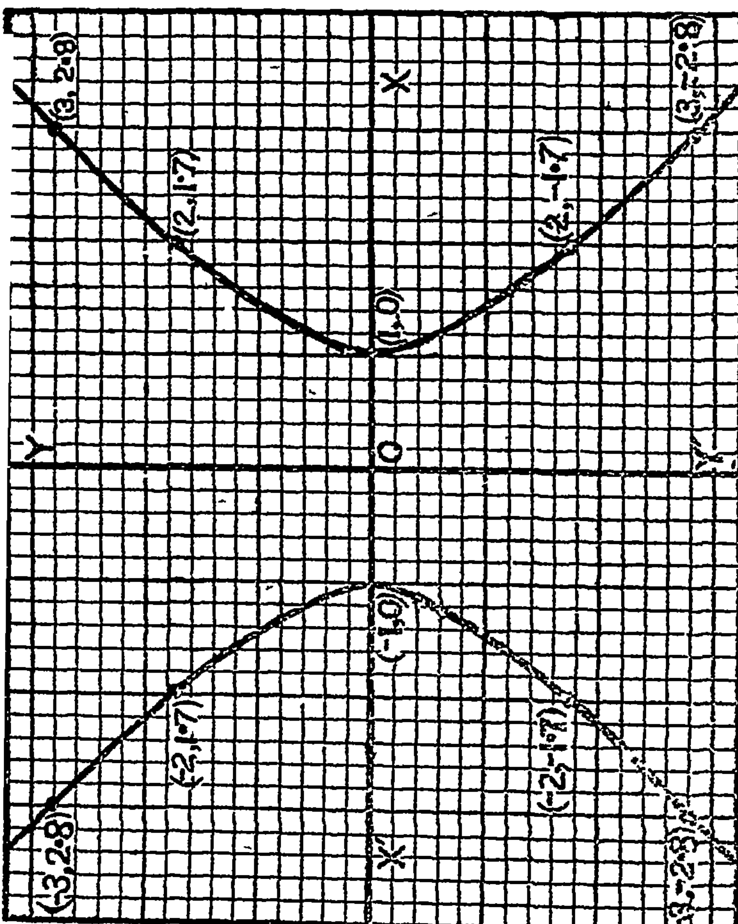
Let us now plot the twelve points as found above (taking 5 times the side of a small square as the unit of length) and draw a free-hand curve through them, as in the diagram on the last page

The curve so drawn is the required graph

Note 1 Evidently the curve is *symmetrical* about the axis of x, *i e*, every chord at right angles to the axis of x is bisected by it Similarly the curve is also *symmetrical* about the axis of y

Note 2 The curve lies entirely within the space enclosed by the four straight lines $x = 3$, $x = -3$, $y = 2$, $y = -2$ A curve of this class is called an Ellipse.

5. Draw the graph of the equation $x^2 - y^2 = 1$.



(1) When $x = 0$, we have $y^2 = -1$ and therefore y is *imaginary* This shews that the graph does not cut the axis of y

(2) When $y = 0$, we have $x^2 = 1$, and therefore $x = \pm 1$. Hence the points (1, 0) and (−1, 0) are on the required graph.

(3) When $x = \pm 2$, we have $y^2 = 3$, and therefore $y = \pm \sqrt{3} = \pm 1732 .. = \pm 1 7$ approximately. Hence the four points (2, 1 7), (2, −1·7), (−2, 1·7) and (−2, −1 7) are on the required graph.

(4) When $x = \pm 3$, we have $y^2 = 8$, and therefore $y = \pm 2\sqrt{2} = \pm 2 \times 1{\cdot}414 = \pm 2\,828 = \pm 2\,8$ approximately. Hence the four points (3, 2 8), (3, −2 8), (−3, 2 8) and (−3, −2 8) are on the required graph.

Let us now plot the ten points as found above (taking 5 times the side of a small square as the unit of length) and draw a free-hand curve through them, as in the diagram on the last page.

The curve so drawn is the required graph.

Note 1 The curve so drawn is evidently symmetrical about the axis of x and also about the axis of y

Note 2 The curve consists of two branches, one lying entirely on the right of the line $x = 1$ and the other lying entirely on the left of the line $x = -1$ A curve of this class is called a Hyperbola

Note 3 From articles 1, 3, 4 and 5 it may be easily seen that if the equation $ax^2 + by^2 = c$ be taken, (1) the graph is *two straight lines passing through the origin* when c is zero, and a and b are of different signs, (2) the graph is a *circle* when $\frac{a}{c}$ and $\frac{b}{c}$ are positive and equal, (3) the graph is an *Ellipse* when $\frac{a}{c}$ and $\frac{b}{c}$ are positive and unequal, and (4) the graph is a *Hyperbola* when $\frac{a}{c}$ and $\frac{b}{c}$ are of different signs (their absolute values being either equal or unequal.)

6 Draw the graph of the equation $y = x^2$, taking the unit for measuring y equal to half that for measuring x

Evidently the following points are on the required graph :—

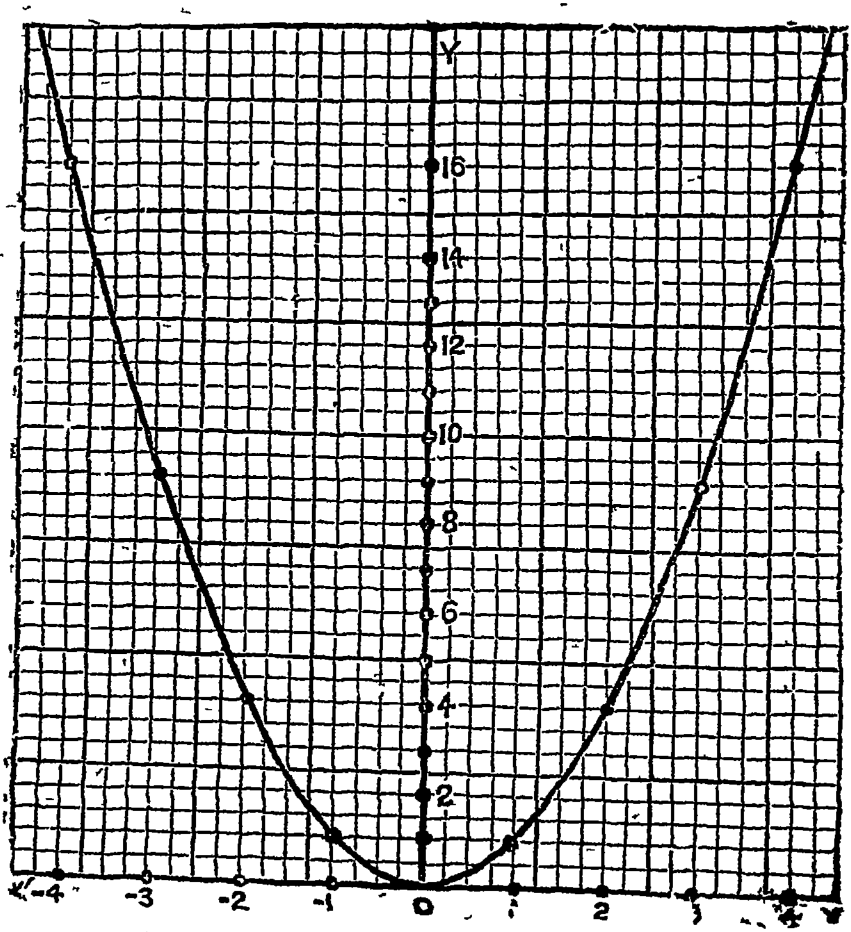
$$\left.\begin{matrix} x = 0 \\ y = 0 \end{matrix}\right\}, \qquad \left.\begin{matrix} x = 1 \\ y = 1 \end{matrix}\right\}, \qquad \left.\begin{matrix} x = -1 \\ y = 1 \end{matrix}\right\},$$

$$\begin{rcases} x = 2 \\ y = 4 \end{rcases}, \quad \begin{rcases} x = -2 \\ y = 4 \end{rcases}, \quad \begin{rcases} x = 3 \\ y = 9 \end{rcases},$$

$$\begin{rcases} x = -3 \\ y = 9 \end{rcases}, \quad \begin{rcases} x = 4 \\ y = 16 \end{rcases}, \quad \begin{rcases} x = -4 \\ y = 16 \end{rcases}.$$

Let four times the sides of a small square (*i.e.* ·4 of an inch) be the unit for measuring x and twice the side of a small square (*i.e.*, ·2 of an inch), the unit for measuring y.

Let us now plot the points found above and draw a curve through them free-hand, as in the following diagram :—



The curve so drawn is the required graph.

Note 1 If the unit for measuring y were the same as that for measuring x (*i e*, 4 of an inch), the curve drawn would be the graph of the equation $y = \frac{1}{2}x^2$, or that of $2y = x^2$.

Note 2 Every chord drawn perpendicular to OY is bisected by it as can be easily verified. Hence the curve drawn above is symmetrical about the axis of y This is also evident from the fact that if the paper be folded about OY the left-hand portion of the curve entirely coincides with the right-hand portion

Note 3 The curve lies entirely above the axis of x, and extends upwards to infinity It is easy to see that the graph of the equation $y = -x^2$ would be a curve lying entirely *below* the axis of x and extending downwards to infinity

Note 4 The abscissa of any point on the curve is evidently the square root of the ordinate Hence when the graph of the equation $y = x^2$ is drawn, by measuring the abscissa of any point on the graph we can determine the ***square root*** of the number which represents the ordinate

Note 5 A curve of this class is called a Parabola.

7. Draw the graph of the expression $3 - 4x - 2x^2$.

The required graph is the same as that of the equation $y = 3 - 4x - 2x^2$.

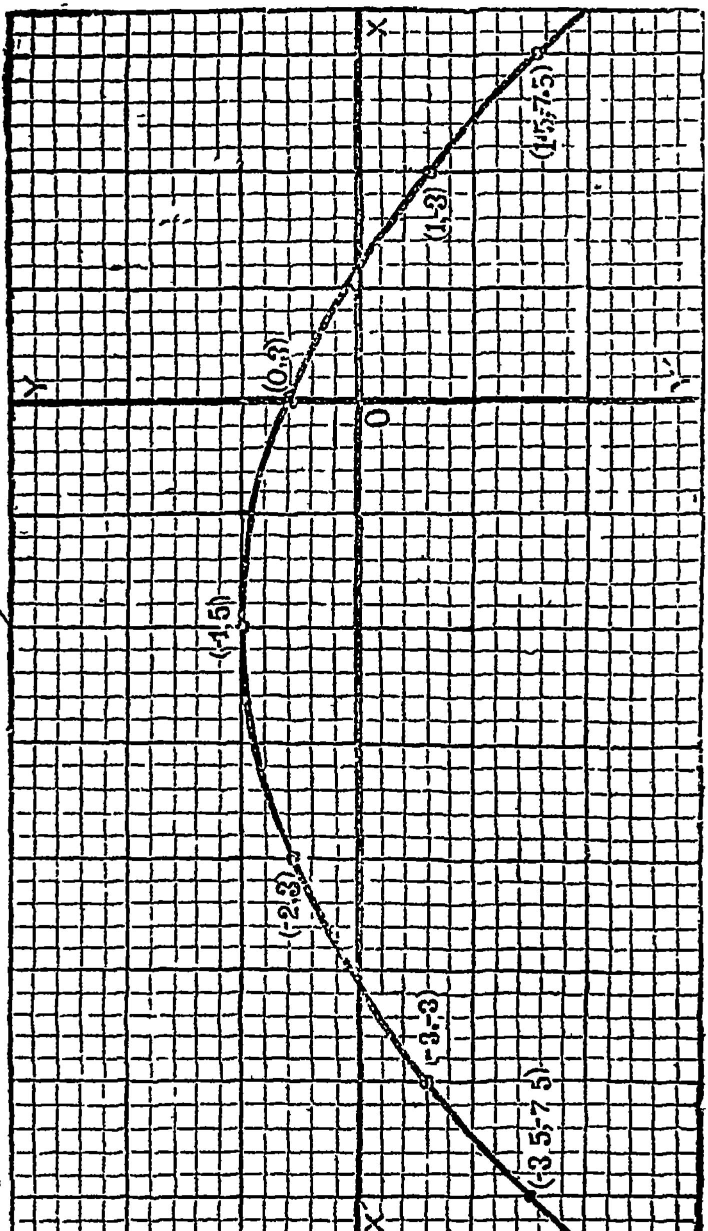
It is easy to see that the following points are on the required graph —

$$\left.\begin{matrix} x = 0 \\ y = 3 \end{matrix}\right\}, \quad \left.\begin{matrix} x = 1 \\ y = -3 \end{matrix}\right\}, \quad \left.\begin{matrix} x = 1\cdot 5 \\ y = -7\cdot 5 \end{matrix}\right\},$$

$$\left.\begin{matrix} x = -1 \\ y = 5 \end{matrix}\right\}, \quad \left.\begin{matrix} x = -2 \\ y = 3 \end{matrix}\right\}, \quad \left.\begin{matrix} x = -3 \\ y = -3 \end{matrix}\right\}, \quad \left.\begin{matrix} x = -3\cdot 5 \\ y = -7\cdot 5 \end{matrix}\right\}$$

Take one inch as the unit for measuring x, and one-tenth of an inch as the unit for measuring y

Let us now plot the above points and draw a curve through them free-hand, as in the following diagram —



The curve so drawn is the required graph.

Note 1 Since $3-4x-2x^2=3-2(x^2+2x)=5-2(x^2+2x+1)$ $=5-2(x+1)^2$, the equation may also be written as

$$y=5-2(x+1)^2,$$

which shows that for all values of x, y is *less than* 5, except when $x=-1$, and in this case $y=5$ This is also clear from the curve drawn Hence, the maximum value of y (*i e*, that of the expression $3-4x-2x^2$) is 5

Note 2 If through the point $(-1, 5)$ a straight line be drawn parallel to the axis of y, it is easy to see that the curve is symmetrical about this straight line

Note 3 From the figure it is evident that $y=0$ when x is approximately equal to $\cdot 6$ or $-2\cdot 6$ Hence, $3-4x-2x^2=0$ when $x=\cdot 6$ or $-2\cdot 6$ approximately, in other words, the roots, of the equation $3-4x-2x^2=0$ are $\cdot 6$ and $-2\cdot 6$ approximately From this it is clear that *the roots of the equation* $3-4x-2x^2=0$ *are the abscissa of the points where the graph of the expression* $3-4x-2x^2$ *cuts the axis of* x

Note 4 The graph of any expression of the form ax^2+bx+c is a *Parabola*

8. Draw the graph of the equation $xy = 1$.

It is easy to see that the following points are on the required graph —

$$\left.\begin{array}{l}x=\cdot 1\\ y=10\end{array}\right\},\quad \left.\begin{array}{l}x=\cdot 2\\ y=5\end{array}\right\},\quad \left.\begin{array}{l}x=\cdot 4\\ y=2\cdot 5\end{array}\right\},$$

$$\left.\begin{array}{l}x=\cdot 5\\ y=2\end{array}\right\},\quad \left.\begin{array}{l}x=\cdot 8\\ y=1\cdot 25\end{array}\right\},\quad \left.\begin{array}{l}x=1\\ y=1\end{array}\right\},\quad \left.\begin{array}{l}x=2\\ y=\cdot 5\end{array}\right\}.$$

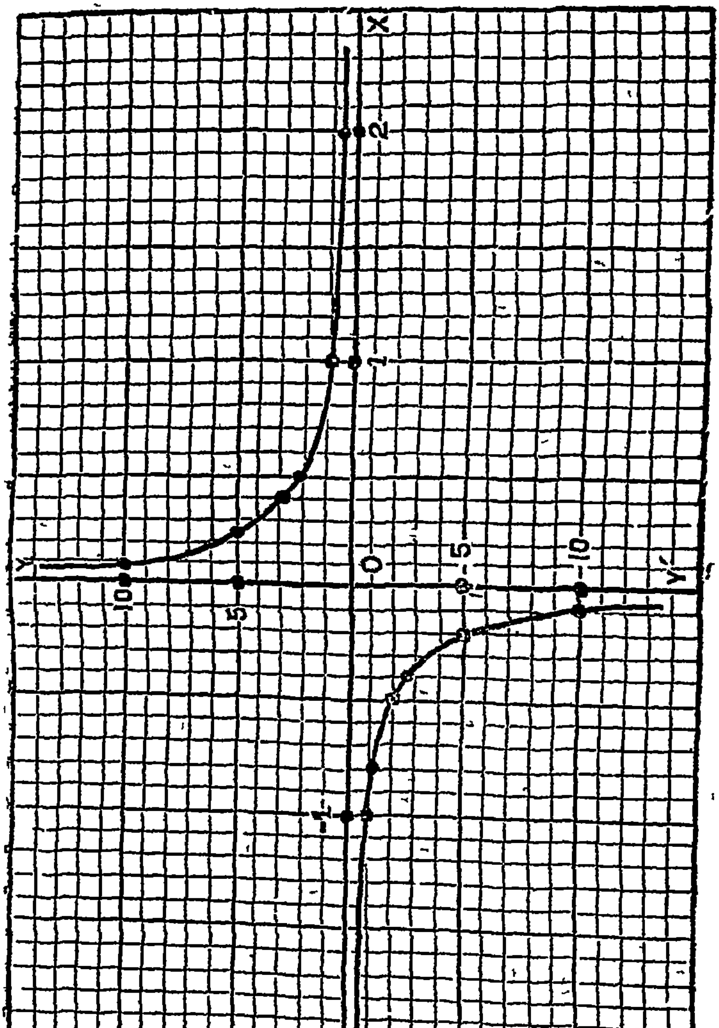
Evidently also the following points are on the required graph —

$$\left.\begin{array}{l}x=-\cdot 1\\ y=-10\end{array}\right\},\quad \left.\begin{array}{l}x=-\cdot 2\\ y=-5\end{array}\right\},\quad \left.\begin{array}{l}x=-\cdot 4\\ y=-2\cdot 5\end{array}\right\},\quad \left.\begin{array}{l}x=-\cdot 5\\ y=-2\end{array}\right\},$$

$$\left.\begin{array}{l}x=-\cdot 8\\ y=-1\cdot 25\end{array}\right\},\quad \left.\begin{array}{l}x=-1\\ y=-1\end{array}\right\},\quad \left.\begin{array}{l}x=-2\\ y=-\cdot 5\end{array}\right\}.$$

Let one inch be the unit for measuring x and one-tenth of an inch the unit for measuring y

Let us now plot the points and draw a curve through them free-hand, as in the following diagram :—



The curve so drawn is the required graph.

Note 1 As x diminishes from 1 to zero, y increases from 1 to finity, and as x diminishes from zero to -1, y increases from gative infinity to -1.

Note 2 As x increases from 1 to infinity, y diminishes from 1 to zero, and as x diminishes from -1 to negative infinity, y increases from -1 to zero

Note 3 The graph consists of two branches, one lying between OX and OY and the other between OX′ and OY′

Note 4 The more we move towards the right or left of O, the nearer does the curve approach the axis of x, whilst the more we move upwards or downwards from O, the nearer does the curve approach the axis of y But in no case does the curve *meet* the axes except at an infinite distance from O Hence, each of the axes is said to be an asymptote to the curve

Note 5 A curve of this kind is called a Rectangular Hyperbola

Exercise (8).

Draw the graphs of the following equations —

1 $x^2+y^2 = 81$ 2. $(x-5)^2+(y-6)^2 = 49$.

3. $(x+6)^2+(y-7)^2 = 100$

4. $x^2+y^2-8x-14y+1 = 0$

5. $x^2+y^2+14x-16y+32 = 0$

6. $x^2+y^2+12x+18y+92 = 0$.

7. $x^2+y^2-10x+16y-55 = 0$. 8. $x^2-y^2 = 0$.

9. $9x^2-4y = 0$ 10. $9y^2 = 16x^2$. 11 $4x^2 = 9$

12. $25y^2 = 16$. 13. $x^2+4y^2 = 4$. 14 $4x^2+9y^2 = 1$.

15. $25x^2+y^2 = 25$. 16 $16x^2+9y^2 = 1$.

17. $x^2-4y^2 = 4$ 18. $y^2-x^2 = 1$

19. $4x^2-y^2 = 16$ 20 $y^2-9x^2 = 9$

21. In one and the same diagram draw the graphs of $4x^2-9y^2 = 0$ and $4x^2-9y^2 = 36$

22. In one and the same diagram draw the graphs of $9y^2-4x^2 = 0$ and $9y^2-4x^2 = 36$

23. Draw the graph of the equation $5y = x^2-10$, taking the unit for measuring y five times as large as that for measuring x

24 Draw the graph of the equation $x^2-4x+2y = 0$, taking the unit for measuring y twice as large as that for measuring x.

25. Draw the graph of the equation $y^2+x=0$, taking the unit for measuring x equal to half that for measuring y

26. Draw the graph of the equation $3y=x^2$, taking the same unit for measuring both x and y

27. Find graphically, correct to the first figure after the decimal point, the square roots of —

(i) 3, (ii) 5, (iii) 7.

28 Find graphically the minimum value of the expression

$$2x^2-6x+7$$

29. Find graphically the maximum value of the expression

$$1+2x-2x^2.$$

30. Draw the graph of the equation $xy=4$.

CHAPTER III

ARITHMETICAL PROGRESSION.

1 **Definition.** Quantities are said to be in Arithmetical Progression when they increase or decrease regularly by a *common difference*

Thus each of the following series of quantities is in Arithmetical progression —

2,	5,	8,	11,	14,	&c.
9,	5,	1,	-3,	-7,	&c
a,	$a+b$,		$a+2b$,	$a+3b$,	&c
a,	$a-b$,		$a-2b$,	$a-3b$,	&c

In the first of the above examples the quantities increase by 3, whereas in the second the quantities decrease by 4, so the common differences in these two cases are said to be 3 and -4 respectively. Similarly in the third example the common difference is b and in the fourth it is $-b$

Note If a be the first term and b the common difference of a series of numbers in Arithmetical Progression, we have the second term $= a+b$, the 3rd term $= a+2b$, the 4th term $= a+3b$, , the 10th term $= a+9b$, , the 21st term $= a+20b$, and so on Hence the nth term $= a+(n-1)b$

Example 1. Find the 19th term of the series 10, 8, 6, 4, &c

The first term $= 10$, and the common difference $= -2$.

Hence the 19th term $= 10+18(-2) = 10-36 = -26$.

Example 2. What term of the series 5, 7, 9, 11, &c. is 25 ?

Let the rth term of the given series be the required term , then we must have

$$25 = 5+(r-1).2$$
$$= 3+2r,$$

whence $r = 11$

Thus the 11th term of the given series $= 25$

Exercise (9).

1. Find the 8th, 20th and $(n-3)$th terms of the series 2, 4, 6, 8, &c

2. What terms of the series 9, 11, 13, 15, &c. are 65, 99, and $6n-13$?

3. The first term of a given series is 3 and the 7th term 39 , find the common difference.

4. If there be 60 terms in *A P.* of which the first term is 8 and the last term 185, find the 31st term.

5. If a be the first term and l the last term of a series of numbers in *A P*, shew that the 5th term from the beginning + the 5th term from the end $= a+l$.

6. In the preceding example, shew that the rth term from the beginning + the rth term from the end $= a+l$.

2. To find the sum of n terms of an Arithmetic series of which the first term is a and the common difference. b.

Let S denote the required sum, and l, the last term (*i.e.*, the nth term).

Then $S=a+(a+b)+(a+2b)+(a+3b)+\&c.+\{a+(n-1)b\}$.
And, by writing the series in the reverse order, we have also $S=l+(l-b)+(l-2b)+(l-3b)+\&c.+\{l-(n-1)b\}$.
Therefore, by addition,

$$2S=(a+l)+(a+l)+(a+l)+\ \&c.\ \ldots\ \ldots \text{ to } n \text{ terms}$$
$$=n(a+l),$$

$$\therefore\quad S=\frac{n}{2}(a+l)\quad \ldots\quad \ldots\quad \ldots\quad (1)$$

Thus the sum of n terms in $A.\ P.$ is n times the semi-sum of the first and last terms, or, in other words, n times the average of the first and last terms

Also, since $l=a+(n-1)b$.

$$\therefore\quad S=\frac{n}{2}\Big[a+\Big\{a+(n-1)b\Big\}\Big]$$
$$=\frac{n}{2}\Big\{2a+(n-1)b\Big\}.\ \ldots\ \ldots\ \ldots\quad (2)$$

N. B.—The formulæ (1) and (2) should be carefully remembered so that they might readily be applied in any suitable case

Example 1. Find the sum to 20 terms of the series 5, $4\frac{1}{3}$, $3\frac{2}{3}$, &c.

The first term $=5$, and the common difference

$$=\frac{13}{3}-5=-\frac{2}{3}.$$

Hence, the required sum $=\dfrac{20}{2}\Big\{2\times 5+\Big(20-1\Big)\times\Big(-\dfrac{2}{3}\Big)\Big\}$

$$=10\Big(10-\frac{19\times 2}{3}\Big)$$
$$=10\Big(-\frac{8}{3}\Big)=26\frac{2}{3}.$$

Example 2. Find the value of $1+2+3+4+\&c.$ to 100 terms,

The last term of the series evidently $=100$.

Hence, the required sum $=\dfrac{100}{2}\Big(1+100\Big)$

$$=50\times 101=5050.$$

Exercise (10).

1. Sum 1, 2, 3, 4, &c to 25 terms.
2. Sum 1, 3, 5, 7, &c to 30 terms.
3. Sum 5, $4\frac{3}{4}$, $4\frac{1}{2}$, &c. to 21 terms.
4. Sum 13, $12\frac{1}{3}$, $11\frac{2}{3}$, &c to 40 terms.
5. Sum 2, 7, 12, &c. to 101 terms.
6. Sum $\frac{n-1}{n}$, $\frac{n-2}{n}$, $\frac{n-3}{n}$, &c to n terms.
7. Sum $\frac{a-b}{a+b}$, $\frac{3a-2b}{a+b}$, $\frac{5a-3b}{a+b}$, &c to n terms.

Application of the formulæ (1) and (2) of the preceding article—The following examples will illustrate some important applications of those formulæ.

Example 1. The first term of a series in $A\ P$, is 17, the last term $-12\frac{3}{8}$ and the sum $25\frac{7}{16}$, find the common difference

Let n = the number of terms, then we must have

$$25\frac{7}{16} = \frac{n}{2}\left\{17+\left(-12\frac{3}{8}\right)\right\}$$

$$= \frac{n}{2}\left(17-12\frac{3}{8}\right)$$

$$= \frac{n}{2}\times 4\frac{5}{8}.$$

$$\text{or,}\quad \frac{407}{16} = \frac{37n}{16},\quad \therefore\quad n = \frac{407}{37} = 11.$$

If then b be the required common difference, we must have $-12\frac{3}{8}$(= the 11th term) $= 17+10b$,

$$\therefore\quad 10b = -12\frac{3}{8}-17$$

$$= -29\frac{3}{8} = -\frac{235}{8},$$

$$b = -\frac{235}{8\times 10} = -\frac{5\times 47}{5\times 2\times 8} = -\frac{47}{16}$$

Example 2. The sum of a series in *A. P* is 72, the first term 17, and the common difference -2, find the number of terms, and explain the double answer.

Let $n =$ the number of terms.

Then we must have

$$72 = \frac{n}{2}\left\{2\times17+\left(n-1\right)\times\left(-2\right)\right\}$$

$$= \frac{n}{2}\left\{34-2\left(n-1\right)\right\} = \frac{n}{2}\left(36-2n\right) = 18n-n^2\ ;$$

$$\therefore\quad n^2-18n+72 = 0\ ,$$

$$\text{or, } (n-6)(n-12) = 0\ ,$$

$$\therefore\quad n = 6 \text{ or, } 12.$$

The double answer shows that there are two sets of numbers, satisfying the conditions of the problem and this can be easily verified For, the series to 6 terms is 17, 15, 13, 11, 9, 7, and to 12 terms it is 17, 15, 13, 11, 9, 7, 5, 3, 1, -1, -3, -5; now since the sum of the last 6 terms of the latter set of numbers$=0$, evidently, therefore, the sum of 6 terms of the series, is exactly the same as that of 12 terms

Example 3. How many terms of the series -8, -6, -4, &c. amount to 52 ?

Let $n =$ the required number.

Then we must have

$$52 = \frac{n}{2}\left\{2\times\left(-8\right)-\left(n-1\right)\times2\right\}$$

$$= \frac{n}{2}(2n-18) = n^2-9n.$$

$$\therefore\quad n^2-9n-52 = 0\ ,$$

$$\text{or, } (n-13)(n+4) = 0\ ;$$

$$\therefore\quad n = 13, \text{ or, } -4.$$

Hence, since the number of terms can only be a positive integer, we must reject the negative value and take 13 to be the answer to the question.

Example 4. The sum of p terms of an A P is q, and the sum of q terms is p, find the sum of $p+q$ terms

Let a be the first term, and b the common difference; then since the sum of p terms $= q$, we must have

$$q = \frac{p}{2}\{2a+(p-1)b\},$$

$$\left.\begin{aligned} \text{or, } 2q &= p\,2a+p(p-1)b \qquad . \qquad . \quad (1) \\ \text{Similarly } 2p &= q\,2a+q(q-1)b \qquad . \qquad . \quad (2) \end{aligned}\right\}$$

Subtracting (2) from (1), we have

$$\begin{aligned} 2(q-p) &= (p-q)\,2a+\{(p^2-q^2)-(p-q)\}b \\ &= (p-q).2a+(p-q)(p+q-1)b\,; \end{aligned}$$

$$\therefore -2 = 2a+(p+q-1)b.$$

Hence, the sum of $p+q$ terms

$$= \frac{p+q}{2}\left\{2a+(p+q-1)b\right\}$$

$$= \frac{p+q}{2}\times(-2) = -(p+q).$$

Exercise (11).

1. The first term of an A P. is 5, the number of terms 30, and their sum 1455; find the common difference.

2. The first term of a series being 2, and the 5th term being 7, find how many terms must be taken so that the sum may be 63

3. What is the common difference when the first term is 1, the last 50, and the sum 204?

4. How many terms of the series 19, 17, 15, &c, amount to 91?

5 The sum of a certain number of terms of the series 21, 19, 17, &c, is 120 Find the last term and the number of terms

6. How many terms of the series 54, 51, 48, &c, must be taken to make 513? Explain the double answer

7. If the sum of 8 terms of an A. P. is 64, and the sum of 19 terms is 361, find the sum of n terms

8. Find the series of which the nth term is $\frac{3+n}{4}$; and also find the sum of the series to 105 terms.

9. Find the series whose rth term is $2r-1$; find the sum of the series to n terms.

10. The sum of n terms of an A P is $3n^2-n$ and the common difference 6, find the first term

11. The sum of n terms of an A P. is 40, the common difference 2, and the last term 13; find n

12. Prove that the latter half of $2n$ terms of any Arithmetical series = $\frac{1}{3}$rd of the sum of $3n$ terms of the same series

13. If $2n+1$ terms of the series 1, 3, 5, 7, 9, &c, be taken, then the sum of the alternate terms 1, 5, 9, &c, will be to the sum of the remaining terms 3, 7, 11, &c, as $n+1$ is to n.

14 Prove that (i) $b = \dfrac{l^2-a^2}{2s-(l+a)}$

and (ii) $s = \dfrac{l+a}{2b}(l-a+b)$.

4. Arithmetic means.—

Definition 1. When three quantities are in Arithmetical Progression the middle one is said to be the **Arithmetic mean** between the other two.

Thus 5 is the Arithmetic mean between 3 and 7.

Definition 2 If A and B be any two quantities and x_1, x_2, x_3, x_4, &c, x_{n-1}, x_n a number of others such that A, x_1, x_2, x_3, &c, x_{n-1}, x_n, B are in Arithmetical Progression, then, x_1, x_2, x_3, &c, are called the **Arithmetic means** between A and B

Thus 3, 4, 5, 6, 7 are Arithmetic means between 2 and 8, and so are the numbers $3\frac{1}{2}$, 5 and $6\frac{1}{2}$. for both the series 2, 3, 4, 5, 6, 7, 8 and 2, $3\frac{1}{2}$, 5, $6\frac{1}{2}$, 8 are in A. P.

Note It is evident from the above example that between any two quantities the number of different sets of Arithmetic means is unlimited

5 To insert a given number of Arithmetic means between two given quantities.

Let a and c be the two given quantities, and n the number of means to be inserted.

Then we have to find out n quantities x_1, x_2, x_3, &c, x_{n-2}, x_{n-1}, x_n such that a, x_1, x_2, x_3, &c., x_{n-1}, x_n, c may be

in A. P Evidently the series [a, x_1, x_2, x_3, &c , x_{n-1}, x_n, c] consists of $n+2$ terms of which a is the first term and c the last.

Hence, if b be the common difference, we must have

$$c = a+(n+1)b,$$

$$\text{whence } b = \frac{c-a}{n+1}$$

$$\text{Hence, } x_1 = a+b = a+\frac{c-a}{n+1},$$

$$x_2 = a+2b = a+\frac{2(c-a)}{n+1},$$

&c. &c. &c

$$x_n = a+nb = a+\frac{n(c-a)}{n+1}.$$

Example 1 Find the Arithmetic mean between any two quantities a and b

Let x = the quantity sought

Then a, x, b are in A P , and ∴ we must have $x-a=b-x$,

$$\text{whence } x = \frac{a+b}{2}$$

Example 2 Insert 4 Arithmetic means between 3 and 18 Let x_1, x_2, x_3, x_4, be the means.

Then 3, x_1, x_2, x_3, x_4, 18 are in A. P.

Hence, if b = the common difference,

we must have $18 = 3+5b$, ∴ $b = 3$.

$$\text{Hence, } \left.\begin{aligned} x_1 &= 3+b = 6 \\ x_2 &= 3+2b = 9 \\ x_3 &= 3+3b = 12 \\ x_4 &= 3+4b = 15 \end{aligned}\right\}.$$

Thus the required means are 6, 9, 12 and 15

Exercise (12).

1. Find the Arithmetic mean between (i) 5 and 8 , (ii) -5 and 21 , (iii) $m-n$ and $m+n$, (iv) $(a+x)^2$ and $(a-x)^2$.

2. Insert 2 Arithmetic means between (i) 8 and 12 ; (ii) —6 and 14

3. Insert 3 Arithmetic means between 117 and 477.

4. Insert 4 Arithmetic means between 2 and—18.

5. Insert 17 Arithmetic means between $3\frac{1}{2}$ and $-41\frac{1}{2}$.

6. There are n Arithmetic means between 1 and 31, such that the 7th mean : $(n-1)$th mean $= 5 : 9$, required n

7. **The Natural Numbers**—The numbers 1, 2, 3, &c., are called the *natural numbers.*

(i) To find the sum of the first n natural numbers.

Let S denote the sum ; then,

$$S = 1+2+3+ \dots +n$$

$$= \frac{n}{2}(1+n) = \frac{n(n+1)}{2} \quad \dots\dots\dots\dots(A)$$

(ii) To find the sum of the first n odd natural numbers

Let S denote the sum ; then

$$S = 1+3+5+7 \quad \dots\dots\dots\dots \text{ to } n \text{ terms},$$

$$= \frac{n}{2}\left\{2+(n-1)\times 2\right\}$$

$$= \frac{n}{2}\times 2n = n^2. \quad \dots\dots\dots\dots(B)$$

(iii) To find the sum of the squares of the first n natural numbers

Let S denote the sum , then,

$$S = 1^2+2^2+3^2+4^2+ \dots\dots +n^2.$$

We have $n^3-(n-1)^3 = 3n^2-3n+1.$

Hence, putting 1, 2, 3, &c , for n, we have

$$1^3-0^3 = 3.1^2-3.1+1,$$

$$2^3-1^3 = 3.2^2-3.2+1,$$

$$3^3-2^3 = 3.3^2-3.3+1,$$

$$4^3-3^3 = 3.4^2-3.4+1.$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$(n-1)^3-(n-2)^3 = 3(n-1)^2-3(n-1)+1,$$

$$n^3-(n-1)^3 = 3n^2-3n+1.$$

Hence, by addition,

$$n^3 = 3(1^2+2^2+3^2+\dots+n^2)-3(1+2+3+\dots+n)+n$$
$$= 3S-3\frac{n(n+1)}{2}+n$$
$$\therefore\ 3S = n^3-n+\frac{3n(n+1)}{2}$$
$$= n(n+1)\left\{(n-1)+\frac{3}{2}\right\},$$
$$\therefore\quad S=\frac{n(n+1)(2n+1)}{6}\ \dots\ \dots\ \dots\quad (C)$$

(iv) To find the sum of the cubes of the first n natural numbers

Let S denote the sum, then

$$S = 1^3+2^3+3^3+\dots\dots+n^3.$$

We have $n^4-(n-1)^4 = 4n^3-6n^2+4n-1$.

Hence, putting 1, 2, 3, &c, for n, we have

$$1^4-0^4 = 4.1^3-6.1^2+4.1-1,$$
$$2^4-1^4 = 4.2^3-6.2^2+4.2-1,$$
$$3^4-2^4 = 4.3^3-6.3^2+4.3-1,$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$(n-1)^4-(n-2)^4 = 4.(n-1)^3-6.(n-1)^2+4(n-1)-1,$$
$$n^4-(n-1)^4 = 4.n^3-6n^2+4n-1$$

Hence, by addition,

$$n^4 = 4(1^3+2^3+3^3+\&c+n^3)-6(1^2+2^2+3^2+\&c.+n^2)$$
$$+4(1+2+3+\&c+n)-n$$
$$= 4S-6.\frac{n(n+1)(2n+1)}{6}+4.\frac{n(n+1)}{2}-n;$$
$$\therefore\ 4S = n^4+n+n(n+1)(2n+1)-2n(n+1)$$
$$= n(n+1)\{(n^2-n+1)+(2n+1)-2\}$$
$$= n(n+1)(n^2+n),$$
$$\therefore\quad S=\frac{n^2(n+1)^2}{4}=\left\{\frac{n(n+1)}{2}\right\}^2\ \dots\ (D)$$

Thus *the sum of the cubes of the first n natural numbers is equal to the square of the sum of these numbers.*

Example 1. Sum the series $1\cdot 2+2\cdot 3+3\cdot 4+$&c to n terms.

The nth term of the series evidently $= n(n+1) = n^2+n$.

Hence, putting $n=1$, the 1st term $=1^2+1$,

„ „ $n=2$, „ 2nd term $=2^2+2$,

„ „ $n=3$, „ 3rd term $=3^2+3$,

... ...

and so on

Hence, if S denote the sum of the given series,

$$\text{we have } S = (1^2+1)+(2^2+2)+(3^2+3)+\text{\&c to } n \text{ terms}$$

$$= (1^2+2^2+3^2+\text{\&c}+n^2)+(1+2+3+\text{\&c.}+n).$$

$$= \frac{n(n+1)(2n+1)}{6}+\frac{n(n+1)}{2}$$

$$= \frac{n(n+1)}{2}\left\{\frac{2n+1}{3}+1\right\} = \frac{n(n+1)(n+2)}{3}.$$

Example 2. Sum the series—

$$1^2+3^2+5^2+7^2+\text{\&c to } n \text{ terms.}$$

Since evidently each term of the given series is equal to the square of the *corresponding* term of the series 1, 3, 5, 7, &c, $\therefore$ the nth term of the given series = the square of the nth term of the series 1, 3, 5, 7, &c,

$$\text{and } \therefore \text{ the } n\text{th term} = \{1+(n-1)\times 2\}^2$$

$$= (2n-1)^2$$

$$= 4n^2-4n+1.$$

Hence, putting $n = 1, 2, 3,$ &c., we have

the 1st term $= 4\cdot 1^2-4\cdot 1+1$,

„ 2nd „ $= 4\cdot 2^2-4\cdot 2+1$,

„ 3rd „ $= 4\cdot 3^2-4\cdot 3+1$.

.. ..

and so on

Hence, if S denote the sum of the given series, we must have

$$S = 4(1^2+2^2+3^2+\text{\&c}+n^2)-4(1+2+3+\text{\&c}+n)+n$$

$$= 4\,\frac{n(n+1)(2n+1)}{6}-4\,\frac{n(n+1)}{2}+n$$

$$= 2n(n+1)\left\{\frac{(2n+1)}{3}-1\right\}+n$$

$$= \frac{2n(n+1)\times 2(n-1)}{3}+n$$

$$= \frac{n}{3}\left\{4\left(n^2-1\right)+3\right\} = \frac{n}{3}\left(4n^2-1\right).$$

Example 3. Sum the series—

$$1^2+(1^2+2^2)+(1^2+2^2+3^2)+\&c \text{ to } n \text{ terms}$$

The nth term of the series—

$$= 1^2+2^2+3^2+\&c.+n^2$$

$$= \frac{n(n+1)(2n+1)}{6} = \frac{n(2n^2+3n+1)}{6}$$

$$= \tfrac{1}{3}n^3+\tfrac{1}{2}n^2+\tfrac{1}{6}n.$$

Hence, the 1st term $= \frac{1}{3}1^3+\frac{1}{2}1^2+\frac{1}{6}1,$

„ 2nd „ $= \frac{1}{3}2^3+\frac{1}{2}.2^2+\frac{1}{6}2,$

„ 3rd „ $= \frac{1}{3}3^3+\frac{1}{2}3^2+\frac{1}{6}3,$

. . . .

and so on

Hence, if S denote the required sum, we must have,

$$S = \tfrac{1}{3}(1^3+2^3+3^3+\&c.+n^3)$$
$$+\tfrac{1}{2}(1^2+2^2+3^2+\&c+n^2)+\tfrac{1}{6}(1+2+3+\&c+n)$$

$$= \tfrac{1}{3}\,\frac{n^2(n+1)^2}{4}+\tfrac{1}{2}\,\frac{n(n+1)(2n+1)}{6}+\tfrac{1}{6}\,\frac{n(n+1)}{2}$$

$$= \frac{n(n+1)}{12}\left\{n(n+1)+(2n+1)+1\right\}$$

$$= \frac{n(n+1)}{12}\left(n^2+3n+2\right) = \frac{n(n+1)^2(n+2)}{12}.$$

Exercise (13).

Sum the series—

1. $2^2+5^2+8^2+\&c$ to n terms.
2. $1\,2^2+2\,3^2+3\,4^2+\&c$ to n terms
3. $1\,3+3\,5+5.7+7\,9+\&c$ to n terms.
4. $1^3+3^3+5^3+\&c$ to n terms
5. $1+(1+2)+(1+2+3)+\&c$ to n terms.

7. Miscellaneous Examples.

Example 1. Prove that if the number of terms of an *A. P.* be *odd*, twice the middle term is equal to the sum of the first and last terms

Since the number of terms is odd, let it be denoted by $2n+1$

Evidently the middle term is one which has n terms on either side of it, hence it is the $(n+1)$th term from the beginning and also the $(n+1)$th term from the end

Hence, putting M for the middle term, we must have

$$M = a+(\overline{n+1}-1)b$$
$$= a+nb \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

and also
$$M = l-(\overline{n+1}-1)b$$
$$= l-nb \quad \ldots \quad \ldots \quad (2)$$

Hence, by addition

$$2M = a+l$$

Example 2. Prove that the sum of an odd number of terms in *A P* is equal to the middle term multiplied by the number of terms

Let $2n+1$ = the number of terms

Then the sum of the terms

$$= \frac{2n+1}{2}\left(a+l\right) = \frac{2n+1}{2}\times 2M \quad \text{[last example]}$$
$$= (2n+1)\times M.$$

Example 3. Find the first five terms of the series of which the sum to n terms $= 5n^2+3n$

Let t_1, t_2, t_3, &c, t_n denote respectively the 1st, 2nd., 3rd, &c., nth terms of the series;

and let s_1, s_2, s_3, &c. s_n denote respectively the sums of 1, 2, 3, &c, n terms of the series.

Evidently then $s_1 = t_1$, $s_2 = t_1+t_2$;

$s_3 = t_1+t_2+t_3$; and so on.

Now, by the question, we have

$$s_n = 5n^2+3n,$$

(*i e*, the sum to *any number* of terms = 5 times the square of *that number* + 3 times *that number*.)

Hence, putting $n = 1$, we have $s_1 = 5+3 = 8$,
" $n = 2$, " $s_2 = 20+6 = 26$,
" $n = 3$, " $s_3 = 45+9 = 54$,
" $n = 4$, " $s_4 = 80+12 = 92$,
" $n = 5$, " $s_5 = 125+15 = 140$
and so on

Hence,

$$\begin{aligned} t_1 &= s_1 &&= 8, \\ t_2 &= s_2 - s_1 &&= 26 - 8 = 18, \\ t_3 &= s_3 - s_2 &&= 54 - 16 = 28, \\ t_4 &= s_4 - s_3 &&= 92 - 54 = 38, \\ t_5 &= s_5 - s_4 &&= 140 - 92 = 48, \end{aligned}$$

and so on

Thus the first five terms of the series are 8, 18, 28, 38 and 48

Example 4. Sum the series—

$$1+5+12+22+35+\&c \text{ to } n \text{ terms}$$

[The peculiarity of the series is that the successive differences of the terms are in A. P]

Let S denote the required sum and let t_n denote the nth term of the series Then we have

$$S = 1+5+12+22+ \;\ldots\; +t_n,$$

also $$S = 0+1+5+12+ \;\ldots\; +t_{n-1}+t_n.$$

Hence, by subtraction,

$$\begin{aligned} 0 &= 1+4+7+10+\&c +(t_n - t_{n-1}) - t_n \\ &= \{1+4+7+10+\&c. \text{ to } n \text{ terms}\} - t_n, \end{aligned}$$

$$\therefore \quad t_n = \frac{n}{2}\left\{2+(n-1)3\right\} = \frac{n(3n-1)}{2},$$

i e, the nth term of the given series $= \frac{3}{2}n^2 - \frac{1}{2}n$

Hence, the 1st term $= \frac{3}{2}\,1^2 - \frac{1}{2}\,1$,
2nd " $= \frac{3}{2}\,2^2 - \frac{1}{2}\,2$,
3rd " $= \frac{3}{2}\,3^2 - \frac{1}{2}\,3$,
and so on.

Hence, $S=\frac{3}{2}(1^2+2^2+3^2+\&c.+n^2)-\frac{1}{2}(1+2+3+\&c.+n)$

$$=\frac{3}{2}\cdot\frac{n(n+1)(2n+1)}{6}-\frac{1}{2}\cdot\frac{n(n+1)}{2}$$

$$=\frac{n(n+1)}{4}.2n=\frac{n^2(n+1)}{2}.$$

Example 5. Sum the series—

$$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\&c \text{ to } n \text{ terms.}$$

Let S denote the sum to n terms,

Now, we have

$$t_1=\frac{1}{1.2}\quad=1-\frac{1}{2},$$

$$t_2=\frac{1}{2.3}\quad=\frac{1}{2}-\frac{1}{3},$$

$$t_3=\frac{1}{3.4}\quad=\frac{1}{3}-\frac{1}{4},$$

&c., &c., &c.,

$$t_n=\frac{1}{n(n+1)}=\frac{1}{n}-\frac{1}{n+1}.$$

Hence, $S=1-\frac{1}{n+1}=\frac{n}{n+1}.$

Example 6 Find three numbers in A. P. whose product $=120$ and whose sum $=15$.

Let $a-\beta$, a and $a+\beta$ be the numbers; then we have

$$\left.\begin{aligned}(a-\beta)a.(a+\beta)&=120 \quad \ldots \quad \ldots \quad (1)\\ \text{and}\quad (a-\beta)+a+(a+\beta)&=15 \quad \ldots \quad \ldots \quad (2)\end{aligned}\right\}$$

From (2), $3a=15$, $\therefore a=5$.

From (1), $a(a^2-\beta^2)=120$,

$$\therefore\quad 5.(25-\beta^2)=120,$$

$$\therefore\quad 25-\beta^2=24,$$

$$\therefore\quad \beta^2=1,\quad\therefore\quad\beta=\pm 1$$

Hence, the numbers are 4, 5, 6.

Example 7. If a^2, b^2, c^2 be in A P, then

$$\frac{1}{b+c},\ \frac{1}{c+a},\ \frac{1}{a+b} \text{ are in A P.}$$

Evidently $\frac{1}{b+c},\ \frac{1}{c+a},\ \frac{1}{a+b}$ are in A P.

$$\text{if } \frac{1}{c+a}-\frac{1}{b+c}=\frac{1}{a+b}-\frac{1}{c+a},$$

i e, if $\frac{b-a}{(c+a)(b+c)}=\frac{c-b}{(a+b)(c+a)}$,

i e, if $(b-a)(b+a)=(c-b)(c+b)$,

i e, if $b^2-a^2=c^2-b^2$,

but this is true by hypothesis,

$$\therefore\ \frac{1}{b+c},\ \frac{1}{c+a},\ \frac{1}{a+b} \text{ are in A. P}$$

Example 8. Determine the relation which must exist between a, b and c, in order that they may be respectively the pth, qth and rth terms of an A P.

Let α denote the first term and β the common difference of the A P of which a, b and c are the pth, qth and rth terms, then we must have

$$\left.\begin{aligned} a &= \alpha+(p-1)\beta \quad \ldots \quad (1) \\ b &= \alpha+(q-1)\beta \quad \ldots \quad (2) \\ c &= \alpha+(r-1)\beta \quad \ldots \quad (3) \end{aligned}\right\}$$

Now we have to eliminate α and β from these three equations

Subtracting (2) from (1), and (3) from (2), we have

$$a-b=(p-q)\beta,$$
$$b-c=(q-r)\beta.$$

Hence, $(a-b)(q-r)=(b-c)(p-q)$,

or, $a(q-r)+b(r-p)+c(p-q)=0$,

which is the relation required

Exercise (14).

1. The $(n+1)$th term of a series in A. P. is $\frac{ma-n}{a-b}$ required the sum of the series to $(2n+1)$ terms

2. Find the first five terms of the series of which the sum to n terms is $2n^2+7n$.

3. The sum to n terms of an A P is $3n^2+10n$, find the first term and the common difference.

4. Find the 35th term of the series of which the sum to n terms is n^2+n.

5. Sum the series—

$$1+3+6+10+15+\&c., \text{ to } n \text{ terms}$$

6. Sum the series—

$$2+5+10+17+\&c., \text{ to } n \text{ terms}$$

7. Sum the series—

$$2+7+14+23+34+\&c., \text{ to } n \text{ terms}$$

8. Sum the series—

$$\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+\&c., \text{ to } n \text{ terms.}$$

9. Find 4 numbers in A P, such that their sum shall be 56, and the sum of their squares 864.

[Let $\alpha-3\beta$, $\alpha-\beta$, $\alpha+\beta$ and $\alpha+3\beta$ be the numbers]

10 The sum of three numbers in A. P is 15, and the sum of the squares of the two extremes is 58 What are the numbers ?

11. There are four numbers in A. P, the sum of the two extremes is 8, and the product of the means is 15. What are the numbers ?

12. Find six numbers in A P. such that the sum of the two extremes may be 16 and the product of the two middle terms 63.

[Let $\alpha-5\beta$, $\alpha-3\beta$ $\alpha-\beta$, $\alpha+\beta$, $\alpha+3\beta$, $\alpha+5\beta$ be the numbers]

13. If $(b-c)^2$, $(c-a)^2$, $(a-b)^2$ are in A.P, shew that

$$\frac{1}{b-c}, \frac{1}{c-a}, \frac{1}{a-b} \text{ are in A. P.}$$

14. Determine the relation which must exist between a, b and c, in order that they may be respectively the sums of p, q and r terms of an A. P.

15. Given P and Q the mth and nth terms of an Arithmetic series, find the rth-term.

16. There are n Arithmetic means between 3 and 54 such that the 8th mean : $(n-2)$th mean $= 3 : 5$; find n.

17. If S_1, S_2, S_3, be the sums of n terms of three Arithmetic series, the first term of each being 1 and the respective common differences 1, 2, 3, prove that $S_1 + S_3 = 2S_2$.

18. If there be r Arithmetical Progressions, each beginning from unity, whose common differences are 1, 2, 3, &c., r, shew that the sum of their nth terms is $= \frac{1}{2}\{(n+1)r^2 + (n+1).r\}$.

19. Sum the series—

$$n.1 + (n-1).2 + (n-2).3 + (n-3).4 + \&c. + 1.n.$$

[The rth term of the series $= \{n-(r-1)\}r = (n+1)r - r^2$.
Hence, the required sum $= (n+1)\{1+2+3+\ldots+n\}$
$-\{1^2+2^2+3^2+\ldots+n^2\} = \&c.$]

20. On the ground are placed n stones; the distance between the first and second is one yard, between the 2nd. and 3rd. three yards, between the 3rd and 4th five yards, and so on. How far will a person have to travel who shall bring them, one by one, to a basket placed at the first stone?

CHAPTER IV.

GEOMETRICAL PROGRESSION.

1. Definition.—Quantities are said to be in Geometrical Progression when each is equal to the product of the preceding and some constant factor.

The constant factor is called the *common ratio* of the series, and it is found by dividing *any* term by that which immediately *precedes it*.

Thus each of the following series forms a Geometrical Progression:—

1,	2,	4,	8,	16,	&c.
1,	$\frac{1}{2}$,	$\frac{1}{4}$,	$\frac{1}{8}$,	$\frac{1}{16}$,	&c.
1,	$-\frac{1}{3}$,	$\frac{1}{9}$,	$-\frac{1}{27}$,	$\frac{1}{81}$,	&c.
a,	ar,	ar^2,	ar^3,	ar^4,	&c.

In the first example the common ratio is 2, in the second $\frac{1}{2}$, in the third $-\frac{1}{3}$, and in the fourth r.

Note. If a be the first term and r the common ratio of a Geometric series we have the 2nd term $= ar$, the 3rd term $= ar^2$, the 4th term $= ar^3$, ... the 10th term $= ar^9$, ... the 21st term $= ar^{20}$, and so on. Hence the nth term $= ar^{n-1}$.

Exercise (15).

1. Find the 8th term of the series 4, 12, 36, &c.

[the common ratio $= \frac{12}{4} = 3$, hence the 8th term $= 4 \cdot 3^7 =$ &c.]

2. Find the 6th term of the series $3\frac{3}{8}$, $2\frac{1}{4}$, $1\frac{1}{2}$, &c.

3. Find the 9th term of the series 1, 4, 16, 64, &c.

4. Find the 6th term of the series 1, -3, 9, -27, &c.

5. Find the 5th term and the $(n-1)$th term of the series $\frac{3}{2}$, -1, $\frac{2}{3}$, &c.

6. Find the 7th term of the series in -21, 14, $-9\frac{1}{3}$, &c.

7. The first two terms of a series in G. P. are 125 and 25, what are the 6th and 7th terms?

2. To find the sum of a number of terms in Geometrical Progression.

Let a be the first term, r the common ratio, n the number of terms and S the sum required.

Then

$$S = a + ar + ar^2 + ar^3 + \text{\&c} + ar^{n-1},$$

$$\therefore\ Sr = \quad ar + ar^2 + ar^3 + \text{\&c} + ar^{n-1} + ar^n.$$

Hence, by subtraction,

$$Sr - S = ar^n - a,$$

$$\therefore\ S(r-1) = a(r^n - 1),$$

$$\therefore\ S = \frac{a(r^n-1)}{r-1} \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

$$\text{or, } = \frac{a(1-r^n)}{1-r} \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

Cor. If l denote the last (or the nth) term of the series, we have $l = ar^{n-1}$, hence from (1),

$$S = \frac{rl-a}{r-1} \qquad . \qquad (3)$$

Note The formula (2) may conveniently be used in all cases *except when r is positive and greater than* 1

Example 1. Find the sum of $\frac{16}{27}-\frac{8}{9}+\frac{4}{3}-$&c to 7 terms.

The common ratio $= -\frac{8}{9} \div \frac{16}{27} = -\frac{8}{9} \times \frac{27}{16} = -\frac{3}{2}$

Hence, by formula (2),

$$\text{the sum} = \frac{\frac{16}{27}\{1-(-\frac{3}{2})^7\}}{1+\frac{3}{2}}$$

$$= \frac{\frac{16}{27}\{1+\frac{2187}{128}\}}{\frac{5}{2}} = \frac{16}{27} \times \frac{2315}{128} \times \frac{2}{5}$$

$$= \frac{463}{108} = 4\frac{31}{108}.$$

Example 2. Find the sum of

$3+4\frac{1}{2}+6\frac{3}{4}+$&c. to 5 terms

The common ratio $= 4\frac{1}{2} \div 3 = \frac{9}{2} \times \frac{1}{3} = \frac{3}{2}$.

Hence, if S denote the required sum, we have by formula (1),

$$S = \frac{3\,\{(\frac{3}{2})^5-1\}}{\frac{3}{2}-1} = \frac{3\,\{\frac{243}{32}-1\}}{\frac{1}{2}} = 3 \times \frac{211}{32} \times 2$$

$$= \frac{633}{16} = 39\frac{9}{16}.$$

Exercise (16).

1. Sum $1+3+9+27+$&c, to 12 terms
2. Sum $81-27+9-$&c., to 8 terms
3. Sum $2-4+8-$&c, to 10 terms.
4. Sum $\frac{4}{9}-\frac{1}{3}+\frac{1}{4}-$&c., to 5 terms.
5. Sum $2-4+8-$&c., to $2r$ terms
6. Sum $2\frac{1}{2}-1+\frac{2}{5}-$&c., to n terms
7. Show that the sum of n terms of a G. P, beginning with the pth term, is r^{p-q} times the sum of an equal number of terms of the same series beginning with the qth term.

3. If n be an integer and r a given proper fraction to prove that r^n diminishes as n increases.

Let $r = \frac{3}{7}$. Now, since $\frac{3}{7}$ of any number is undoubtedly less than that number,

$(\frac{3}{7})^2$ is less than $\frac{3}{7}$, because $(\frac{3}{7})^2 = \frac{3}{7}$ of $\frac{3}{7}$;
$(\frac{3}{7})^3$ is less than $(\frac{3}{7})^2$, because $(\frac{3}{7})^3 = \frac{3}{7}$ of $(\frac{3}{7})^2$;
$(\frac{3}{7})^4$ is less than $(\frac{3}{7})^3$, because $(\frac{3}{7})^4 = \frac{3}{7}$ of $(\frac{3}{7})^3$;
and so on.

Hence, it is clear that in the series $\frac{3}{7}$, $(\frac{3}{7})^2$, $(\frac{3}{7})^3$, $(\frac{3}{7})^4$, each term is less than the preceding ; which is briefly expressed by saying that $(\frac{3}{7})^n$ diminishes as n increases

Similarly the proposition may be proved for any other value of r which is less than 1.

Hence, generally speaking, if r has a given value less than 1, r^n diminishes as n increases.

Note From above it is quite clear that if r be a proper fraction r^n is very small when n is infinitely large

4. Geometrical series continued to infinity.

Let us consider the series a, ar, ar^2, ar^3, &c.

If S denote the sum to n terms, we have

$$S = \frac{a(1-r^n)}{1-r}$$

$$= \frac{a}{1-r} - \frac{ar^n}{1-r}.$$

If then r be a *proper fraction*, the larger n is, the smaller will (r^n and $\therefore$) $\frac{ar^n}{1-r}$ be ; hence by sufficiently increasing the value of n we can make $\frac{ar^n}{1-r}$ less than any assigned quantity, however small , and therefore *by sufficiently increasing the value of n, the sum of n terms of the series can be made to differ from $\frac{a}{1-r}$ by as small a quantity as we please.*

This statement is usually put thus :—*the sum of an infinite number of terms of the Geometrical Progression is* $\frac{a}{1-r}$;

or, more briefly, *the sum to infinity is* $\frac{a}{1-r}$.

Let us apply all these remarks to a particular example.

Consider the series 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, &c

Here $a = 1$, $r = \frac{1}{2}$, hence the sum to n terms

$$= \frac{1}{1-\frac{1}{2}}\left(1-\frac{1}{2^n}\right) = 2\left(1-\frac{1}{2^n}\right) = 2-\frac{1}{2^{n-1}}.$$

Now by taking n large enough, 2^{n-1} can be made as large as we please, and therefore $\frac{1}{2^{n-1}}$ as small as we please. Hence we may say that *by taking n large enough, the sum of n terms of the series can be made to differ from 2 by as small a quantity as we please*, or, briefly, *the sum of an infinite number of terms of this series is 2.*

N B It must be borne in mind that the sum of n terms of a Geometrical Progression *approaches a fixed limit as n increases indefinitely only when r is less than unity If r be greater than unity there is no such fixed limit*

Example 1. Prove that in a decreasing Geometrical Progression continued to infinity each term bears a constant ratio to the sum of all which follow it

Let the series be a, ar, ar^2, ar^3, &c., where r is less than unity.

Then the nth term $= ar^{n-1}$ and the sum of all the terms which follow this

$$= ar^n(1+r+r^2+r^3+\text{\&c. to infinity})$$

$$= ar^n.\frac{1}{1-r}.$$

Hence the ratio of the n^{th} term to the sum of all which follow it

$$= \left(ar^{n-1} \div \frac{ar^n}{1-r}\right) = \frac{1-r}{r}.$$

Now this is constant *whatever value n may have*, which proves the proposition.

Example 2. Sum to infinity $\frac{3}{2}-\frac{2}{3}+\frac{8}{27}-$&c.

Here $a=\frac{3}{2}$, and $r=-\frac{2}{3}\div\frac{3}{2}=-\frac{4}{9}$.

Hence, the required sum $=\dfrac{\frac{3}{2}}{1+\frac{4}{9}}$

$=\frac{3}{2}\times\frac{9}{13}$

$=\frac{27}{26}$.

Exercise (17).

Sum to infinity each of the following series :—

1. $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+$&c 2. $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+$&c.

3. $\frac{5}{8}+\frac{1}{2}+\frac{2}{5}+\frac{8}{25}+$&c 4. $1-\frac{2}{3}+\frac{4}{9}-$&c.

5. $3\frac{3}{8}+2\frac{1}{4}+1\frac{1}{2}+$&c

6. $\dfrac{2}{5}+\dfrac{3}{5^2}+\dfrac{2}{5^3}+\dfrac{3}{5^4}+$&c. [Split this up into two series.]

7. $\dfrac{4}{7}+\dfrac{5}{7^2}+\dfrac{4}{7^3}+\dfrac{5}{7^4}+$&c. 8. $\sqrt{3}+\dfrac{1}{\sqrt{3}}+\dfrac{1}{3\sqrt{3}}+$&c.

9. $(\sqrt{2}+1)+1+(\sqrt{2}-1)+$&c.

10. Find the common ratio of a G P. continued to infinity in which each term is ten times the sum of all the terms which follow it.

5. Recurring Decimals.—Recurring decimals furnish a good illustration of infinite Geometrical Progressions.

Thus, for example, $\cdot 2\dot{3}\dot{4}=\cdot 2343434\ldots\ldots\ldots$

$$=\left.\begin{array}{l}\cdot 2\\ +\cdot 034\\ +\cdot 00034\\ +\cdot 0000034\\ +\text{\&c., \&c.,}\end{array}\right\}=\frac{2}{10}+\frac{34}{10^3}+\frac{34}{10^5}+\frac{34}{10^7}+\text{\&c}$$

Here the terms after $\frac{2}{10}$ constitute a Geometrical Progression, of which the first term is $\frac{34}{10^3}$, and the common ratio $\frac{1}{10^2}$.

Hence, we may take $.2\dot{3}\dot{4} = \frac{2}{10} + \frac{34}{10^3} \div \left\{1 - \frac{1}{10^2}\right\}$

$$= \frac{2}{10} + \frac{34}{990} = \frac{232}{990},$$ which

agrees with the value found by the usual Arithmetical rule.

6. Geometric means *Definition 1* When three quantities are in Geometrical Progression the middle one is called the **Geometric mean** between the other two.

Definition 2 When any number of quantities x_1, x_2, x_3, &c are such that a, x_1, x_2, x_3, &c b are in G P, then x_1, x_2, x_3, &c. are called **Geometric means** between a and b

(i) To find the Geometric mean between two given quantities

Let a and b be the two given quantities; G the Geometric mean

Then since a, G, b are in G. P, we must have $\frac{G}{a} = \frac{b}{G}$, each being equal to the common ratio.

$\therefore \quad G^2 = ab$, and $\therefore \quad G = \sqrt{ab}$

(ii) To insert a given number of Geometric means between two given quantities

Let a and b be the two given quantities, and x_1, x_2, x_3, x_4, &c, x_n, the n means to be inserted

Then a, x_1, x_2, x_3, &c, x_n, b are in G P.

Let r denote the common ratio of the series;

then b = the $(n+2)$th term $= ar^{n+1}$,

$$\therefore \quad r^{n+1} = \frac{b}{a},$$

$$\text{and} \quad r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}.$$

Hence $x_1 = a\left(\frac{b}{a}\right)^{\frac{1}{n+1}}$, $x_2 = a\left(\frac{b}{a}\right)^{\frac{2}{n+1}}$, $x_3 = a\left(\frac{b}{a}\right)^{\frac{3}{n+1}}$

and so on.

Example. Insert 3 Geometric means between $\frac{1}{2}$ and 128.

Let x_1, x_2, x_3 be the means

Then $\frac{1}{2}, x_1, x_2, x_3$, are in G. P.

Hence if r be the common ratio of the series,
we must have $128 =$ the 5th term $= \frac{1}{2} r^4$,

$$\therefore r^4 = 256, \text{ whence } r = 4$$

$$\text{Hence, } \left.\begin{aligned} x_1 &= \tfrac{1}{2}\, 4 = 2 \\ x_2 &= \tfrac{1}{2}\, 4^2 = 8 \\ x_3 &= \tfrac{1}{2}\, 4^3 = 32 \end{aligned}\right\}$$

Exercise (18).

1. Insert 2 Geometric means between 3 and 24.
2. Insert 3 Geometric means between $2\frac{1}{4}$ and $\frac{4}{9}$.
3. Insert 4 Geometric means between $\frac{2}{3}$ and $-5\frac{1}{16}$.
4. Insert 5 Geometric means between $3\frac{5}{9}$ and $40\frac{1}{2}$.

7. Miscellaneous Examples.

Example 1 If $x<1$, sum the series

$$1+2x+3x^2+4x^3+\&c \text{ to infinity}$$

Let S denote the required sum, then,

$$S = 1+2x+3x^2+4x^3+\&c.$$

$$\text{and } \therefore Sx = \quad x+2x^2+3x^3+\&c.$$

Hence, by subtraction,

$$S(1-x) = 1+x+x^2+x^3+\&c \text{ to infinity}$$

$$= \frac{1}{1-x}.$$

$$\therefore S = \frac{1}{(1-x)^2}.$$

Example 2. Sum to n terms $5+55+555+\&c.$

Let S denote the required sum, then,

$$\begin{aligned} S &= 5+55+555+\&c \text{ to } n \text{ terms} \\ &= 5\{1+11+111+\&c \text{ to } n \text{ terms}\} \\ &= \tfrac{5}{9}\times 9\{1+11+111+\&c. \text{ to } n \text{ terms}\} \\ &= \tfrac{5}{9}\{9+99+999+\&c. \text{ to } n \text{ terms}\} \\ &= \tfrac{5}{9}\{(10-1)+(10^2-1)+(10^3-1)+\&c. \text{ to } n \text{ terms}\} \\ &= \tfrac{5}{9}\{(10+10^2+10^3+\&c. \text{ to } n \text{ terms})-n\} \end{aligned}$$

$$= \frac{5}{9}\left\{\frac{10(10^n-1)}{10-1}-n\right\}$$
$$= \frac{50}{81}\left(10^n-1\right)-\frac{5n}{9}.$$

Example 3 Sum to n terms $1+5+13+29+$&c

Let t_n denote the nth term of the series, and S the required sum, then

$$S = 1+5+13+29+\&c+t_n;$$
also $$S = 0+1+5+13+\&c+t^{n-1}+t_n$$

Therefore, by subtraction,

$$0 = (1+4+8+16+\&c \text{ to } n \text{ terms})-t_n,$$
$$\therefore\ t_n = 1+\{4+8+16+\&c. \text{ to } (n-1) \text{ terms}\}$$
$$= 1+\frac{4(2^{n-1}-1)}{2-1}$$
$$= 1+2^2\left(2^{n-1}-1\right) = 2^{n+1}-3.$$

Hence, the 1st. term $= 2^2-3$,

„ 2nd. „ $= 2^3-3$,

„ 3rd. „ $= 2^4-3$,

and so on

Hence, $S = (2^2-3)+(2^3-3)+(2^4-3)+\&c+(2^{n-1}-3).$
$$= (2^2+2^3+2^4+\&c. \text{ to } n \text{ terms})-3n$$
$$= \frac{2^2(2^n-1)}{2-1}-3n$$
$$= 4(2^n-1)-3n.$$

Example 4. If a, b, c, d be in G P, shew that
$$(a-d)^2 = (b-c)^2+(c-a)^2+(d-b)^2.$$

We have $\frac{b}{a} = \frac{c}{b} = \frac{d}{c}$, each of them being equal to the common ratio

$$\therefore\ b^2 = ac,\ c^2 = bd, \text{ and } bc = ad. \quad \ldots \quad (a)$$

Hence, $(b-c)^2+(c-a)^2+(d-b)^2$
$$= (b^2+c^2-2bc)+(c^2+a^2-2ca)+(d^2+b^2-2db)$$
$$= 2(b^2-ac)+2(c^2-bd)+a^2+d^2-2bc$$
$$= 2\times0+2\times0+a^2+d^2-2ad \quad [\text{by } a]$$
$$= (a-d)^2.$$

Example 5. If a, b, c, d be in G. P., shew that
$a^2-b^2,\ b^2-c^2,\ c^2-d^2$ are in G P.

Evidently $a^2-b^2,\ b^2-c^2,\ c^2-d^2$ are in G. P.,
if $(a^2-b^2)(c^2-d^2)=(b^2-c^2)^2$.

Now, since a, b, c, d are in G. P., we have

$$\frac{b}{a}=\frac{c}{b}=\frac{d}{c},$$

$$\therefore \quad ac=b^2,\ bd=c^2 \text{ and } ad=bc.$$

Hence,
$$\begin{aligned}(a^2-b^2)(c^2-d^2)&=a^2c^2-b^2c^2-a^2d^2+b^2d^2\\&=b^4-b^2c^2-b^2c^2+c^4\\&=b^4-2b^2c^2+c^4\\&=(b^2-c^2)^2.\end{aligned}$$

$\therefore\ a^2-b^2,\ b^2-c^2,\ c^2-d^2$ are in G. P.

Example 6. The continued product of three numbers in G P, is 216, and the sum of the products of them in pairs is 156, find the numbers

Let $\frac{a}{r}, a, ar$ be the numbers,

then, by the conditions given, we must have

$$\left.\begin{aligned}\frac{a}{r}\,a\,ar&=216 \quad \ldots (1)\\ \text{and}\quad \frac{a}{r}a+\frac{a}{r}.ar+a\,ar&=156 \quad \ldots (2)\end{aligned}\right\}$$

From (1) $\quad a^3=216, \quad \therefore\ a=6.$

Hence, from (2), $\quad \frac{1}{r}+1+r=\frac{156}{36}=\frac{13}{3}.$

$$\begin{aligned}\therefore\quad 3(1+r+r^2)&=13r,\\ \text{or,}\quad (3r^2-10r+3)&=0,\\ \text{or,}\quad (r-3)(3r-1)&=0,\\ \therefore\quad r&=3, \text{ or } \tfrac{1}{3}.\end{aligned}$$

Hence the numbers are 2, 6, 18.

Exercise (19).

1. Find by the method of summation of infinite Geometric series the value of ,—

(i) $\cdot 0\dot{2}\dot{7}$, (ii) $1\cdot 1\dot{4}\dot{5}$, (iii) $2150\dot{1}$, (iv) $\cdot\dot{1}4285\dot{7}$.

2. Sum $1+3x+5x^2+7x^3+$&c to infinity.

3. Sum $1\cdot 2x+2\cdot 4x^2+3\cdot 8x^3+$&c to infinity.

4. Sum $1\cdot 3x+4\cdot 9x^2+7\cdot 27x^3+$&c to infinity

5. Sum $a+2a^2+3a^3+4a^4+$&c to n terms.

6. Sum $1-3x+5x^2-7x^3+$&c to infinity

7. Sum $\frac{1}{3}+\frac{3}{9}+\frac{5}{27}+$&c to infinity

8. Sum $1+\dfrac{2}{2}+\dfrac{3}{2^2}+\dfrac{4}{2^3}+$&c to n terms

9 Find the nth term, and the sum to n terms, of the series —

$$1\cdot 1,\ 2\cdot 3,\ 4\cdot 5,\ 8\cdot 7,\ \&c$$

10. Sum $1+\dfrac{2}{5}+\dfrac{3}{5^2}+\dfrac{4}{5^3}+$&c to n terms

11. Sum to n terms $4+44+444+$&c

12. Sum the series $\cdot 9+\cdot 99+\cdot 999+$&c to n terms

13. Sum the series $1+3+7+15+$&c to n terms

14. Sum to n terms $-6-4+0+8+24+$&c.

15. Find the sum of $6+9+21+69+261+$&c to n terms.

16. If a, b, c, d be in G P, shew that

$$(a^2+b^2+c^2)(b^2+c^2+d^2) = (ab+bc+cd)^2.$$

[We have $\dfrac{a}{b}=\dfrac{b}{c}=\dfrac{c}{d}=k$ (say), thus $a=bk$, $b=ck$, $c=dk$, hence $a^2+b^2+c^2=k^2(b^2+c^2+d^2)$ and also $a^2+b^2+c^2=k(ab+bc+cd)$]

17. If a, b, c, d are in G P, prove that

(i) $(b+c)(b+d)=(c+a)(c+d)$;

(ii) $(a+d)(b+c)-(a+c)(b+d)=(b-c)^2$

18. Three numbers whose sum is 15 are in A P., if 1, 4 and 19 be added to them respectively, the results are in G P Determine the numbers

[Let $\alpha-\beta$, α, $\alpha+\beta$ be the numbers]

19. Three numbers whose product is 512 are in G. P.; if 8 be added to the first and 6 to the second, the numbers are in A P. Find the numbers

20. The sum of three quantities in G. P. is $24\frac{3}{4}$, and their product is 64, find them.

21. Find the relation between a, b, c that they may be the pth, qth and rth terms of a Geometric series.

22. If P and Q be the pth and qth terms of a Geometric series, find the nth term.

23. If S be the sum, P the product and R the sum of the reciprocals of n terms in G. P, prove that $P^2 = \left(\frac{R}{S}\right)$.

24. Find the sum of n terms of the series, the rth term of which is $(2r+1)2^r$.

25. In a G P. shew that the product of any two terms equidistant from a given term is always the same

[If the nth term be taken as the given term, it can be easily shown that the product of the $(n+p)$th and $(n-p)$th terms is independent of p]

26. If there be n terms in G P, prove that the nth root of their product is equal to the square root of the product of the first and last terms

27. If n Geometrical means be found between two quantities a and c, shew that their product will be $(ac)^{\frac{n}{2}}$.

28. If a, b, c, d are in G P, shew that the reciprocals of a^2-b^2, b^2-c^2, c^2-d^2 are also in G P.

29. If S_1, S_2, S_3, &c, S_n are the sums of infinite Geometric series, whose first terms are 1, 2, 3, &c, n, and whose common ratio are $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, &c, $\frac{1}{n+1}$ respectively, prove that

$$S_1+S_2+S_3+\&c.+S_n = \frac{n}{2}(n+3)$$

30. Find the sum of the infinite series—

$$1+(1+a)r+(1+a+a^2)r^2+(1+a+a^2+a^3)r^3+$$

&c, r and a being proper fractions.

CHAPTER V.

VARIATION.

1. Definition One quantity is said to *vary directly* as another when the two quantities are so related that if one of them be changed the other is changed *in the same ratio*; or, in other words, if a, a' be *any two* values of a quantity A, and b, b' the *corresponding* values of a second quantity B, then A is said to vary directly as B when $a : a' = b : b'$.

For instance, suppose the measure of the area of a triangle is a, when that of the base is b; now if, the height remaining unchanged, the base be increased to $2b$, then as we know from Geometry the area will become $2a$; if the base becomes $3b$, the area will be $3a$, and so on. Thus the height remaining the same if the base is doubled, trebled, quadrupled, &c., the area also becomes doubled, trebled, quadrupled, &c., (*i e*, the area changes *in the same ratio* as the base) and so we say that if the height of a triangle remains unaltered, the area *varies directly* as the base

Note 1 The word *directly* is often omitted, so that when we say A *varies as* B it is implied that A varies directly as B

Note 2 The symbol $\propto$ is used to express variation, thus $A \propto B$ stands for "A varies as B"

1. If A varies as B, then the numerical measure of *any* value of A and that of the *corresponding* value of B are in a constant ratio.

Let a_1, a_2, a_3, &c, be the measure of a series of values of A, and let b_1, b_2, b_3, &c, be the measure of the corresponding values of B

Then, by definition,

$$\frac{a_1}{a_2} = \frac{b_1}{b_2}, \frac{a_2}{a_3} = \frac{b_2}{b_3}, \frac{a_3}{a_4} = \frac{b_3}{b_4}, \text{ and so on.}$$

Hence, $\dfrac{a_1}{b_1} = \dfrac{a_2}{b_2} = \dfrac{a_3}{b_3} = \dfrac{a_4}{b_4} =$ &c., which proves the proposition

Note Putting m for each of the above ratios, we have $a_1 = mb_1$, $a_2 = mb_2$, $a_3 = mb_3$, and so on Thus when A varies as B, the numerical measure of *any* value of A is equal to that of the corresponding value of B multiplied by a constant This result is briefly expressed as follows —"If $A \propto B$, then $B = mB$ where m is a constant"

3. Definitions.—(1) One quantity A is said to *vary inversely* as another B, when A varies directly as the reciprocal of B.

Thus if A varies inversely as B, $A = \dfrac{m}{B}$, where m is constant.

Illustration .—If 20 men do a certain work in 4 hours, 10 men would do it in 8 hours, 40 men in 2 hours , and so on. Thus when the number of men *diminishes*, the time *proportionately increases* ; and *vice-versa*. This is expressed by saying that if the amount of work to be done remains constant, the number of men varies inversely as the time.

(2) One quantity is said to *vary jointly* as a number of others, when it varies directly as their product. Thus if A varies jointly as B and C, $A = m\,BC$, where m is constant.

Illustration :—The monthly income of a day labourer varies jointly as his daily earning and the number of days he works in a month.

(3) A is said to vary directly as B and inversely as C when A varies jointly as B and the reciprocal of C, that is, when $A = m.\dfrac{B}{C}$, where m is constant.

Illustration —The time of travelling a distance varies directly as the distance and inversely as the speed of travelling.

4. An Important Theorem—

If A varies as B when C is constant, and A varies as C when B is constant, then will A vary as BC when both B and C vary

Suppose a_1 is the value of A when b_1 is that of B, and c_1 that of C. Suppose also that a_2 is the value of A when b_2 is that of B, and c_2 that of C. Then the proposition will be proved if we can show that $a_1 : a_2 = b_1c_1 : b_2c_2$.

Now, the change of A from a_1 to a_2, is due to *two* causes, namely,

(1) the change of B from b_1 to b_2, and (2) the change of C from c_1 to c_2.

Hence, it is clear that if *one* only of these causes be present (*i e*, if either B or C *alone* undergoes the supposed change), A will change from a_1 to some value which is *different* from a_2. Let therefore a' be the value of A when b_2 is that of B, and c_1 that of C

Thus we have the value of A

$= a_1$ when those of B and C are respectively b_1 and c_1 (1)

$= a'$, when those of B and C are respectively b_2 and c_1 (2)

$= a_2$ when those of B and C are respectively b_2 and c_2 (3)

Hence, from (1) and (3), we see that A changes from a_1 to a', when B changes from b_1 to b_2, C *remaining constant* (*i e*, retaining the value c_1), and therefore, by hypothesis,

$$\frac{a_1}{a'} = \frac{b_1}{b_2}, \quad \ldots \quad \ldots \qquad (\alpha)$$

and from (2) and (3) we see that A changes from a' to a_2 when C changes from c_1 to c_2, B *remaining constant* (*i e*, retaining the value b_2), and therefore, by hypothesis,

$$\frac{a'}{a_2} = \frac{c_1}{c_2} \quad \ldots \quad \ldots \quad \ldots \qquad (\beta)$$

Hence, from (α) and (β) $\frac{a_1}{a'} \times \frac{a'}{a_2} = \frac{b_1}{b_2} \times \frac{c_1}{c_2}$

$$\text{or,} \quad \frac{a_1}{a_2} = \frac{b_1 c_1}{b_2 c_2}$$

which proves the proposition

Illustration —(1) Suppose that a number of plants have to be watered the quantity of water bestowed evidently varies directly as the number of men employed *if the time for watering remains unchanged*, and also it varies directly as the number of hours for which the men can work, *if the number of men engaged remains the same*, hence if the number of men and the number of hours be both variable, the quantity of water will vary as the product of the number of men and the number of hours

(2) The area of a triangle varies directly as the base when the height is constant, and it also varies directly as the height when the base is constant, hence when both the base and the height are variable the area varies as the product o the numbers which express the base and the height

Cor If there be any number of quantities B, C, D, &c., each of which varies as another A when the rest are constant, then if they are all variable, A varies as their product.

5. Some Results worth remembering—

(1) If $A \propto B$ and $B \propto C$, then $A \propto C$

For, let $A = mB$, and $B = nC$, where m and n are constants, then $A = mnC$, and $\therefore$ as mn is constant, $A \propto C$.

(2) If $A \propto C$, and $B \propto C$, then $A \pm B \propto C$, and $\sqrt{AB} \propto C$

For, let $A = mC$, and $B = nC$, where m and n are constants, then $A + B = (m+n)C$, and $A - B = (m-n)C$;
$\therefore$ $(A \pm B) \propto C$

Also $\sqrt{AB} = \sqrt{mn\,C^2} = C\sqrt{mn}$, $\therefore$ $\sqrt{AB} \propto C$.

(3) If $A \propto BC$, then $B \propto \dfrac{A}{C}$, and $C \propto \dfrac{A}{B}$.

For, let $A = mBC$, then $B = \dfrac{1}{m} \cdot \dfrac{A}{C}$; $\therefore$ $B \propto \dfrac{A}{C}$.

Similarly $C \propto \dfrac{A}{B}$

(4) If $A \propto B$, and $C \propto D$, then $AC \propto BD$

For, let $A = mB$, and $C = nD$, then $AC = mnBD$;
$\therefore$ $AC \propto BD$.

(5) If $A \propto B$, then $A^n \propto B^n$.

For, let $A = mB$, then $A^n = m^n B^n$;
$\therefore$ $A^n \propto B^n$,

(6) If $A \propto B$, then $AP \propto BP$, where P is *any* quantity *variable* or *invariable*.

For, let $A = mB$, then $AP = mBP$; $\therefore$
$AP \propto BP$.

6 Examples—Application of the principles explained in some of the preceding articles will be illustrated by the following examples.

Example 1. If y varies as x, and $y = 5$ when $x = 12$, find the value of y when $x = 18$.

By supposition $y = mx$ where m is constant.

Putting $y = 5, x = 12$, we have

$$5 = m\,12, \quad \therefore \quad m = \frac{5}{12}.$$

Hence, x and y are connected by the relation $y = \frac{5}{12}x$.

Hence, when $x = 18$, we have $y = \frac{5}{12}18 = \frac{15}{2} = 7\frac{1}{2}$

Example 2. If z varies as $px+y$, and if $z = 3$ when $x = 1$ and $y = 2$, and $z = 5$ when $x = 2$ and $y = 3$, find p.

By supposition $z = m(px+y)$ where m is constant.

Putting $z = 3$, $x = 1, y = 2$, we have

$$3 = m(p+2) \quad \ldots \quad \ldots \quad (1)$$

Again putting $z = 5, x = 2, y = 3$, we have

$$5 = m\,(2p+3) \quad \ldots \quad (2),$$

Hence, from (1) and (2), by division,

$$\frac{3}{5} = \frac{p+2}{2p+3}, \text{ whence } p = 1.$$

Example 3. If y = the sum of 3 quantities, of which the 1st. $\propto x^2$, the 2nd $\propto x$, and the 3rd is constant; and when $x = 1, 2, 3$, $y = 6, 11, 18$, respectively, find the *equation* between x and y

By suppoition $y = mx^2+nx+p$, where m, n, p are constants

Now since $y = 6$ when $x = 1$, we have

$$\left.\begin{aligned} 6 &= m+n+p \quad \ldots \quad \ldots \quad (1)\\ \text{Similarly,}\quad 11 &= 4m+2n+p \quad \ldots \quad \ldots \quad (2)\\ \text{and}\quad 18 &= 9m+3n+p \quad \ldots \quad \ldots \quad (3) \end{aligned}\right\}$$

From (1) and (2) by subtraction,

$$3m+n = 5 \quad \ldots \quad \ldots \quad (4)$$

Similarly from (2) and (3),

$$5m+n = 7 \quad \ldots \quad \ldots \quad (5)$$

Now subtracting (4) from (5), we have

$$2m = 2, \quad \therefore \quad m = 1,$$

hence from (4), $n = 2$, $\therefore$ from (1), $p = 3$

Hence, the equation between x and y is $y = x^2+2x+3$.

Example 4. If $a+b \propto a-b$, prove that $a^2+b^2 \propto ab$; and if $a \propto b$, prove that $a^2-b^2 \propto ab$.

(i) By supposition, $a+b = m(a-b)$ where m is constant.

Hence, $$(a+b)^2 = m^2(a-b)^2,$$

or, $$a^2+b^2+2ab = m^2(a^2+b^2-2ab)$$

$$\therefore \quad (m^2-1)(a^2+b^2) = 2ab(1+m^2),$$

$$\therefore \quad a^2+b^2 = \frac{2(m^2+1)}{m^2-1}\,ab.$$

But $\dfrac{2(m^2+3)}{m^2-1}$ is constant, $\therefore$ $a^2+b^2 \propto ab$

(ii) Since $a = mb$,

multiplying both sides by a, we have

$$a^2 = m.ab \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

and also multiplying both sides by b, we have

$$b^2 = \frac{ab}{m} \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

Subtracting (2) from (1)

$$a^2-b^2 = \left(m-\frac{1}{m}\right)ab, \text{ where } \left(m-\frac{1}{m}\right) \text{ is constant,}$$

$$\therefore \quad a^2-b^2 \propto ab.$$

Example 5. The wages of 5 men for 6 weeks being £14 5*s*, how many weeks will 4 men work for £19 ?

Let x denote the wages (in pounds), earned by y men in z weeks.

Then evidently $x \propto y$ when z is constant,

and also $x \propto z$, when y is constant ;

$\therefore$ when y and z are both variable, $x \propto yz$,

i.e., $x = m.yz$, where m is constant.

Now, since $x = 14\frac{1}{4}$, when $y = 5$ and $z = 6$,

$$\therefore \quad 14\tfrac{1}{4} = m \times 5 \times 6 \quad \ldots \quad \ldots \quad (1)$$

Also, if z_1, denote the required number of weeks, then, since the corresponding values of x and y are respectively 19 and 4, we have

$$19 = m \times 4 \times z_1 \quad \ldots \quad (2)$$

Hence, dividing (1) by (2),

$$\frac{3}{4} = \frac{5 \times 6}{4 \times z_1}, \text{ whence } z_1 = 10;$$

i e, the required time = 10 weeks.

Example 6. Assuming that the quantity of work done varies as the cube root of the number of agents when the time is the same, and varies as the square root of the time when the number of agents is the same, find how long 3 men would take to do one-fifth of the work which 24 men can do in 25 hours

Let x denote the quantity of work done by y men in z hours

Then by supposition,

$$x \propto y^{\frac{1}{3}} \text{ when } z \text{ and } \therefore z^{\frac{1}{2}} \text{ is constant;}$$

and also, $x \propto z^{\frac{1}{2}}$ when y and $\therefore y^{\frac{1}{3}}$ is constant.

Hence, when both y and z and $\therefore y^{\frac{1}{3}}$ and $z^{\frac{1}{2}}$ are variable,

$$x \propto y^{\frac{1}{3}}z^{\frac{1}{2}}, \text{ i.e., } x = k.y^{\frac{1}{3}}z^{\frac{1}{2}}, \text{ where } k \text{ is constant.}$$

Now, since by the problem,

$$x = 1, \text{ where } y = 24 \text{ and } z = 25,$$

$$\therefore \quad 1 = k\sqrt[3]{24}.\sqrt{25} \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

Also, if z_1 be the required number of hours, since the corresponding values x and y are respectively $\frac{1}{5}$ and 3, we have

$$\tfrac{1}{5} = k\sqrt[3]{3}\ \sqrt{z_1} \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

Hence, dividing (1) by (2),

$$5 = \frac{\sqrt[3]{24} \times 5}{\sqrt[3]{3} \times \sqrt{z_1}} = \frac{\sqrt[3]{8} \times 5}{\sqrt{z_1}}$$

$\therefore \quad \sqrt{z_1} = 2$, and $\therefore \quad z_1 = 4$,

i.e., the required time = 4 hours

Example 7. A sphere of metal is known to have a hollow space about its centre in the form of a concentric sphere, and its weight is $\frac{7}{8}$ of the weight of a solid sphere of the same substance and radius; compare the inner and outer radii, having given that the weights of spheres of the same substance $\propto$ (radii)^3

Let R be the outer radius and W the weight of a solid sphere of the given metal of radius R, also let r be the inner radius (i.e., radius of the spherical cavity), and w the weight of a solid sphere of the given metal of radius r

Then by hypothesis,

$W = KR^3$, and $w = Kr^3$, where K is constant,

Now, since $(W - w)$ is the weight of the given sphere, we have, by the question $W - w = \frac{7}{8} W$, hence we must have

$$K(R^3 - r^3) = \frac{7}{8} KR^3.$$

$$\therefore \frac{1}{8}R^3 = r^3 \text{ whence } \frac{r}{R} = \frac{1}{2}$$

Exercise (20).

1. If $y \propto x$, and $y = 5$ when $x = 15$, find the *equation* between x and y.

2. If $y \propto x$, and $y = 10$ when $x = 25$, find y when $x = 35$

3. If P varies inversely as Q, and $Q = 10$ when $P = 2$, what will P become when $Q = 8$?

4. If $P \propto QR$, and the three corresponding values of P, Q, R be 6, 9, 10 respectively find the values of P when $Q = 5$ and $R = 3$.

5. If the square of x vary as the cube of y, and $x = 2$ when $y = 3$, find the *equation* between x and y.

6. Given that y varies as the sum of two quantities, one of which varies as x directly, the other as x inversely, and that $y = 4$ when $x = 1$, and $y = 5$ when $x = 2$, find

7. If $xy \propto x^2+y^2$, and $y = 4$ when $x = 3$, find the *equation* between x and y.

8 Given that y is equal to the sum of two quantities one of which varies as x, and the other varies inversely as x^2, and when $x = 1, 2$, $y = 6, 5$ respectively. Find the *equation* between x and y.

9 If $y =$ the sum of 3 quantities, of which the 1st is constant, the 2nd $\propto x$, and the 3rd $\propto x^2$, also when $x = 3, 5, 7$, $y = 0, -12, -32$ respectively, find the *equation* between x and y

10. Given that $y^2 \propto a^2-x^2$ and when $x = \sqrt{a^2-b^2}$, $y=\frac{b^2}{a^2}$, find the *equation* between x and y.

11. If $y = r+s$, whilst $r \propto x$, and $s \propto \sqrt{x}$, and if when $x = 4$, $y = 5$, and when $x = 9$, $y = 10$, shew that $6y = 5(x+\sqrt{x})$.

12. Assuming that the time of oscillation of a pendulum varies as the square root of its length, if the length of a pendulum which oscillates once in a second be 39 2 inches, find the length of one which oscillates 56 times in a minute.

13 If 13 men earn £7 in 15 days of 8 hours each what will be the wages of 52 men for $12\frac{1}{2}$ days of 9 hours each ?

14 Given that the volume of a sphere varies as the cube of its radius, prove that the volume of a sphere whose radius is 6 inches is equal to the sum of the volumes of three spheres whose radii are 3, 4, 5 inches

15 The volume of a pyramid varies jointly as its height and the area of its base, and when the area of the base is 60 square feet and the height 14 feet, the volume is 280 cubic feet What is the area of the base of a pyramid whose volume is 390 cubic feet and whose height is 26 feet ?

16 Given that the area of a circle varies as the square of its radius, and that the area of a circle is 154 square feet when the radius is 7 feet, find the area of a circle whose radius is 10 feet 6 inches

17 If the volume of a cone whose height is 12 inches and base 30 square inches be 120 cubic inches, find the

volume of another whose height is 20 inches and base 1 square foot the volume of a cone varying as the height and base jointly.

18 The volume of a circular cylinder varies as the square of the radius of the base when the height is the same, and as the height when the base is the same The volume is 88 cubic feet when the height is 7 feet, and the radius of the base is 2 feet, what will be the height of a cylinder on a base of radius 9 feet, when the volume is 396 cubic feet ?

19 Two circular gold plates, each an inch thick, the diameters of which are 6 inches and 8 inches respectively, are melted and formed into a single circular plate one inch thick Find its diameter, having given that the area of a circle varies as the square of its diameter.

20 Given that the illustration from a source of light, *varies inversely* as the *square* of the distance, how much further from a candle must a book, which is now three inches off, be removed, so as to receive just half as much light ?

21. A solid spherical mass of glass, 1 inch in diameter, is blown into a shell bounded by two concentric spheres, the diameter of the outer one being 3 inches Calculate the thickness of the shell (The volume of a sphere varies directly as the cube of its diameter).

22. When a body falls from rest, its distance from the starting point varies as the square of the time it has been falling, if a body falls through $402\frac{1}{2}$ feet in 5 seconds, how far does it fall in 10 seconds ? Also how far does it fall in the 10th second ?

23 If 10 men can reap a field of $7\frac{1}{2}$ acres, in 3 days of 12 hours each, how long will it take 8 men to reap 9 acres, working 16 hours a day ?

24 If $x+y \propto x-y$, shew that $x^2+y^2 \propto xy$ and $x^3+y^3 \propto xy(x \pm y)$.

25 If $x \propto \dfrac{1}{y}$, prove that $x+y$ is least when $x = y$.

ANSWERS TO EXERCISES IN THE APPENDIX

1. [Page 502.]

2. Take BE equal to AD; by guess let F be the middle point of DE. Then F is very approximately the middle point of AB, the error, if any, being indefinitely small.

7. 2·56, 1·68, 3·79, 2·39, 1·40.

2. [Pages 505, 506.]

1. $6\frac{2}{3}$ units of length. 2. $7\frac{4}{5}$ feet. 3. $7\frac{1}{7}$ yards.
4. 3·5 inches. 5. 3·6 feet. 6. $\frac{7}{10}$ ft. 7. 5 yards.
8. 65 feet. 9. 17 feet. 10. 28·3 feet.

4. [Pages 510—512.]

1. (i) $(11, 7)$, $(-9, 13)$, $(-5, -7)$; $(8, -10)$.
   (ii) $(2·2, 1·4)$, $(-1·8, 2·6)$, $(-1, -1·4)$; $(1·6, -2)$.
2. $(3\frac{2}{3}, 2\frac{1}{3})$; $(-3, 4\frac{1}{3})$, $(-1\frac{2}{3}, -2\frac{1}{3})$, $(2\frac{2}{3}, -3\frac{1}{3})$.
5. 20. 6. 13. 7. 50. 8. 11, −13.
9. 17·5, 36. 10. 12, 8.

5. [Pages 518, 519.]

7. (1) $6x-5y=0$, (2) $5x+7y=35$;
   (3) $x+y+2=0$, (4) $21x-5y+124=0$;
   (5) $5x+9y+55=0$.

6. [Page 520.]

1. $x=5, y=4$. 2. $x=7, y=-5$. 3. $x=8, y=6$. 4. $x=9, y=11$. 5. $x=10, y=13$.

6. [Take ten times the side of a small square as the unit of length.] $x=1·2$.

7. $x=7$. 8. $x=7$.

7. [Pages 527, 528.]

1. 13 as. 3 pies, 2 seers 11 chattacks.
2. 1 Re. 9 as. 6 p., 19. 3. $3\frac{1}{4}$ hours, 19 miles.
4. $8\frac{5}{8}$ feet, $4\frac{1}{2}$ cubits. 5. $2\frac{1}{2}$ hours after A starts, $7\frac{1}{2}$ miles from the place of starting.

6. 4 hours after starting, 12 miles from A.

7. 1 Re. 8 as, 39 8. 5. 11. At 4-30 P. M, $13\frac{1}{2}$ miles from B

9. [Page 542.]

1. 16, 40, $2n-6$ 2. 29th, 46th, $(3n-10)$th. 3. 6.
4. 98

10. [Page 544]

1. 325. 2. 900. 3. $52\frac{1}{2}$. 4. 0. 5. 25452

6. $\frac{1}{2}(n-1)$. 7. $\dfrac{n}{a+b}\left\{na-\dfrac{n-1}{2}b\right\}$.

11. [Pages 546, 547.]

1. 3 2. 9. 3. 7. 4. 13 or 7. 5. Last term 8, or -1, number of terms 10 or 12.

6. 18 or 19. 7. n^2. 8. 1, $\frac{5}{4}$, $\frac{3}{2}$, $\frac{7}{4}$, 2, &c., 1470.
9. 1, 3, 5, 7, &c., n^2. 10. 2. 11. 4 or 10.

12. [Pages 548, 549]

1. $6\frac{1}{2}$; 8; m, a^2+x^2. 2. $9\frac{1}{3}$, $10\frac{2}{3}$, $\frac{2}{3}$, $7\frac{1}{3}$
3. 207, 297, 387 4. -2, -6, -10, -11.
5. 1, $-1\frac{1}{2}$, &c., -39.

13. [Page 552]

1. $\dfrac{n}{2}(6n^2+3n-1)$. 2. $\dfrac{n(n+1)(n+2)(3n+5)}{12}$.

3. $\dfrac{n}{3}(4n^2+6n-1)$. 4. $n^2(2n^2-1)$

5. $\dfrac{n(n+1)(n+2)}{6}$

14. [Pages 556, 558.]

1. $\dfrac{ma-nb}{a-b}(2n-1)$. 2. 9, 13, 17, 21, 25

3. 13; 6 4. 70. 5. $\dfrac{n(n+1)(n+2)}{6}$.

6. $\dfrac{n}{6}(2n^2+3n+7)$. 7. $\dfrac{n}{6}(2n^2+9n+1)$

8 $\dfrac{n}{3(2n+3)}$ 9 8, 12, 16, 20.

10 3, 5, 7 **11.** 1, 3, 5, 7. **12.** 3, 5, 7, 9, 11, 13.

14 $\dfrac{a}{p}(q-r)+\dfrac{b}{q}(r-p)+\dfrac{c}{r}(p-q) = 0.$

15 $\dfrac{P(r-n)+Q(m-r)}{m-n}$ **16.** 16.

19 $\dfrac{n(n+1)(n+2)}{6}$. 20 $\frac{1}{3}(n-1).n(2n-1)$ yards

15. [Page 559]

1 8748 **2.** $\frac{4}{9}$ **3.** 65536. **4.** -243

5. $\dfrac{8}{27}$, $\pm\dfrac{2^{n-3}}{3^{n-3}}$, + or − according as n is even or odd.

6. $-\frac{448}{243}$ **7.** $\frac{1}{25}$, $\frac{1}{125}$

16. [Page 560.]

1. 265720. **2.** $60\frac{20}{27}$. **3.** -682

4. $\frac{181}{576}$ **5.** $\frac{2}{3}(1-1^{2r})$

6. $\dfrac{1}{14}\,\dfrac{5^n \mp 2^n}{5^{n-2}}$, − or + according as n is even or odd.

17. [Page 563]

1. 1. **2.** $\frac{2}{3}$ **3.** $3\frac{1}{8}$ **4** $\frac{3}{5}$. **5.** $10\frac{1}{8}$.

6 $\dfrac{13}{24}$. **7.** $\dfrac{33}{48}$ **8.** $\dfrac{3\sqrt{3}}{2}$ **9.** $\frac{1}{2}(4+3\sqrt{2})$.

10 $\dfrac{1}{11}$.

18. [Page 565.]

1. 6, 12. **2.** $\frac{3}{2}$, 1, $\frac{2}{3}$ **3.** -1, $\frac{3}{2}$, $-\frac{9}{4}$, $\frac{27}{8}$

4. $\frac{16}{3}$, 8, 12, 18, 27

19. [Pages 568, 569.]

1 $\dfrac{1}{36}$. $1\dfrac{8}{55}$, $\dfrac{358}{1665}$, $\dfrac{1}{7}$ **2.** $\dfrac{1+x}{(1-x)^2}$

3. $\dfrac{2x}{(1-2x)^2}$. 4 $\dfrac{(1+6x)3x}{(1-3x)^2}$

5. $\dfrac{a(1-a^n)}{(1-a)^2}-\dfrac{na^{n+1}}{1-a}$. 6. $\dfrac{1-x}{(1+x)^2}$. 7. 1

8. $4-\dfrac{n+2}{2^{n-1}}$. 9. $2^{n-1}(2n-1)$; $2^n(2n-3)+3$.

10. $\dfrac{5^{n+1}-5-4n}{16.5^{n-1}}$ 11. $\dfrac{40}{81}(10^n-1)-\dfrac{4n}{9}$.

12 $n-\dfrac{1}{9}.\left(1-\dfrac{1}{10^n}\right)$. 13 $2^{n+1}-2-n$

14. $2(2^n-1-4n)$ 15. $\frac{1}{3}(4^n-1+15n)$

18. 2, 5, 8. 19. 4, 8, 16.

20. $\frac{4}{5}$, 4, 20. 21. $a^{q-r}\,b^{r-p}\,c^{p-q}=1$.

22. $\left(\dfrac{P^{n-q}}{Q^{n-p}}\right)^{\frac{1}{p-q}}$ 24. $n\,2^{n+2}-2^{n+1}+2$.

30. $\dfrac{1}{(1-r)(1-ar)}$.

20 [Pages 579—582.]

1. $3y=x$. 2. 14. 3. $2\frac{1}{2}$ 4 1. 5 $27x^2=4y^3$.

6. $y=2x+\dfrac{2}{x}$. 7 $xy=\dfrac{12}{25}\left(x^2+y^2\right)$.

8. $y=2x+\dfrac{4}{x^2}$. 9. $y=3+2x-x^2$.

10. $y=\dfrac{b}{a^2}\sqrt{a^2-x^2}$.

12 45 inches. 13. £26 5s 15. 45 sq ft.
16. $346\frac{1}{2}$ sq. ft. 17. 960 cubic ins 18. $1\frac{5}{9}$ ft
19. 10. 20. 1 2426 ins nearly. 21. ·01875 in
22. 1610 feet , 305 9 feet. 23. 3 days, 6 hours.

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